Entanglement Entropy at 2D quantum critical points, topological fluids and quantum Hall fluids

Talk at the 2nd INSTANS Summer Conference
Exact Results in Low-Dimensional Quantum Systems
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Collaborators and References

- Stefanos Papanikolaou, Kumar Raman, Benjamin Hsu, Shiying Dong, Robert G. Leigh and Sean Nowling (UIUC),
- Michael Mulligan and Eun-Ah Kim (Stanford), Joel E. Moore (University of California Berkeley), Paul Fendley (Virginia)

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    Evidence for $q = e/4$ vortex.
    Shot noise @ point contact in a 2DEG (Heiblum)
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- The effective action that respects the symmetries is Abelian Chern-Simons gauge theory $U(1)_m$ for the hydrodynamic gauge field $\mathcal{A}_\mu$

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- Matching the correlation functions of the RK and Lifshitz models, one finds $\kappa = \frac{1}{8\pi}$.

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- Universal $O(1)$ term in *topological phases* in 2D

$$S = \alpha\ L - \gamma + O(L^{-1}),$$

Kitaev and Preskill, Levin and Wen

α is non universal and γ depends only on topological invariants

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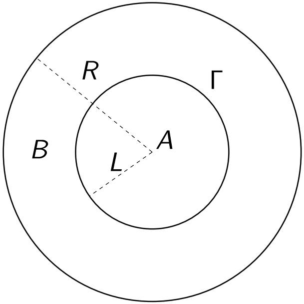

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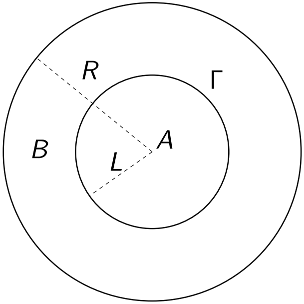

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with Shying Dong, Sean Nowling and Rob Leigh (2008)

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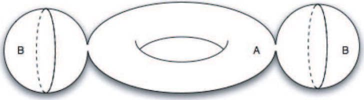

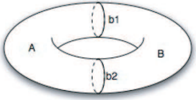

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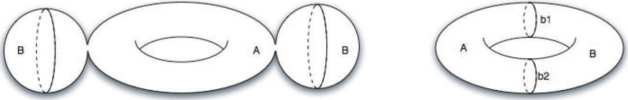

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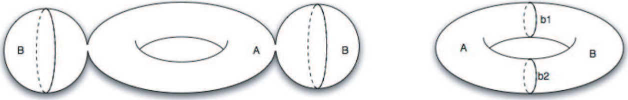

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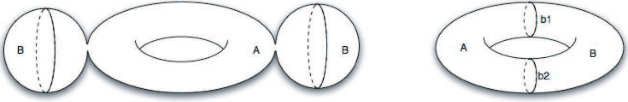

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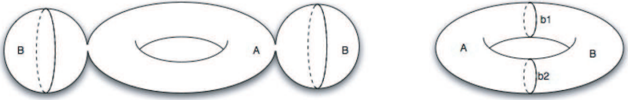

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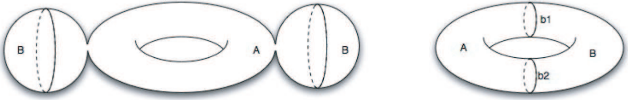

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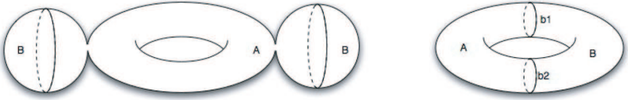

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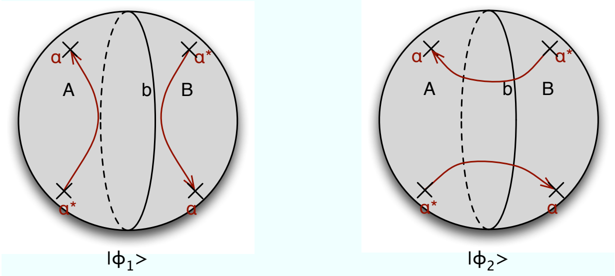

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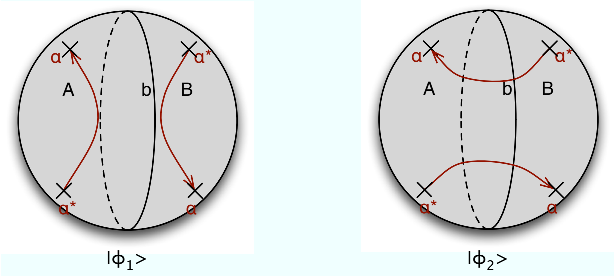

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Reduced density matrix: $\rho_A = aa^*\rho_{11} + ab^*\rho_{12} + a^*b\rho_{21} + bb^*\rho_{22}$

Entanglement Entropy: $S_A = \ln \mathcal{S}_0{}^0 - \lambda_1 \ln \lambda_1 - (d_{\hat{\alpha}}^2 - 1)\lambda_2 \ln \lambda_2$

$$\lambda_1 = \frac{|ad_{\hat{\alpha}} + b|^2}{|ad_{\hat{\alpha}} + b|^2 + (d_{\hat{\alpha}}^2 - 1)|b|^2}, \quad \lambda_2 = \frac{|b|^2}{|ad_{\hat{\alpha}} + b|^2 + (d_{\hat{\alpha}}^2 - 1)|b|^2}.$$

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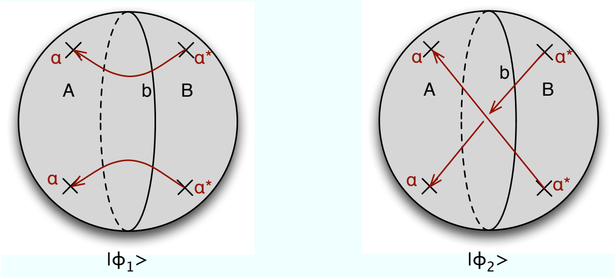

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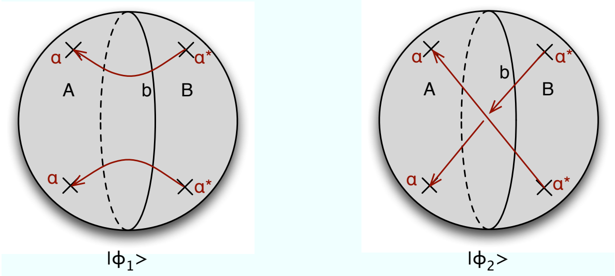

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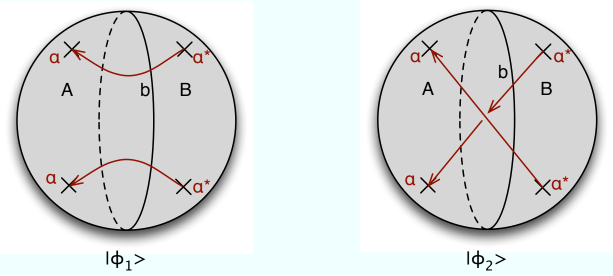

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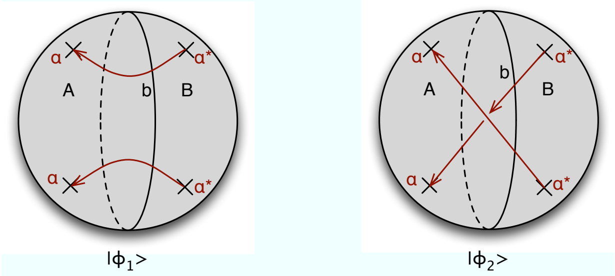

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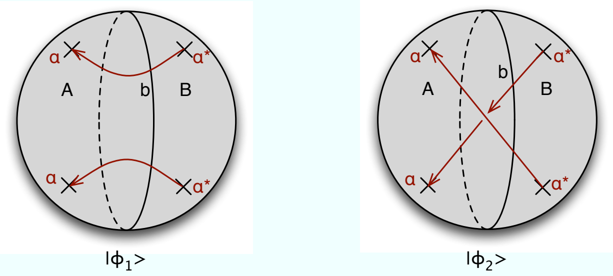

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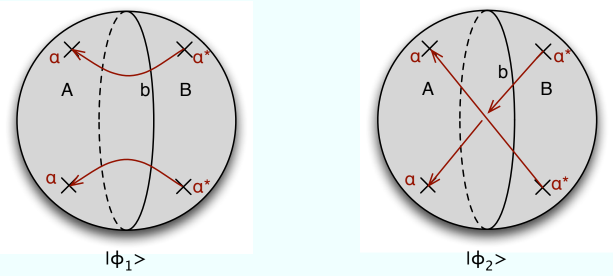

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