Theory of Oblique Topological Insulators

Eduardo Fradkin
University of Illinois

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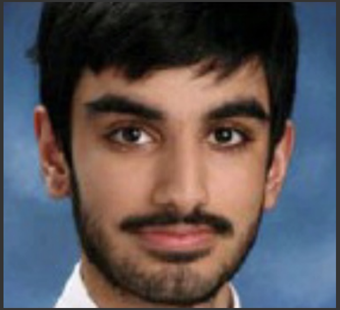
Collaborators

Benjamin Moy
Illinois

Hart Goldman
MIT



Raman Sohal
Princeton

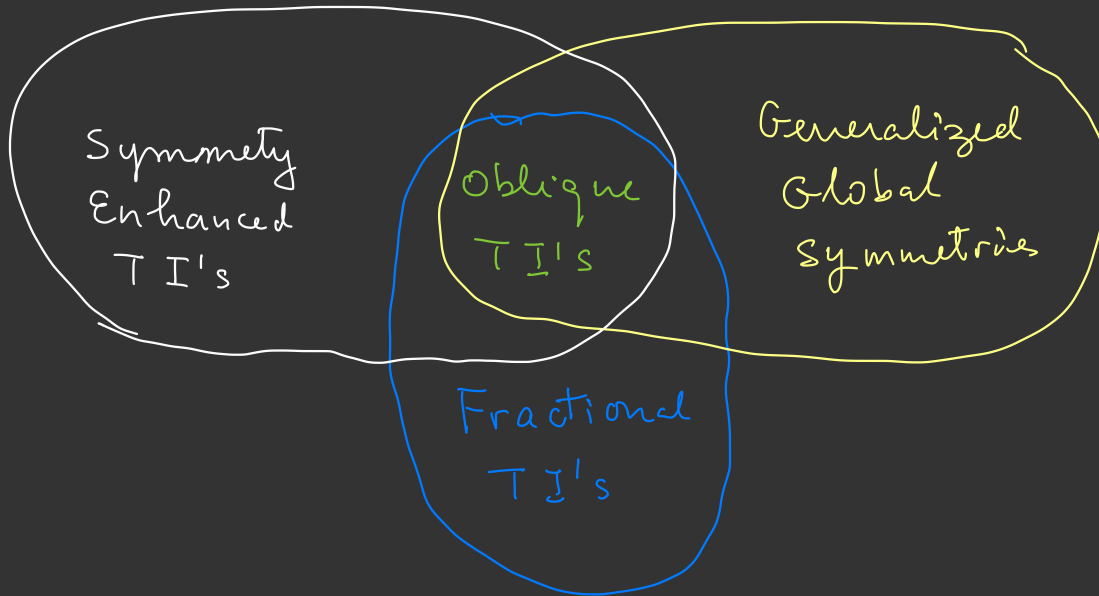
* Much of what we know about 3D ("3+1") TI's is based on time-reversal-invariant free fermion topological insulators.
* Extensions of these concepts to systems with bulk topological order often involve fine-tuned lattice models and/or parton constructions (which often require uncontrolled approximations).
* The relation between bulk and edge topological orders (and their types) is poorly understood and their universal responses are not known.

This Talk

* We consider a simple $\mathbb{Z}_N$ lattice gauge theory in 3+1 dimensions with a rational θ angle

* We examine the topological orders of its oblique confined phases, both bulk and boundary

* We construct effective hydrodynamic TQFT's of these phases and of their gapped surface states



Symmetry Enhanced TI's
Oblique TI's
Generalized Global symmetries
Fractional TI's

Phases of Gauge Theories

Confined

electric charges are confined

Wilson Loops obey the area law

$$W_\Gamma = \langle e^{i\oint_\Gamma A} \rangle \sim e^{-\sigma_W \text{Area}(\Gamma)}$$

'tHooft loops obey a Perimeter Law

T_Γ : "disorder operator" that inserts the fundamental flux unit

$$T_\Gamma \sim e^{-\lambda_T L(\Gamma)}$$

Deconfined

magnetic monopoles are deconfined

Wilson Loops obey the Perimeter Law

$$W_\Gamma \sim e^{-\lambda_W L(\Gamma)}$$

'tHoof Loops obey an Area Law

$$T_\Gamma \sim e^{-\sigma_T \text{Area}(\Gamma)}$$

'tHooft: θ - angles induce <u>oblique confinement</u> through <u>dyon condensation</u>

Generalized Global Symmetries

* Global continuous symmetries have associated conserved currents

e.g. $\phi'(x) = e^{i\alpha}\phi(x) \Leftrightarrow U(1) \Leftrightarrow j^\mu / \partial_\mu j^\mu = 0$

Noether's Thm. conservation law: $Q = \int d^3x\, j_0(x)$ is a constant of motion

* Gauge theories have global symmetries

e.g. $U(1)$ gauge Theory (compact in the sense of flux quantization)

Eqn of motion: $\partial_\mu F^{\mu\nu} = j^\nu$ ✓

If $j^\nu = 0 \Rightarrow \partial_\mu F^{\mu\nu} = 0 \Rightarrow J^{\mu\nu} \propto F^{\mu\nu}$ conserved

$Q_E^i = \int d^3x\, F^{0i}$ is conserved

counts # of electric field lines

Bianchi Identity $\partial_\mu F^{\mu\nu *} = 0$, $F^{\mu\nu *} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$

(no monopoles) $Q_M^i = \int d^3x\, F^{0i *}$ # of magnetic field lines is conserved

⑦

$U(1)_E$ global symmetry $\qquad W_\Gamma \to W_{\Gamma'} = e^{i\alpha_E} W_\Gamma$ ✓

$U(1)_M$ global symmetry $\qquad T_\Gamma \to T'_\Gamma = e^{i\alpha_M} T_\Gamma$ ✓

Zero form symmetry: $\quad \phi \to \phi' = e^{i\alpha}\phi \qquad U(1)$

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$$

✓

one form symmetry $\quad A_\mu \to A'_\mu = A_\mu + \lambda_\mu$

(two-form field) $\quad B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$

$$\oint_\Gamma dx_\mu \, \lambda^\mu = \alpha_E$$

(loosely speaking) W_Γ and T_Γ behave as "order parameters"

(subtleties omitted)

$\mathbb{Z}_N$ gauge theories with rational θ-angle

(Cardy & Rabinovici 1982)

Use the Villain form of the $\mathbb{Z}_N$ gauge theory

$$Z = \mathrm{Tr}\, \exp\left[-\frac{1}{4g^2}\sum_{(x,\mu\nu)} \Gamma_{\mu\nu}^2 - \frac{iN\theta}{32\pi^2}\sum_{x,\tilde{x}} f(x-\tilde{x})\, \varepsilon_{\mu\nu\lambda\rho}\Gamma_{\mu\nu}\Gamma_{\lambda\rho} + iN\sum_{(x,\mu)} n_\mu a_\mu\right]$$

dual

breaks the symmetry to $\mathbb{Z}_N$

$\Gamma_{\mu\nu} \equiv \Delta_\mu a_\nu - \Delta_\nu a_\mu - 2\pi S_{\mu\nu}$; $a_\mu \in \mathbb{R}$

$S_{\mu\nu} = -S_{\nu\mu} \in \mathbb{Z}$, $f(x-\tilde{x})$ is short-ranged

$n_\mu \in \mathbb{Z}$ "particle worldlines" with charge N

* $a'_\mu = a_\mu + 2\pi P_\mu$; $S'_{\mu\nu} = S_{\mu\nu} + \Delta_\mu P_\nu - \Delta_\nu P_\mu$ (periodicity)

* Global one-form symmetry $a_\mu \to a_\mu + \frac{2\pi}{N}\eta_\mu$, $\eta_\mu \in \mathbb{Z}$, $\Delta_\mu \eta_\nu - \Delta_\nu \eta_\mu = 0$

(electric) $W_\Gamma \longrightarrow \omega\, W_\Gamma$ / $\omega = \exp\left(\frac{2\pi i k}{N}\right)$

Coulomb Gas Picture: charges and monopoles

Integrating a_μ out

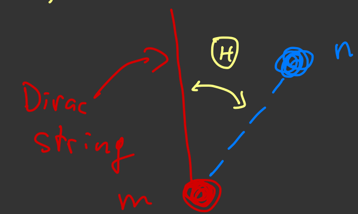
$$Z = \mathrm{Tr}\,\exp\Big[-\frac{2\pi^2}{g^2}\sum_{\tilde{x},\tilde{x}'} m_\mu(\tilde{x})\, G(\tilde{x}-\tilde{x}')\, m_\mu(\tilde{x}')$$

$$-\frac{1}{2}N^2 g^2 \sum_{x,x'}\Big(n_\mu(x)+\frac{\theta}{2\pi}m_\mu(x)\Big)\, G(x-x')\,\Big(n_\mu(x')+\frac{\theta}{2\pi}m_\mu(x')\Big)$$

$$+\, iN\sum_{\tilde{x},x} m_\mu(\tilde{x})\, \Theta_{\mu\nu}(\tilde{x}-x)\, n_\nu(x)$$

n_μ: particle worldlines

$m_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\Delta_\nu S_{\lambda\rho}$: monopole worldlines



(n, m) dyon of electric charge $N\left(n+\frac{\theta}{2\pi}m\right)$ and magnetic charge m

witten effect

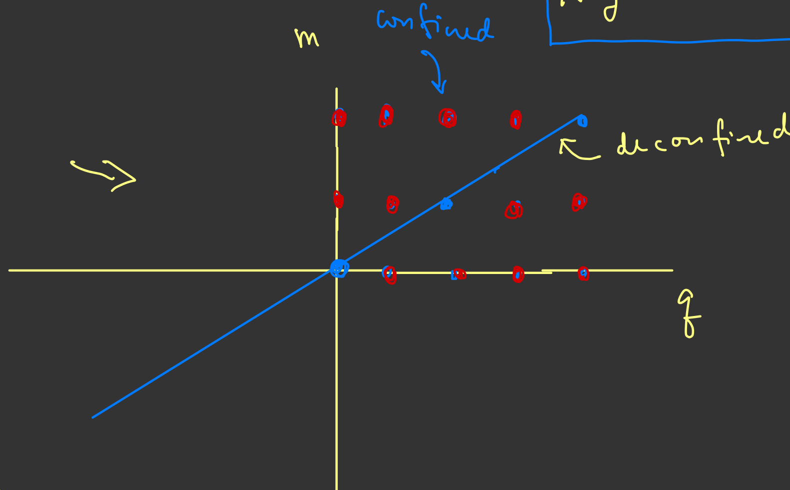
Dyon Condensation and Phase Diagram

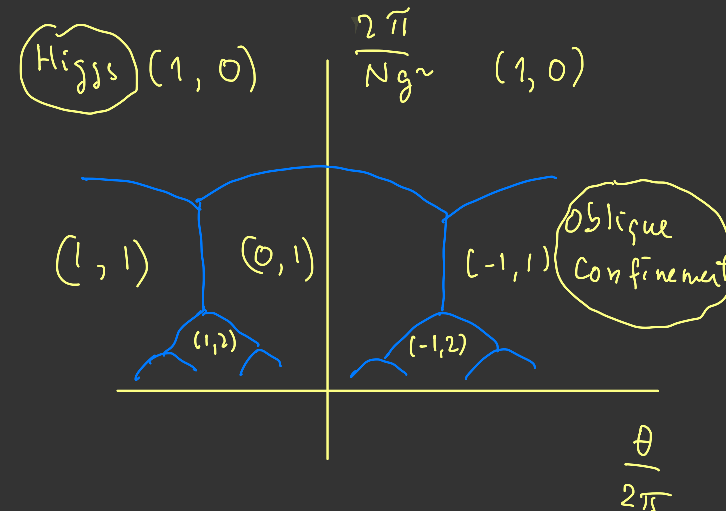
(n,m) dyon condenses if

$$\boxed{\frac{2\pi}{Ng^2}m^2 + \frac{Ng^2}{2\pi}\left(n + \frac{\theta}{2\pi}m\right)^2 < \frac{C}{N}}$$

Cardy Rabinovici

C : non-universal constant





Problem

The statistics of an (n,m) bosonic anyon is $(-1)^{Nnm}$

Fermions cannot condense ! $\Rightarrow$ Only bosonic dyons can condense $\Rightarrow$ (n,m) condenses if Nnm is even

or fermion pairing $\Rightarrow$ $(2n, 2m)$

UV Duality and Modular Symmetry

* Define: $\tau = \frac{\theta}{2\pi} + i\frac{2\pi}{Ng^2}$

* S duality: $(n,m) \longrightarrow (-m,n)$; $\tau \to -\frac{1}{\tau}$ (duality)
* T duality: $(n,m) \longrightarrow (n-m,n)$; $\tau \longrightarrow \tau+1$ (periodicity)

* S and T generate $PSL(2,\mathbb{Z})$ $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$ Modular Transf.

$a,b,c,d \in \mathbb{Z}$; $ad-bc \neq 0$

$$Z[\tau] = Z\left[-\frac{1}{\tau}\right] \quad \text{and} \quad Z[\tau] = Z[\tau+1]$$

* Generalization of Kramers–Wannier duality
* This symmetry generates the phase diagram

Observables of the Oblique Confined Phases

* Wilson loops $\langle W[\Gamma]\rangle$ (Γ: closed curve) (worldline of electric charges)
* 't Hooft loops $\langle T[\Gamma]\rangle$ worldline of magnetic monopole

* dyons $\langle W[\Gamma]^{q'}\, T[\Gamma]^{m'}\rangle$
* In general they obey the area law $\Longrightarrow$ confined
* For $m'Nn - q'm = 0$ they obey a perimeter law $\Rightarrow$ deconfined

* # of inequivalent deconfined dyons is $L = \gcd(Nn, m)$

$\langle W[\Gamma]^{Nnk/L}\, T[\Gamma]^{mk/L}\rangle$; $k = 0, \dots, L-1$ (deconfined $\mathbb{Z}_L$ gauge theory)

* There are also surface operators : worldsheets of flux tubes

* electric flux tube $\Phi_E(\Sigma) \Rightarrow e^{i\Phi_E \oint_\Sigma F}$ ← measures the magnetic flux
* magnetic flux tube $\Phi_M(\Sigma) \Rightarrow e^{i\Phi \oint_\Sigma F^*}$ measures the electric flux
* # of inequivalent surface ops. in the (n, m) phase is L

Cardy - Rabinovici with a two-form probe field $B_{\mu\nu}$

* define a (background) $\mathbb{Z}_N$ two-form probe field $B_{\mu\nu} \equiv \frac{2\pi}{N} s_{\mu\nu}$, $s_{\mu\nu} \in \mathbb{Z}$

locally flat: $\frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} \Delta_\nu s_{\lambda\rho} = 0$

* shift $\Gamma_{\mu\nu} \to \Gamma_{\mu\nu} + \frac{2\pi}{N} s_{\mu\nu}$ in the CR action

global one-form symmetry: $a_\mu \to a_\mu + \frac{2\pi}{N} \xi_\mu$ $\quad (\xi_\mu \in \mathbb{Z})$

$$s_{\mu\nu} \to s_{\mu\nu} + \Delta_\mu \xi_\nu - \Delta_\nu \xi_\mu$$

We will see that the $\mathbb{Z}_N$ gauge theory has a universal "magnetoelectric" response to $B_{\mu\nu}$

Effective Hydrodynamic Theory of Oblique Confinement

* Formally go to $g^2 \to \infty$ and set $\theta = -2\pi \frac{n}{m}$ $\quad (\gcd(n,m)=1)$

$$\Rightarrow \quad \frac{n}{m} m_\mu = n_\mu$$

$\to$ m_μ: monopole current; n_μ: electric current $\qquad l_\mu(x) \in \mathbb{Z}$

$$Z \simeq \operatorname{Tr} \exp\left[-\frac{2\pi i}{m} \sum_x m_\mu(x)\, \underline{l_\mu(x)} + i \frac{Nnm}{4\pi} \sum_x \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho} \frac{2\pi S_{\mu\nu}(x)}{m} \frac{2\pi S_{\lambda\rho}(x)}{m} \right]$$

* Continuum limit $\quad \frac{2\pi l_\mu}{m} \to \underline{\tilde{a}_\mu} \quad ; \quad \frac{2\pi S_{\mu\nu}}{m} \to \underline{\tilde{b}_{\mu\nu}}$

(Kapustin – Seiberg)

TQFT: $\quad Z = \int \mathcal{D}\tilde{a}\, \mathcal{D}\tilde{b}\; e^{-S_{dual}(\tilde{a},\tilde{b})}$

$$S_{dual}(\tilde{a},\tilde{b}) = -\frac{im}{2\pi} \int \tilde{b} \wedge d\tilde{a} - \frac{iNnm}{4\pi} \int \tilde{b} \wedge \tilde{b}$$

$\tilde{a}_\mu$: one-form $U(1)$ gauge field ; $\tilde{b}_{\mu\nu}$: two-form $U(1)$ gauge field

Gauge Invariance and IR electromagnetic duality

$$* \begin{cases} \tilde{b} \to \tilde{b} - d\lambda \\ \tilde{a} \to \tilde{a} + Nn\lambda \end{cases} \qquad \lambda: \text{one-form field}$$

* S_{dual} changes by $\quad 2\pi i m \int \frac{d\lambda}{2\pi} \wedge \frac{d\tilde{a}}{2\pi} + i N n m \int \frac{d\lambda}{2\pi} \wedge \frac{d\lambda}{2\pi}$

* On a closed manifold $\begin{cases} \text{the 1st term is } 2\pi i \times \text{integer} \\ \text{the 2nd term is } 2\pi i \times \text{integer } \underline{\text{if}} \ \underline{Nnm \text{ is even}} \end{cases}$ (i.e. (n,m) is a boson)

* Duality: $\quad S = -\frac{im}{2\pi} \int \tilde{b}\, d\tilde{a} + i\frac{Nnm}{2\pi} \int b \wedge \tilde{b} + i\frac{Nnm}{4\pi} \int b \wedge b$

$$b \to b + d\lambda$$

Integrate $\underline{\tilde{a}_\mu}$ out $\Rightarrow \quad S = \frac{iNn}{2\pi} \int b\, da + \frac{iNnm}{4\pi} \int b \wedge b$

$$\underset{(Nn,\, m)}{\frac{iNn}{2\pi} \int b \wedge da + \frac{iNnm}{4\pi} \int b \wedge b} \longleftrightarrow \underset{(m,\, -Nn)}{-\frac{im}{2\pi} \int \tilde{b}\, d\tilde{a} - i\frac{Nnm}{4\pi} \int \tilde{b} \wedge \tilde{b}}$$

Global Symmetries of the TQFT

One-form $\mathbb{Z}_{Nn}$

$$a \to a + \frac{1}{Nn}\epsilon^{(1)}\,, \quad d\epsilon^{(1)} = 0\,, \quad \oint \epsilon^{(1)} = 2\pi\mathbb{Z} \quad \text{(electric)}$$

$$\tilde{a} \to \tilde{a} + \frac{1}{m}\tilde{\epsilon}^{(1)}\,, \quad d\tilde{\epsilon}^{(1)} = 0\,, \quad \oint \tilde{\epsilon}^{(1)} = 2\pi\mathbb{Z}$$

Two-form

$$a \to a - \frac{m}{J}\hat{\epsilon}^{(1)}\,, \quad \tilde{a} \to \tilde{a} + \frac{Nn}{J}\hat{\epsilon}^{(1)}\,, \quad f \to f - \frac{m}{J}\epsilon^{(2)} \qquad \epsilon^{(2)} = d\epsilon^{(1)}$$

$$b \to b + \frac{1}{J}\hat{\epsilon}^{(2)}\,, \quad \tilde{b} \to \tilde{b} - \frac{1}{J}\epsilon^{(2)}\,, \quad \tilde{f} \to \tilde{f} + \frac{Nn}{J}\epsilon^{(2)}$$

x On a spin manifold $\Delta S = 0 \pmod{2\pi i}$ <u>if</u> $J = \gcd(Nn, m) = L$

$\Rightarrow$ $\mathbb{Z}_L$ two-form global symmetry

* on a non-spin manifold $J = \frac{L}{2}$ and $\mathbb{Z}_{L/2}$ (fermions)

Bulk Topological Order

Observables involve Wilson loops attached to surface operators

Σ surface with boundary $\Gamma = \partial \Sigma$

$$\Rightarrow W[\Sigma]^q = \exp\left(i q \oint_\Gamma a + i m q \int_\Sigma b \right) \qquad q \in \mathbb{Z}$$

(two-form gauge inv.)

$\Rightarrow$ a particle (dyon) of charge q is attached to the end of a string of magnetic flux $\Phi_M = \frac{mq}{Nn}$ $\Rightarrow$ q has magnetic charge $\frac{mq}{Nn}$

These operators have non-trivial linking

However $W[\Gamma] = W[\Sigma]^{\frac{Nn}{L}} = \exp\left(i \frac{Nn}{L} \oint_\Gamma a + i \frac{Nnm}{L} \int_\Sigma b \right)$ $\quad \frac{Nn}{L}$

$L: \gcd(Nn, m)$ $\Rightarrow$ the surface op. is invisible (quantization)

$\Rightarrow$ These are the genuine ops. $\Rightarrow$ global $\mathbb{Z}_{Nn} \longrightarrow \mathbb{Z}_{Nn/L}$ (electric)

$\mathbb{Z}_m \rightarrow \mathbb{Z}_{m/L}$ (magnetic)

* **Closed surface operators**

$$U(\Sigma)^k = e^{ik\oint_\Sigma b}$$

cycles of b are $\frac{2\pi}{Nn} \times \mathbb{Z} \Rightarrow k \sim k + Nn$

If $k = m \times \mathbb{Z} \Rightarrow k \sim k + m$ [when $k \propto m$, the surface op. can be obeyed and terminate on a dyon (as in the lattice model)

$\Rightarrow$ # of ineq. ops. is $k = 0, 1, \ldots, L-1$

Braiding: * $W[\Gamma]$ has statistics $(-1)^{Nnm/L^2}$

* braiding between particles and loops

$$\langle W[\Gamma]^q \, U(\Sigma)^k \rangle = \exp\left(2\pi i \frac{q}{L} k \, l(\Gamma, \Sigma)\right)$$

$l(\Gamma, \Sigma)$: linking #

$\Rightarrow$ dyon $W[\Gamma]^q$ circling once around $U(\Sigma)^k$ has an Aharonov-Bohm phase $e^{i\frac{2\pi}{L} qk}$ $\Rightarrow$ bulk topological order is the same as in $\mathbb{Z}_L$

Surface Topological Order and Anomaly Inflow

* If the manifold M has a boundary $\partial M \Rightarrow$ gauge anomaly
* There are at least two surface TQFT's that cancel the bulk anomaly

The bulk TQFT action changes by a boundary contribution

$$\Delta S_{dual} = \frac{i Nnm}{4\pi} \int_{\partial M} \lambda \, d\lambda + \frac{i m}{2\pi} \int_{\partial M} \lambda \wedge d\tilde{a}$$

Introduce a one-form gauge field c_μ at ∂M / $c_\mu \to c_\mu - \lambda_\mu$

$$S_{\partial M} = \frac{i Nnm}{4\pi} \int_{\partial M} c \wedge dc + \frac{i m}{2\pi} \int_{\partial M} c \wedge d\tilde{a}$$

$\Rightarrow S_{dual} + S_{\partial M}$ is invariant

$$S = \int_M \left(\frac{im}{2\pi} d\tilde{b} \wedge \tilde{a} - \frac{i\, Nmn}{4\pi} \tilde{b} \wedge \tilde{b} \right) + \int_{\partial M} \left(\frac{i\, Nm}{4\pi} c \wedge dc + \frac{im}{2\pi} c \wedge d\tilde{a} - \frac{im}{2\pi} \tilde{b} \wedge \tilde{a} \right)$$

$\Rightarrow$ EOM of $\tilde{a}$ in M $\Rightarrow$ $d\tilde{b} = 0$ locally

while globally $\oint \tilde{b} = \frac{2\pi}{m}\mathbb{Z}$; This boundary state is equivalent to $\underline{\tilde{b}|_{\partial M} = 0}$

$$\tilde{U}(\Sigma) = \exp\left(i\oint_\Gamma c - i\int_\Sigma \tilde{b}\right), \quad \tilde{W}(\Gamma) = \exp\left(i\oint_\Gamma \tilde{a} + iNn\oint_\Gamma c\right)$$

$\Gamma = \partial\Sigma$ lies in ∂M while Σ lies in M

$\Rightarrow$ genuine operators $\tilde{U}(\Sigma)^m$ (s.t. $\int_\Sigma \tilde{b}$ is invisible)

but EOM for $\tilde{a}$ $\Rightarrow$ $\tilde{b}|_{\partial M} = dc$ $\Rightarrow$ $\tilde{U}(\Sigma)^m$ is trivial

$\Rightarrow$ The <u>only</u> genuine ops. at ∂M are $\tilde{W}(\Gamma)$

$\tilde{W}(\Sigma)$ is a magnetic flux tube ($\Phi_M = 1$) that terminates @ ∂M

It represents a monopole of electric charge $\frac{Nn}{m}$ (+ Witten effect)

Statistics: $\langle \tilde{W}(\Gamma)^k \, \tilde{W}(\Gamma')^{k'} \rangle = \exp\left(2\pi i \frac{kk'Nn}{m} \, l(\Gamma, \Gamma')\right)$

Surface Topological Order

For $0 \leq k \leq \frac{m}{L} - 1 \Rightarrow \tilde{W}(\Gamma)$ is an anyon at ∂M

$\tilde{W}(\Gamma)^{m/L}$ has charges $\left(\frac{Nn}{L}, \frac{m}{L}\right) \Rightarrow$ same as the bulk $W(\Gamma)$

Other boundary ops. are fusion of a bulk g.p. and an anyon with $0 \leq k \leq \frac{m}{L} - 1$

There are L bulk Wilson loops $\Rightarrow$ # boundary loop ops. is $\frac{m}{L} \times L = m$

$\Rightarrow$ there are fewer types of anyons @ ∂M than naively expected due to the one-form symmetry and the undetectability of the surface op.

$\Rightarrow$ Also the # of surface anyons is m and has $\tilde{b}|_{\partial M} = 0$

This violates the CR duality

$\Rightarrow$ There is a alternative surface theory with Nn anyons with $\tilde{a}|_{\partial M} = 0$

CR duality revisited

✓ Under CR duality $S \to \tau \to -\frac{1}{\tau}$ and $(n,m) \to (-m,n)$

$T \to \tau \to \tau+1$ and $(n,m) \to (n-m,m)$

✓ These transf. map Oblique Confined phases with ≠ topological orders

✓ Bulk topological order is determined by $L = \gcd(Nn, m)$, $(-1)^{Nnm/L^2}$

✓ S is a symmetry of Z but not of the correlators

Want a genuine S duality that exchanges $m \leftrightarrow nN$ (not just n)

IR EM duality

IR duality

$\tilde{S}: (Nn, m) \to (m, -Nn)$ preserve $\underline{h} = \gcd(Nn, m)$

$\tilde{T}: (Nn, m) \to (Nn - m, m)$

✓ $\tilde{S}$ preserves the statistics while $\tilde{T}$ bosons $\leftrightarrow$ fermions
but $\tilde{T}^2$ leaves their statistics invariant

✓ same as bosons have a Θ angle periodicity of 4π
and fermions of 2π

✓ Bulk Topological Order invariant under $\tilde{S}$

✓ Boundary Topological Order in <u>not</u> invariant

✓ $\tilde{S}$ maps $\neq$ boundary states into each other

Bulk Higer Form Magnetoelectric Effect

* Response to a background flat 2-form field $B_{\mu\nu}$

$$S_{eff}(B) = -i \frac{Nn}{\underbrace{4\pi m}} \int (da + B) \wedge (da + B) \leftarrow$$

$\Rightarrow$ vortex current $\Sigma = \frac{Nn}{2\pi m} * (da + B)$

$\Rightarrow$ $\frac{Nn}{2\pi m}$ is an universal topological index of the oblique confined phase (n, m)

References

* The phase diagram of $\mathbb{Z}_N$ gauge theories (without a θ term) was studied by Elitzur, Pearson, and Shigemitsu (1979) and by Ukawa, Windey and Guth (1980)
* Oblique confinement was introduced by 't Hooft (1981)
* Cardy and Rabinovici studied the oblique phases of $\mathbb{Z}_N$ gauge theories with a rational θ angle (1982)
* von-Keyserlingk and Burnell discussed the relation with Walker-Wang models (2015)
* Gaiotto, Kapustin, Seiberg and Willett introduced generalized global symmetries (2015)
* Qi, Hughes, Zhang introduced the magnetoelectric effect as a feature of a TI (2008)

.... and many more!