Lectures on Duality in Condensed Matter Physics

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Outline

- Electromagnetic Duality and the duality of forces
- Ising Models: Kramers - Wannier dualities
- Gauge theory and duality
- Vortices and Monopoles
- Particle - Vortex duality
- Boson - Fermion mappings as duality
- Duality and the Fractional Quantum Hall Effect

Electromagnetic Duality (Dirac, 1931)

$$\vec{\nabla}\cdot\vec{E} = \rho\,, \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \vec{j}\,, \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

* EM Duality : $\vec{E} \longleftrightarrow \vec{B}$ (if $\rho = 0$ and $\vec{j} = 0$)

* electric charge e $\longleftrightarrow$ magnetic monopole m

* Dirac quantization: $em = 2\pi$

* $$\underset{\substack{\uparrow \\ \text{field tensor}}}{\partial_\mu F^{\mu\nu}} = \underset{\substack{\nwarrow \\ \text{charge current}}}{j^\nu}\;; \qquad \underset{\substack{\uparrow \\ \text{Bianchi Identity}}}{\partial^\mu F^*_{\mu\nu} = 0}\;; \qquad \underset{\substack{\uparrow \\ \text{dual field}}}{F^*_{\mu\nu}} \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$$

* Bianchi Identity $\Longleftrightarrow$ absence of magnetic monopoles

Duality of Forms

- Electromagnetic duality is geometric
- In Differential Geometry a vector field A_μ is a 1-form
- The field tensor $F_{\mu\nu}$ is a 2-form
- The dual field $F^*_{\mu\nu}$ is also a 2-form
- In d-dimensions a p-form (antisymmetric tensor of rank p) is dual to a d-p form
- In $d = 4$ dimensions a 2-form is dual to a 2-form
- In $d = 2$ dimensions a 1-form is dual to a 1-form

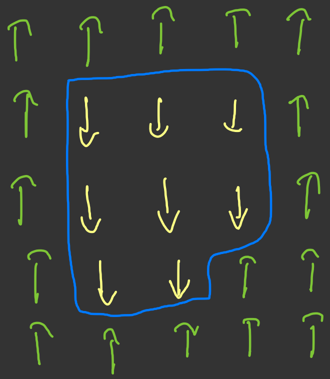
$\partial_\mu \phi = \varepsilon_{\mu\nu} \partial^\nu \psi$ Cauchy-Riemann!

Duality in Ising Models

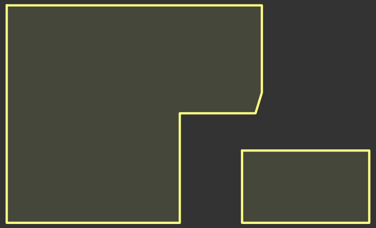
- 2D: the Ising model is (1941) self-dual (Kramers-Wannier)
- Low T: Z is an expansion in closed domain wall loops; weight ~ $e^{-2/T} \times$ length
- High T: Z is an expansion in closed loops weight ~ $\tanh(1/T) \times$ length
- maps high T ⟷ low T
  disorder ⟷ order
- self-dual: $e^{-2/T_c} = \tanh(1/T_c)$

$$\Longrightarrow \quad \frac{1}{T_c} = \frac{1}{2} \ln(\sqrt{2}+1)$$

(Onsager, 1944)



domain walls (low T)

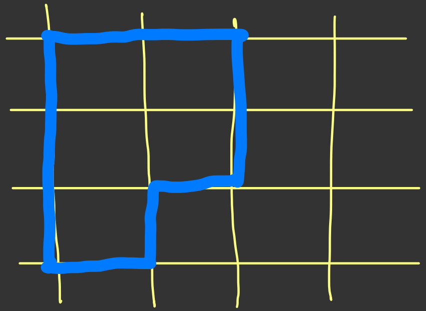


high T expansion diagrams

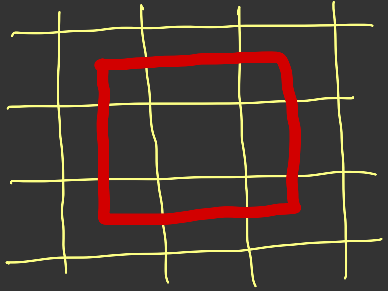
$$Z = \sum_{[\sigma]} e^{\frac{1}{T} \sum_{\langle x,x' \rangle} \sigma(x)\sigma(x')}$$

$$Z_{DW}\left[e^{-\frac{2}{T}}\right] = Z_{\text{loops}}\left[\tanh \frac{1}{T}\right]$$

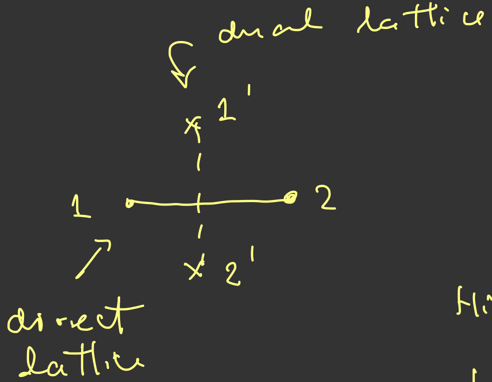
- The high-T_c loops live on the direct lattice
- The low-T_c loops (domain walls) live on the dual lattice



direct loops



dual loops (domain walls)



In 2d $\begin{cases} \text{links are dual to links} \\ \text{sites are dual to plaquettes} \end{cases}$

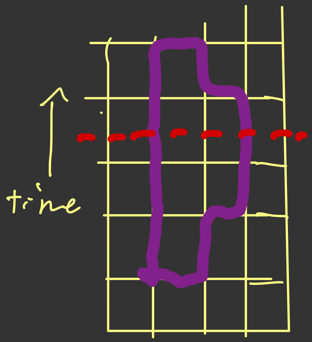
This is the same as the duality of forms

High T: $Z = \sum_{\{loops\}} \left(\tanh \frac{1}{T}\right)^{L(loops)}$ #

Low T: $Z = \sum_{\substack{domain \\ walls}} \left(e^{-\frac{2}{T}}\right)^{L(DW)}$ #

Field Theory Interpretation

- * We can regard the P. F. Z as a sum over histories of spin configurations from one row to the next row
- * Path integral in a discretized imaginary time
- * High T expansion loops ⟺ processes in which pairs of particles are created and destroyed



- * Low T expansion loops ⟺ processes in which pairs of domain walls are created and destroyed
- * Analog of the (imaginary) time evolution operator is the transfer matrix
- * The classical d-dimensional Ising Model is equivalent to a quantum Ising Model in d-1 dimensions

The Quantum Ising Model (EF & L. Susskind)

$\boxed{d=1}$ $$H = -\sum_j \sigma_1(j) - \lambda \sum_j \sigma_3(j)\,\sigma_3(j+1)$$ PBC's

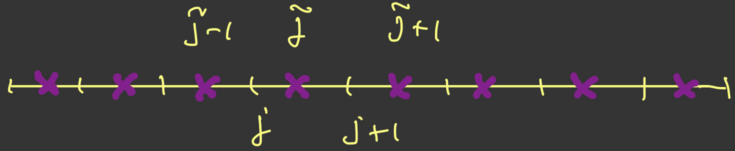
ⓐ λ small <=> T high <=> disorder

ⓑ λ large <=> T low <=> order

Global $\mathbb{Z}_2$ symmetry: $Q = \prod_j \sigma_1(j)$; $[Q, H] = 0$

flipping all spins

Duality:



dual lattice: midpoints

$$\tau_1(\tilde{j}) = \sigma_3(j)\,\sigma_3(j+1) \,; \quad \tau_3(\tilde{j}) = \prod_{n \le j} \sigma_1(n)$$ ← creates domain walls

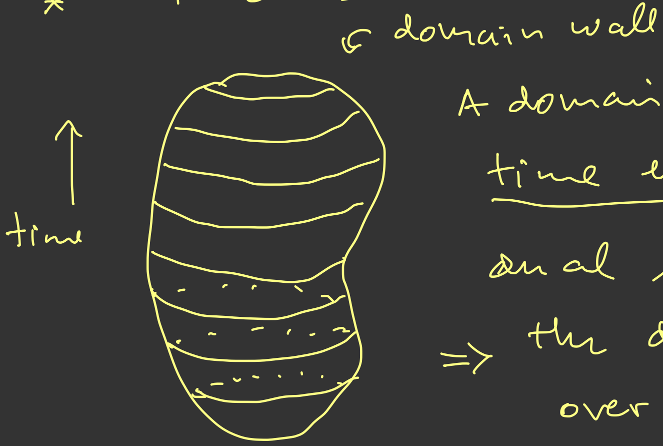
$$\tau_3(\tilde{j}-1)\,\tau_3(\tilde{j}) = \sigma_1(j) \quad ; \quad \tau_1^2 = \tau_3^2 = 1 \,, \quad \{\tau_1, \tau_3\} = 0$$

$$H = -\sum_{\tilde{j}} \tau_3(\tilde{j})\,\tau_3(\tilde{j}+1) - \lambda \sum_{\tilde{j}} \tau_1(\tilde{j})$$

duality $\lambda \Longleftrightarrow \frac{1}{\lambda} \Longrightarrow$ self duality $\boxed{\lambda = 1}$ (Onsager)

Dual of the D = 3 Classical Ising Model

- The high T expansion is a sum over loop configurations
- The low T expansion is a sum over domain wall configs.
- In D = 3 the domain walls are closed surfaces



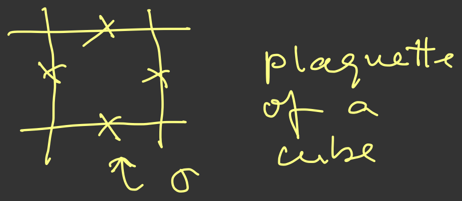
A domain wall configs. can be pictured as the time evolution of a closed string on the dual lattice

⇒ the dual model at low $\tilde{T}$ is a sum over loops and at high $\tilde{T}$ is a sum over closed surfaces

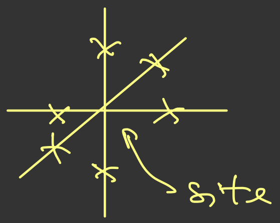
The D=3 Ising Gauge Theory (Wegner 1971)

$\mathbb{Z}_2$ gauge fields on the links $\{\sigma_\mu\}$ $\quad \sigma_\mu = \pm 1$
Interactions on plaquettes

$$Z = \sum_{\{\sigma_\mu\}} \exp\left(\frac{1}{T} \sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma \right)$$



Gauge invariance: flip all $\mathbb{Z}_2$ gauge fields that share a site

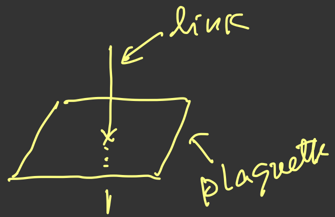


High T ⇒ sum over closed surfaces with a weight $\left(\tanh \frac{1}{T}\right)^{\text{surface}}$

Low T ⇒ sum over closed loops of the dual lattice

This is the dual of the D=3 Ising Model

In D=3 links are dual to plaquettes (1 form ⇔ 2 form)



Observables

3D Ising Model

Correlator $\langle \sigma(x)\, \sigma(x') \rangle$

* high T $\sim e^{-|x-x'|/\xi}$

disorder

* low T $\sim |\langle\sigma\rangle|^2 + e^{-|x-x'|/\xi}$

Long range order

Wilson Loop: creates an open domain wall (defect)

Area law in the ordered phase

Perimeter law in the disordered phase

3D $\mathbb{Z}_2$ gauge theory

Wilson loop $\langle \prod_\gamma \sigma \rangle$; $\partial\Sigma = \gamma$

γ: closed loop

* high T^* $\sim \exp - \text{Area}(\Sigma))$

confinement

* low T^* $\sim \exp(-\text{length}(\gamma))$

deconfinement

Correlator: creates an open $\mathbb{Z}_2$ flux tube ending at two "monopoles"

confinement $\Leftrightarrow$ monopole condensation

(EF & L. Susskind, '78) Quantum Version (2+1 dimensions)

Ising Model

$$H = -\sum_{\vec r} \sigma_1(\vec r) - \lambda \sum_{\langle \vec r, \vec r' \rangle} \sigma_3(\vec r)\,\sigma_3(\vec r')$$

Global $\mathbb{Z}_2$ symmetry

$$Q = \prod_{\text{sites}} \sigma_1(\text{sites})$$

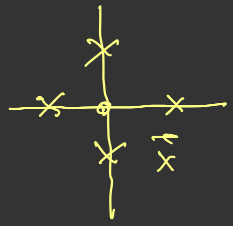
$\mathbb{Z}_2$ gauge Theory

$$H = -\sum_{\text{links}} \sigma_1(\text{link}) - g \sum_{\text{plaquettes}} \sigma_3\sigma_3\sigma_3\sigma_3$$

Local (gauge) $\mathbb{Z}_2$ symmetry

$$Q(\vec x) = \prod_{\substack{\text{links} \\ \text{that} \\ \text{share } \vec x}} \sigma_1(\text{links})$$

("star")



$$[Q(\vec x), Q(\vec x')] = 0$$

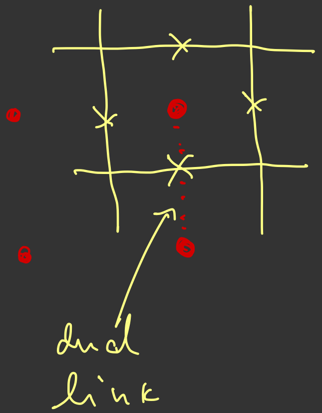
$$[Q(\vec x), H] = 0$$

Gauge Invariant states

$$Q(\vec x)\,|\text{Phys}\rangle = |\text{Phys}\rangle$$

Gauss Law

Quantum version of Duality



$$\prod_{\text{plaquette}} \sigma_3 = \tau_1 \text{ (dual site)}$$

$$\sigma_1 \text{(link)} = \tau_3 \tau_3 \quad \text{dual link}$$

$$\prod_{\text{star}} \sigma_1 = 1$$

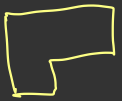
$\Rightarrow$ The gauge invariant sector ($Q(\vec{x}) = 1$) of the $\mathbb{Z}$ gauge theory with coupling g maps onto the 2+1 dim. Ising model with coupling $\lambda = \frac{1}{g}$

$$H = -\sum_{\text{links}} \sigma_1 - g \sum_{\text{plaquettes}} \sigma_3 \sigma_3 \sigma_3 \sigma_3 \xleftrightarrow[\text{duality}]{} H = -\sum \tau_3 \tau_3 - \lambda \sum \tau_1$$

sites are dual to plaquettes, links are dual to links

Physical Picture

Confined phase ($g < g_c$) (use σ_1 eigenstates)

* $|\mathrm{Gnd}\rangle = \sum_{\text{loops}} \#\ \Big|$  $\Big\rangle$

↑ loops created by the plaquette operator

Z_2 electric loops are small for g small

* At g_c the loops proliferate

* For $g \gg g_c$ we approximate $H = -\sum_{\text{plaquettes}} \sigma_3 \sigma_3 \sigma_3 \sigma_3$

  $Q(x) = 1$ ("Toric Code")

* On a torus it has a 4-fold degeneracy (Topological Phase)

* The dual is $H_{\text{Ising}} = -\sum_{\text{sites}} \tau_1 \Rightarrow$ disordered phase (Kitaev, 1997) (unique state)

Vortices and Monopoles

We will discuss models with a (compact) $U(1)$ symmetry

Complex scalar $\phi(x) = |\phi(x)|\, e^{i\theta(x)}$ (θ defined mod 2π)

Order parameter of an XY classical spin, superfluid or an incommensurate CDW

Global symmetry $\phi(x) \to \phi(x)\, e^{i\alpha} \Leftrightarrow \theta(x) \to \theta(x) + \alpha$

Ordered phase (T low) $|\phi(x)| \approx \phi_0$

$$Z \approx \int \mathcal{D}\theta(x) \exp\left(-\int d^2x \, \frac{1}{2g} (\vec{\nabla}\theta)^2\right) \qquad \theta \cong \theta + 2\pi$$

$g = T / J|\phi_0|^2$, $\kappa = J|\phi_0|^2$ = stiffness

Vortices

C: closed oriented path

Total change of phase on the closed path C : $\frac{(\Delta\Theta)_C}{2\pi}$

$$\frac{(\Delta\Theta)_C}{2\pi} = \frac{1}{2\pi}\oint_C d\vec{x}\cdot\vec{\nabla}\Theta(x) = i\int_0 \frac{d\varphi}{2\pi}\, e^{i\Theta(\varphi)}\,\partial_\varphi e^{-i\Theta(\varphi)} \equiv \underset{\text{vorticity}}{m}$$

m: topological invariant under smooth deformations of C

$\Theta(x)$ is a map of C $\rightarrow$ phase field $e^{i\Theta}$

$$\Rightarrow \quad \Theta(x) : \underset{\substack{\uparrow \\ \text{base}}}{S_1} \longrightarrow \underset{\substack{\uparrow \\ \text{target}}}{S_1} \qquad \pi_{\underset{\substack{\uparrow \\ \text{base}}}{1}}(\underset{\substack{\uparrow \\ \text{target}}}{S_1}) \cong \mathbb{Z}$$

homotopy class

m : winding number

Superfluid current $j_\mu = \partial_\mu \theta$

Vorticity $\omega(x) = \underset{\substack{\uparrow \\ \text{Levi-Civita}}}{\varepsilon_{\mu\nu}} \partial_\mu j_\nu = \varepsilon_{\mu\nu} \partial_\mu \partial_\nu \theta(x)$

$\Rightarrow$ $\theta(x)$ has a branch cut singularity accross which it jumps by $2\pi m$

* set of vortices at locations $\{\vec{x}_j\}$ with topological charges $\{m_j\}$

$$\Rightarrow \quad \omega(\vec{x}) = 2\pi \sum_j m_j \, \delta^2(\vec{x} - \vec{x}_j)$$

$z = x_1 + i x_2$

$$\equiv 2\pi \sum_j m_j \, \text{Im} \ln(z - z_j)$$

Define ϑ, the Cauchy-Riemann dual $\partial_\mu \vartheta = \varepsilon_{\mu\nu} \partial_\nu \theta$

$$\Rightarrow -\nabla^2 \vartheta = \omega(x)$$

$$\Rightarrow \quad \theta(\vec{x}) = \int d^2y \, G(|\vec{x}-\vec{y}|)\, \omega(\vec{y})$$

$$-\nabla^2 G(\vec{x}-\vec{y}) = \delta^2(\vec{x}-\vec{y}) \qquad \text{Green Function}$$

$$G(|\vec{x}-\vec{y}|) = \frac{1}{2\pi} \ln\left(\frac{a}{|\vec{x}-\vec{y}|}\right)$$

UV cutoff

s.t.

$G(|x-y|) = 0$

if $|\vec{x}-\vec{y}| < a$

<u>Energy of the Config:</u>

$$E[\theta] = \frac{J\phi_0^2}{2} \int d^2x \, (\vec{\nabla}\theta)^2 = \frac{J\phi_0^2}{2} \int d^2x \int d^2y \, \omega(x)\, G(x-y)\, \omega(y)$$

$$= 2\pi J \phi_0^2 \sum_{j<k} m_j m_k \ln\left(\frac{a}{|x_j - x_k|}\right)$$

which is IR divergent <u>unless</u> $\sum_j m_j = 0$ (zero total vorticity)

$$Z_{XY} \approx Z_{\substack{\text{Coulomb}\\ \text{Gas}}} = \sum_{\{m_j\}}{}' \exp\left(-\frac{2\pi |\phi_0|^2 J}{T} \sum_{j<k} m_j m_k \ln\left(\frac{a}{|x_j - x_k|}\right)\right)$$

Kosterlitz - Thouless Transition

At low T the vortices are bound in neutral pairs

The free energy of a vortex is

$$F_{vortex} = E_{vortex} - T\, S_{vortex}$$

$$E_{vortex} = \pi J |\phi_0|^2 \ln\left(\frac{L}{a}\right) \leftarrow \text{Energy}$$ (L: linear size of the system)

$$S_{vortex} = \ln\left(\frac{L}{a}\right)^2 \leftarrow \text{Entropy}$$

$$F_{vortex}(T_{KT}) = 0 \iff T_{KT} = \frac{\pi}{2} J \phi_0^2$$

$T < T_{KT}$ vortices are suppressed

$T > T_{KT}$ vortices proliferate

Alternative Picture

Let A_μ be a background U(1) gauge field

$$Z[A] = \int \mathcal{D}\theta \; e^{-\frac{1}{2g}\int d^2x \, (\partial_\mu \theta - A_\mu)^2}$$

Let A_μ represent a vortex field $\epsilon_{\mu\nu} \partial_\mu A_\nu = \omega(x)$

Hubbard-Stratonovich:

$$Z[A] = \int \mathcal{D}\theta \, \mathcal{D}a_\mu \; e^{-\frac{g}{2}\int d^2x \, a_\mu^2 + i\int d^2x \, a_\mu (\partial_\mu \theta - A_\mu)} \Rightarrow a_\mu = \epsilon_{\mu\nu} \partial_\nu \vartheta$$

$$Z[A] = \# \int \mathcal{D}\vartheta \; e^{-\frac{g}{2}\int d^2x \, (\partial_\mu \vartheta)^2 + i\int d^2x \, \vartheta \, \omega}$$ with $\int d^2x \, \omega = 0$

Duality: $\theta \longleftrightarrow \vartheta$ and $g \longleftrightarrow \frac{1}{g}$

Summing over vortices with core energy $u m^2$

$$Z = \sum_{\{m_j\}}' \int \mathcal{D}\vartheta \, \exp\left[-\frac{g}{2}\int d^2x \, (\partial_\mu \vartheta)^2 + i\sum_j 2\pi m_j \vartheta(x_j) - \frac{u}{T}\sum_j m_j^2\right.$$

$z = e^{-u/T} \ll 1 \Rightarrow$ only $m_j = 0, \pm 1$ contribute

$$\Rightarrow Z = \int \mathcal{D}\vartheta \, \exp\left[-\int d^2x \left[\frac{g}{2}(\partial_\mu \vartheta)^2 - v\cos(2\pi\vartheta)\right]\right.$$

Sine-Gordon

$v = 2z/a^2$

vortex correlator: $\langle e^{2\pi i \vartheta(x)} e^{-2\pi i \vartheta(y)} \rangle = \frac{\text{const.}}{|x-y|^{2\pi/g}}$

$\Rightarrow$ scaling dimension $\Delta_{\text{vortex}} = \pi/g$

$\Rightarrow$ vortices are $\underline{\text{relevant}}$ if $\Delta_{\text{vortex}} < d = 2 \Rightarrow g_c = \frac{\pi}{2}$

$\Rightarrow$ KT transition

$* \quad \langle e^{i\theta(x)} e^{-i\theta(y)} \rangle = \frac{\text{const}}{|x-y|^{g/2\pi}} \Rightarrow \frac{g}{2\pi} = \frac{T}{2\pi J \phi_0^2} \leq \frac{1}{4}$ for all $T < T_{KT}$

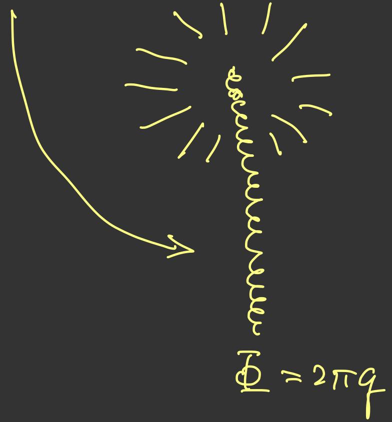
Power law decay

Magnetic monopoles in Compact QED $d=3$

(Euclidian spacetime)

Dirac monopole $$B_i(x) = \frac{q}{2}\,\frac{x_i}{2} - 2\pi q\, \delta_{i,3}\, \delta(x_1)\, \delta(x_2)\, \Theta(-x_3)$$

$\uparrow$ Dirac string



Lattice model (Polyakov, 1977)

$$Z = \prod_{\text{links}} \int_0^{2\pi} \frac{dA_\mu}{2\pi} \exp\left(\frac{1}{4e^2} \sum_{\text{plaquettes}} \cos F_{\mu\nu}\right)$$

$F_{\mu\nu}$: flux through a plaquette

$F_{\mu\nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu$

Gauge invariance: $A_\mu \rightarrow A_\mu + \Delta_\mu \Phi(x)$

Periodicity: $A_\mu \longrightarrow A_\mu + 2\pi\, l_\mu \,, \quad l_\mu \in \mathbb{Z}$

$$\prod_{\substack{\text{cube}\\ \text{faces}}} e^{iF_{\mu\nu}} = 1 \qquad \text{(Bianchi Identity)}$$

$\Rightarrow$ allows for monopoles with $q \in \mathbb{Z}$

We will follow the same approach we used with vortices

⇒ $B_{\mu\nu}$ is a background 2-form field ($B_{\mu\nu} = -B_{\nu\mu}$)
(Kalb-Ramond)

$$Z[B_\mu] = \int \mathcal{D}A_\mu \exp\left(-\frac{1}{4e^2}\int d^2x \left(F_{\mu\nu} - B_{\mu\nu}\right)^2\right)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; $A_\mu \to A_\mu + \partial_\mu \Phi$; $B_{\mu\nu} \to B_{\mu\nu}$

we also have $A_\mu \to A_\mu + a_\mu$, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu a_\nu - \partial_\nu a_\mu$
(2-form gauge transf.)

Monopole density: $\mathcal{M}(x) = 2\pi \sum_j m_j \delta^3(x - x_j)$

$$\mathcal{M}(x) = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\mu B_{\nu\lambda}$$

$$Z[B] = \int \mathcal{D}A_\mu \mathcal{D}b_{\mu\nu} \exp\left(-\frac{e^2}{2}\int d^3x\, b_{\mu\nu}^2 + i\int d^3x\, \frac{1}{2} b_{\mu\nu}\left(F_{\mu\nu} - B_{\mu\nu}\right)\right)$$

⇒ $\partial_\nu b_{\mu\nu} = 0$ ⇒ $b_{\mu\nu} = \varepsilon_{\mu\nu\lambda}\partial_\lambda \vartheta$ (compact scalar)

invariant under $\vartheta \to \vartheta + \alpha$ (α: const mod 2π)

$$\Rightarrow \; Z[B] = \int \mathcal{D}\vartheta \, \exp\left(-\frac{e^2}{2}\int d^3x \, (\partial_\mu \vartheta)^2 + 2\pi i \sum_j m_j \vartheta(j)\right)$$

$$\Rightarrow \; Z = \sum_{\{m_j\}}{}' Z[m_j] \equiv \int \mathcal{D}\vartheta \, \exp\left(-\int d^3x \left(\frac{e^2}{2}(\partial_\mu \vartheta)^2 - v \cos 2\pi\vartheta\right)\right)$$

Sine-Gordon!
but $\underline{d=3}$

$v = 2\exp(-u)/a^3$ (u: core energy)

Monopole correlator

$$\langle e^{i 2\pi \vartheta(x)} e^{-i 2\pi \vartheta(y)} \rangle = \exp\left(\frac{\pi}{2e^2}\left[\frac{1}{R} - \frac{1}{a}\right]\right) \qquad R = |x-y|$$

$$\xrightarrow[R\to\infty]{} \text{Const}$$

$\Rightarrow$ Monopoles proliferate <u>for all</u> $e^2 \neq 0$

$\Rightarrow$ In $d=3$ the energy $< \infty$ but the entropy $\sim \ln\left(\frac{L}{a}\right)^3 \to \infty$

$\Rightarrow$ confinement by monopole condensation

Wilson loop: $W_\gamma = \langle e^{i\oint_\gamma dx_\mu A_\mu} \rangle$ has an area law $\Rightarrow$ confinement

(Polyakov, 1977)

Higgs, Confinement and Topology $(d=3)$

Consider a theory of complex order parameter of charge $n \in \mathbb{Z}$ coupled to a dynamical $U(1)$ (compact) gauge field

Order Parameter field $e^{i\Theta(x)}$, A_μ gauge field

$n = 2$ is (with some caveats) the case of a superconductor

Lattice model: $\Theta(x)$ on sites and A_μ on links

$$Z = \prod_{\text{sites}} \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{\text{links}} \int_0^{2\pi} \frac{dA_\mu}{2\pi} \exp\left(S(\theta, A_\mu) \right)$$

$$S = \beta \sum_{\text{links}} \cos\left(\Delta_\mu \theta - n A_\mu\right) + \frac{1}{g^2} \sum_{\text{plaquettes}} \cos\left(F_{\mu\nu}\right)$$

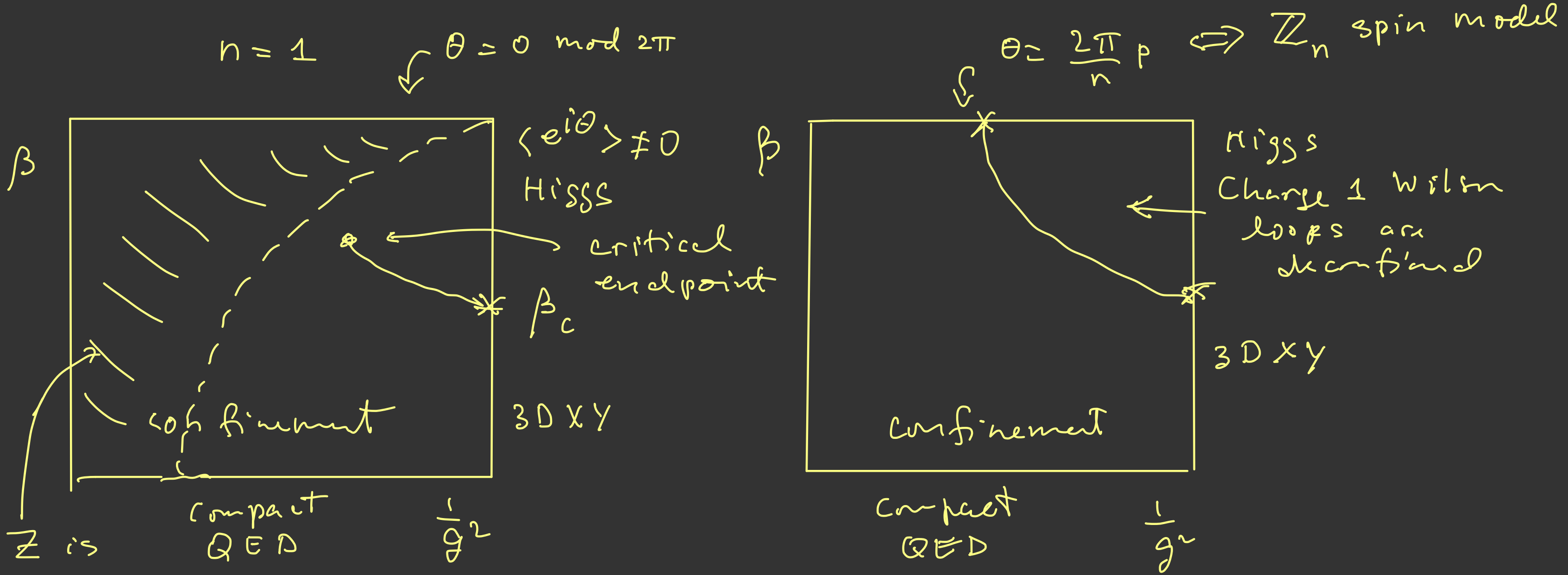
$g \to 0 \Rightarrow F_{\mu\nu} \to 0 \pmod{2\pi} \Rightarrow$ 3D XY model $\Rightarrow$ "Higgs" for β large

$\beta \to 0 \Rightarrow$ Polyakov's QED $\Rightarrow$ confinement

Q1: How are the Higgs and confinement limits related?

Q2: Are they different phases

Answer: it depends on $\textcircled{n}$ (EF & S. Shenker 1979)



Z is analytic
No phase transition

Higgs and Confinement are the same phase

Consider the deconfined phase $n>1$

$$Z = \int \mathcal{D}\theta\, \mathcal{D}A_\mu \exp\left(-\int d^3x \left[\frac{\beta}{2}\left(\partial_\mu \theta - n A_\mu\right)^2 - \frac{1}{4g^2} F_{\mu\nu}^2\right]\right)$$

for $\beta \gg 1$

Hubbard - Stratonovich

$$Z = \int \mathcal{D}\theta\, \mathcal{D}A_\mu\, \mathcal{D}a_\mu\, e^{-\int d^3x \frac{1}{2\beta} a_\mu^2 + i\int a_\mu\left(\partial_\mu \theta - n A_\mu\right) - \int \frac{1}{4g^2} F_{\mu\nu}^2}$$

Integrate θ out $\Rightarrow$ $\partial_\mu a_\mu = 0 \Rightarrow a_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda$

$$Z = \int \mathcal{D}b_\mu\, \mathcal{D}A_\mu \exp\left(i n \int d^3x\, A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda - \int d^3x \frac{1}{4\beta} f_{\mu\nu}^2 - \int d^3x \frac{1}{4g^2} F_{\mu\nu}^2\right)$$

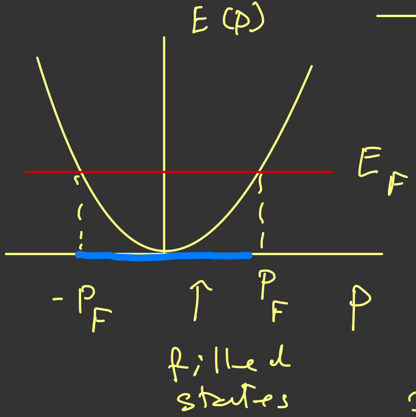
$\beta \to \infty$ and $g \to 0$ only the "BF" term survives

This is a topological term $\Rightarrow$ The deconfined phase is topological $\neq$ Higgs

For $n=2 \Rightarrow$ Toric Code

There is never a Higgs phase

Fermions in one-dimension



$$E(p) \approx v_F(p-p_F) - v_F(p+p_F) + \cdots$$

$$\Psi(x) \simeq \psi_R(x)\, e^{ip_F x} + \psi_L(x)\, e^{-ip_F x}$$

density $\rho(x) = \Psi^\dagger(x)\Psi(x)$

$$\simeq \bar{\rho} + \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L + \psi_R^\dagger \psi_L\, e^{i2p_F x} + \psi_L^\dagger \psi_R\, e^{-i2p_F x}$$

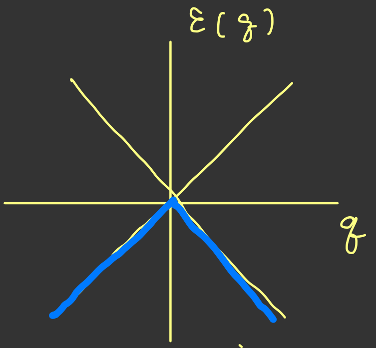
slowly varying density: $j_0 \equiv \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L \equiv \rho_R + \rho_L$

slowly varying current: $j_1 \equiv \psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L \equiv \rho_R - \rho_L$

$\psi_R^\dagger(x)\psi_L(x) \equiv$ Order parameter of a CDW with $Q = 2p_F$; complex field

If $\langle \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \rangle \neq 0 \implies$ energy gap at $p = \pm p_F$

The density, current and the CDW O.P. are invariant under the global U(1) gauge transformation $\Psi(x) \to \Psi'(x) = e^{i\theta}\,\Psi(x)$



filled Dirac sea

$q = p - p_F$ etc. $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ Dirac spinor

$$\mathcal{H} = -v_F \left(\psi_R^+ i \partial_x \psi_R - v_F \psi_L^+ i \partial_x \psi_L \right)$$

2x2 Dirac γ-matrices

$$\gamma_0 = \sigma_1 \,, \quad \gamma_1 = -i\sigma_2 \,, \quad \gamma_5 = \sigma_3 \,, \{ \gamma_\mu, \gamma_\nu \} = 2 g_{\mu\nu} ; \, g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Lagrangian density $\mathcal{L} = \bar{\Psi} i \gamma_\mu \partial^\mu \psi$; $\bar{\Psi} = \psi^+ \gamma_0$

It has two continuous symmetries

* $\psi(x) \longrightarrow \psi'(x) = e^{i\theta} \psi(x) \Rightarrow \psi_R' = e^{i\theta} \psi_R, \psi_L' = e^{i\theta} \psi_L$ (gauge)
* $\psi(x) \to \psi'(x) = e^{i\gamma_5 \theta} \psi(x) \Rightarrow \psi_R' = e^{+i\theta} \psi_R, \psi_L' = e^{-i\theta} \psi_L$ (chiral)

Q: How come we have two symmetries?

Chiral Anomaly

* The massless Dirac theory has two global symmetries whereas the microscopic model has only one: gauge invariance.

* In the massless Dirac theory ψ_R and ψ_L are separately conserved

* Two currents: the gauge current $j_\mu = \bar{\psi}\gamma_\mu\psi$, $\partial_\mu j^\mu = 0$ and the chiral current $j^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$; $\partial^\mu j^5_\mu = 0$

$$j_\mu = (\rho_R + \rho_L, \rho_R - \rho_L) \,, \quad j^5_\mu = (\rho_L - \rho_R, \rho_R + \rho_L)$$

* If we couple the theory to an uniform electric field E

$$\Rightarrow \quad \frac{dN_R}{dt} = \frac{e}{2\pi}E \quad \text{and} \quad \frac{dN_L}{dt} = -\frac{e}{2\pi}E \Rightarrow \frac{dQ}{dt} = 0 \Rightarrow \text{electric charge is conserved}$$

But $$\frac{dQ_5}{dt} = \frac{e}{\pi}E \Rightarrow \quad \text{chiral charge is not conserved}$$

(following H. Nielsen and Y. Ninomiya 1982)

This is the chiral anomaly

Chiral Anomaly and Bosonization

The normal-ordered gauge charge density j_0 and current j_1 satisfy the equal-time commutator

$$[j_0(x_1), j_1(x_1')] = -\frac{i}{\pi}\partial_1\delta(x_1-x_1')$$

$$[j_0(x_1), j_0(x_1')] = [j_1(x_1), j_1(x_1')] = 0$$

Mattis & Lieb, 1965
Luther & Emery, 1974
S. Coleman, 1975
E. Witten, 1978

Let ϕ be a real scalar field and Π the conjugate momentum

$$\Rightarrow [\phi(x), \Pi(x')] = i\delta(x-x')$$

$\Rightarrow$ we identify $j_0(x) \equiv \frac{1}{\sqrt{\pi}}\partial_1\phi$, $j_1(x) \equiv -\frac{1}{\sqrt{\pi}}\Pi(x) = -\frac{1}{\sqrt{\pi}}\partial_0\phi$

$\Rightarrow j_\mu(x) = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi$ (duality!) $\Leftrightarrow \partial^\mu j_\mu = 0$ ✓

But $j^5_\mu = \epsilon_{\mu\nu}j^\nu \Rightarrow \partial^\mu j^5_\mu = -\frac{1}{\sqrt{\pi}}\partial^2\phi \Rightarrow \partial^\mu j^5_\mu = 0 \Leftrightarrow \phi$ is a free field!

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu\partial_\mu\Psi \longleftrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2$$

Coupling to a gauge field

$$\mathcal{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - e A^\mu \bar{\Psi} \gamma_\mu \Psi$$

In the bosonic theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{e}{\sqrt{\pi}} A^\mu \varepsilon_{\mu\nu} \partial^\nu \phi$$

$\Rightarrow$ Eq. of motion $\quad -\partial^2 \phi = \frac{e}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial^\nu A^\mu \equiv \frac{e}{\sqrt{\pi}} F^*$ (dual "tensor")

$\Rightarrow \quad \partial^\mu j^5_\mu = -\frac{1}{\sqrt{\pi}} \partial^2 \phi = \frac{e}{2\pi} F^*$ chiral anomaly!

Total fermion # $\quad N_F \in \mathbb{Z}$

$$N_F = \int_0^L dx_1 \, j_0(x_0, x_1) = \frac{1}{\sqrt{\pi}} \int_0^L dx_1 \, \partial_1 \phi(x_0, x_1) = \frac{1}{\sqrt{\pi}} \left(\phi(x_0, L) - \phi(x_0, 0) \right)$$

$\Rightarrow \quad \phi(x_1 + L) = \phi(x_1) + \sqrt{\pi} N_F \quad \Rightarrow$ ϕ is a compactified scalar

$\phi(x + L) = \phi(x) + 2\pi R N \Rightarrow \quad R = 1/\sqrt{4\pi}$ compactification radius

Operator Mappings

* compactification requires that the observables be invariant under $\phi \to \phi + 2\pi n R$, with $n \in \mathbb{Z}$

$\Rightarrow$ $V_\alpha(\phi) = \exp(i\alpha\phi)$ is allowed if $\alpha = \frac{n}{R} = n\sqrt{4\pi}$

scaling dimension $= n^2$

$$\phi = \phi_R + \phi_L \;, \quad \vartheta = -\phi_R + \phi_L \;, \quad \vartheta(x_0, x_1) = \int_{-\infty}^{x_1} dx_1' \, \Pi(x_0, x_1')$$

$$\Rightarrow \quad \partial_\mu \phi = \varepsilon_{\mu\nu} \partial^\nu \vartheta$$ (Cauchy-Riemann)

Fermion Operators (Mandelstam, 1975)

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a}} :\exp(i\sqrt{4\pi}\,\phi_R): \;, \quad \psi_L(x) = \frac{1}{\sqrt{2\pi a}} :\exp(-i\sqrt{4\pi}\,\phi_L)$$

mass operators :

$$\left.\begin{aligned} \bar\psi\psi &\equiv \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \longleftrightarrow \frac{1}{2\pi a} :\cos(\sqrt{4\pi}\,\phi): \\ i\bar\psi\gamma_5\psi &\equiv i(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R) \longleftrightarrow \frac{1}{\sqrt{2\pi a}} :\sin(\sqrt{4\pi}\,\phi): \end{aligned}\right\}$$ scaling dimension 1

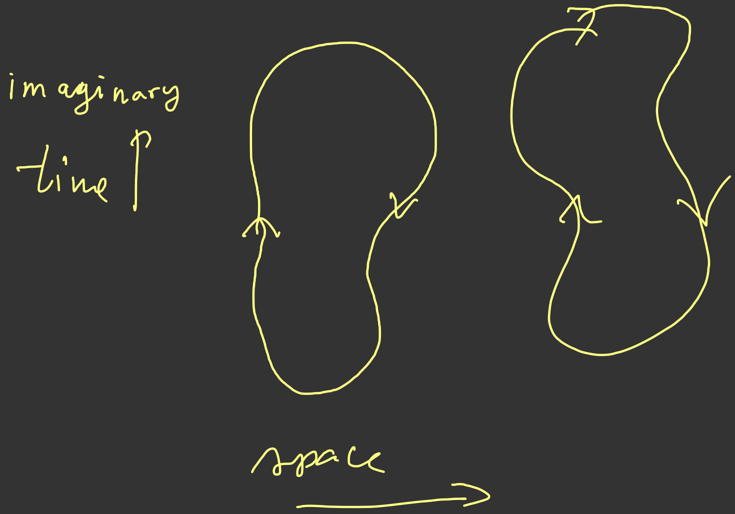
$g = m/(2\pi a)$

$$\mathcal{L} = \bar\psi i \gamma^\mu \partial_\mu \psi - m \bar\psi\psi \longleftrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - g\cos(\sqrt{4\pi}\,\phi)$$ Sine-Gordon!

3D: Particle-Vortex duality

(Peskin; Stone & Thomas; Dasgupta, Halperin) (~ 1978 - 1981)

* Global U(1) symmetry
* 3D XY model (superfluid)
* High T: gas of closed loops with short-range interactions (i.e. particle worldlines)
* Low T: closed vortex loops w/ Biot-Savart interactions
* Particle - vortex duality



Simple derivation of 3D Particle-Vortex Duality

- We will follow the same procedure we used in 2D
- $\Theta(x)$: phase field of a 3D complex field
- $A_\mu(x)$: background gauge field that will create vortices

$$Z[A_\mu] = \int \mathcal{D}\Theta \, \exp\left(-\frac{1}{2g}\int d^3x \, (\partial_\mu \Theta - A_\mu)^2\right)$$

vorticity $\omega_\mu(x) = \varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \equiv 2\pi \sum_k l^k_\mu(x) \, \delta^3(x - x_k)$

↑ vortex loops

$$\partial_\mu \omega_\mu = 0 \iff \partial_\mu l^k_\mu = 0$$

Hubbard-Stratonovich

$$Z[A] = \int \mathcal{D}\Theta \, \mathcal{D}b_\mu \, \exp\left(-\int d^3x \, \frac{g}{2} b_\mu^2 + i\int d^3x \, b_\mu (\partial_\mu \Theta - A_\mu)\right)$$

$$\equiv \int \mathcal{D}b_\mu \, \delta(\partial_\mu b_\mu) \, \exp\left(-\int d^3x \, \frac{g}{2} b_\mu^2 + i\int d^3x \, b_\mu A_\mu\right)$$

$$\partial_\mu b_\mu = 0 \Rightarrow b_\mu = \varepsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \; ; \; f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

$$\Rightarrow Z[A] = \int \mathcal{D}a_\mu \exp\left(-\int d^3x \, \frac{g}{4} f_{\mu\nu}^2 + i\int d^3x \; a_\mu \underbrace{\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda}_{\omega_\mu}\right)$$

Note: These steps contain the statement that in 3D the dual of a Goldstone field (θ) is a gauge field (a_μ)
compactification of θ $\Longleftrightarrow$ charge quantization

Next we sum over vortex configurations

$$Z = \sum_{\{l^k_\mu\}} \delta(\partial_\mu l^k_\mu) \int \mathcal{D}a_\mu \exp\left(-\int d^3x \, \frac{g}{4} f_{\mu\nu}^2 + i\sum_k l^k_\mu(x_k)\, a_\mu(x_k)\right)$$

upon adding an energy/length to the vortices and a short range repulsion (no crossing)

$$\Rightarrow Z = \int \mathcal{D}a_\mu \, \mathcal{D}\phi \, \mathcal{D}\phi^* \exp\left(-\int d^3x \left[\frac{g}{4} f_{\mu\nu}^2 + |D_a\phi|^2 + \overset{\overset{m^2>0}{\downarrow}}{m^2}|\phi|^2 + \lambda|\phi|^4\right.\right)$$

* The theory we derived is the 3D Abelian-Higgs model

This is the same as a superconductor ϕ coupled to a fluctuating e.m. field a_μ

* Introduce a probe field B_μ $\Rightarrow$ $A_\mu \to A_\mu + q B_\mu$

This leads in the dual theory to an extra term

$$\exp\left(-i \int d^3x \, q \, B_\mu \, \varepsilon_{\mu\nu\lambda} \partial_\nu a_\lambda\right) \qquad (q \in \mathbb{Z} \text{ charge})$$

$\Rightarrow$ current $j_\mu \longleftrightarrow q \, \varepsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$

* For $m^2 < 0$ we can run the duality backwards and map the Higgs-Superconducting phase to the unbroken phase of the XY model

Field Theory Picture of Particle-Vortex Duality

* Theory A

$$\mathcal{L} = |D_A \phi|^2 - m^2 |\phi|^2 - u|\phi|^4 \quad , \quad D_A \equiv \partial - i \overset{\substack{\text{background}\\ \text{field}}}{\overset{\downarrow}{A}}$$

* Theory B

$$\mathcal{L} = |\underset{\substack{\uparrow \\ \text{dynamical}\\ \text{field}}}{D_a} \varphi|^2 + m^2 |\varphi|^2 - u|\varphi|^4 + \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \overset{\substack{\text{external field}\\ \downarrow}}{A^\mu} \partial^\nu a^\lambda - \frac{1}{4e^2} f_{\mu\nu}^2$$

$$j_\mu \longleftrightarrow \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda \qquad (\text{particle-vortex})$$

Duality maps the unbroken phase of Ⓐ to the Higgs phase of Ⓑ
broken phase of Ⓐ to the unbroken phase of Ⓑ

* Wilson-Fisher Fixed Points are mapped into each other

Generalization: Web of Dualities

① **Particle-Vortex duality** (Peskin, 1978; Dasgupta & Halperin, 1981)

$$\rightarrow \; |D_{\underset{\text{external}}{A}}\Phi|^2 - m^2|\Phi|^2 - u|\Phi|^4 \longleftrightarrow |D_b\varphi|^2 + m^2|\varphi|^2 - u|\varphi|^4 + \frac{1}{2\pi}A\,d\overset{\text{dynamical}}{b} + \text{Maxwell}$$

$$\boxed{J_\mu \longleftrightarrow \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\nu b^\lambda}$$

; vortices ⟷ particles

② **Bosonization** (Fradkin & Schaposnik, 1994; Seiberg, Senthil, Wang & Witten, 2016)

$$\bar{\Psi}(i\not{D}_A - \underline{M})\Psi - \underline{\frac{1}{8\pi}}AdA \longleftrightarrow |D_a\phi|^2 - m^2|\phi|^2 - u|\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA$$

Dirac fermion ⟷ monopole ;

$$\boxed{\bar{\Psi}\gamma^\mu\Psi \leftrightarrow \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda}$$

③ **Fermion Particle-Vortex duality** (Son, 2015; Metlitski & Vishwanath, 2016)

$$\bar{\Psi}(i\not{D}_A - \underline{M})\Psi - \frac{1}{8\pi}AdA \Leftrightarrow \bar{\chi}(i\not{D}_a + \underline{M})\chi + \frac{1}{8\pi}ada - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA$$

"QED_3"

- In general dimension duality often maps theories with ≠ character and symmetry
- In $D = 4$ ⇒ gauge theory ⟺ gauge theory
- There are many other dualities
- AdS / CFT ⟺ gauge / gravity duality
- S and T duality in String Theory

(S duality is related to particle - vortex duality)

- Conjectured web of dualities in 2+1 dimensions
- Fermion ⟺ Boson duality
- Can we "derive" these conjectures ?

Strategy for a derivation (Goldman & EF 2018)

- * We will use generalized loop models near criticality but still in the gapped phases
- * Generalization of the particle-vortex duality
- * We consider loop models in 2+1 dimensions
- * Assume that the loops cannot intersect
- * Include phase factors for linking numbers
- * Frame the loops and include self-linking and Berry phase factors ⇒ fractional spin

Loop Models in 2+1 Dimensions

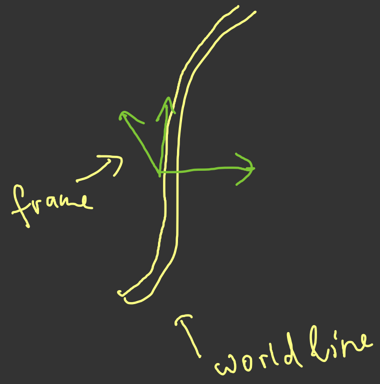
$$Z[A] = \sum_{\{\ell_\mu\}}' \delta(\partial_\mu \ell_\mu)\, e^{-S[\ell] + i\pi\Phi[\ell]}$$

background field

loop configurations ("conserved currents")

weight per unit length + interactions

$\Phi[\ell]$ = linkig number + self-linking number + Berry phase of the frame



Linking # of two loops ℓ_1 and ℓ_2

Φ = 2 × linking # of ℓ_1 with ℓ_2 +

$+ W[\ell_1] + W[\ell_2]$

$W[\ell] = SL[\ell] - T[\ell] \simeq$ "writhe"

self-linking

twist

Twist
$$T[\ell] = \frac{1}{2\pi}\int_0^1 ds \int_0^1 du \;\; \underset{\substack{\uparrow \\ \text{tangent}}}{\hat{e}} \cdot \partial_s \hat{e} \times \partial_u \hat{e}$$

$T[\ell]$ in general is <u>not</u> quantized and depends on the metric } Berry phase of the frame

Linking # of ℓ_1 and ℓ_2

$$\left\langle e^{i\oint_{\ell_1 \cup \ell_2} dz_\mu \cdot a_\mu} \right\rangle_{\text{Chern Simons}} \equiv e^{\frac{i\pi}{k} \underset{\substack{\uparrow \\ \text{linking \#}}}{N_\gamma}} \leftarrow \text{topological invariant}$$

wilson loop in Chern-Simons Theory (Witten 89')

$$S = \frac{k}{4\pi}\int d^3x \; a\, d\, a$$

$$k \in \mathbb{Z}$$

Example (~ Polyakov 89')

$$* \quad Z_{fermion} = \underset{\substack{\uparrow \\ \text{loop} \\ \text{representation}}}{\text{Det}(i\not{D} - M) \cong} \int \mathcal{D}j_\mu \, \delta(\partial_\mu j^\mu) \, e^{-|M| L[j] - i\pi \, \text{sgn}(M) \, \Phi[j]}$$

linking #

+ spin factor

$k=1$

$$* \quad \mathcal{L}_{boson} = |D_a \phi|^2 - m^2 |\phi|^2 - u |\phi|^4 + \frac{1}{4\pi} a\,da + \frac{1}{2\pi} a\,dA$$

$$Z[A] = \int \mathcal{D}j_\mu \, \mathcal{D}a_\mu \, \delta(\partial_\mu j_\mu) \, e^{-|m| L[j] + i S[j, a, A]}$$

$$S[j, a, A] = \int d^3x \left[j_\mu (a_\mu - A_\mu) + \frac{1}{4\pi} a\,da - \frac{1}{4\pi} A\,dA + \cdots \right]$$

Integrating over $\underline{a_\mu}$ $\Rightarrow$ $-\pi \Phi[j] + \int d^3x \left(j_\mu A_\mu - \frac{1}{4\pi} A\,dA \right)$

$$\Rightarrow \quad \mathcal{L}_{\text{fermion}} = \bar{\Psi}\left(i \not{D}_A - M\right)\Psi - \underset{\substack{\uparrow \\ \text{anomaly } (\eta \text{ invariant})}}{\frac{1}{8\pi} A dA} \qquad \text{with} \quad M<0$$

$$Z_{\text{fermion}}\left[A, M<0\right] \underbrace{e^{-\frac{i}{\underset{\uparrow}{2}} S_{CS}[A]}} = \int \mathcal{D}j\, \delta(\partial \cdot j)\, e^{-|M| L[j]}\, e^{i S_{\text{fermion}}[j, A, M<0]}\, e^{-\frac{0}{2} S_{CS}[A]}$$

$$S_{\text{fermion}}[j, A, M<0] = \int d^3x \left[j \cdot A - \frac{1}{8\pi} A dA \right] - \pi \Phi[j]$$

* To get the bosonization identity for $M>0$ one uses bosonic particle-vortex duality

* In the fermionic theory $M<0 \Leftrightarrow M>0 \Rightarrow \sigma_{xy} = 0 \longleftrightarrow \frac{e^2}{h}$

* In the bosonic theory this is the transition from broken to the unbroken phase

Fermion Particle-Vortex Duality

- Duality from a free Dirac fermion ⟷ QED_3 with a quantized CS term

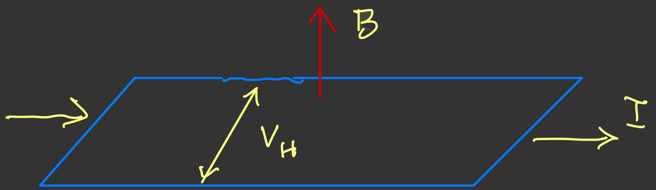
$$\begin{tikzcd}
\bar{\Psi}(i\not{D}_A + M)\Psi - \frac{1}{8\pi} A\,dA \arrow[r, leftrightarrow] \arrow[d] & \begin{array}{l} \bar{\chi}(i\not{D}_a - M)\chi + \frac{1}{8\pi} a\,da \\ - \frac{1}{2\pi} a\,db + \frac{2}{4\pi} b\,db - \frac{1}{2\pi} b\,dA \end{array} \arrow[d] \\
\begin{array}{l} \text{loop model} \\ \to \int d^3x \, j_\mu A^\mu + \pi \bar{\Phi}[j] \end{array} & \begin{array}{l} -\pi \Phi[j] + \int d^3x \Big[j\cdot a - \frac{1}{2\pi} a\,db + \frac{2}{4\pi} b\,db \\ \qquad - \frac{1}{2\pi} b\,dA \Big] \end{array} \arrow[l, "\text{integrate out } b_\mu \text{ and } a_\mu"']
\end{tikzcd}$$

$$Z_{\text{fermion}}[A, M] = Z_{QED_3}[A, -M] \; ; \; Z_f[A, -M] = Z_{QED_3}[A, M]$$

Currents: $\bar{\Psi}\gamma^\mu\Psi \longleftrightarrow \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

Application: Fractional Quantum Hall States

In the beginning .... two-dimensional electron gases in large magnetic fields



$\sigma_{xy} = \nu \frac{e^2}{h}$, $\sigma_{xx} \to 0$ $(T \to 0)$

no dissipation

Laughlin: $\Psi_m(z_1, \dots, z_N) \sim \prod_{i<j} (z_i - z_j)^m \; e^{-\frac{1}{4 l_0^2} \sum_{j=1}^{N} |z_j|^2}$ (1983)

filling fraction $\nu = \frac{1}{m}$; $\{z_j\}$: electron coordinates $(z = x + iy)$

l_0: magnetic length

(m odd)

Jain: composite fermion : electron + (m-1) fluxes

FQH state : IQH state of composite fermions

$\to \nu_\pm(p, s) = \frac{p}{2sp \pm 1}$ $p = 1, 2, \dots$ $s = 0, 1, 2 \dots$ (Laughlin: $p = 1$, + ; $m = 2s+1$)

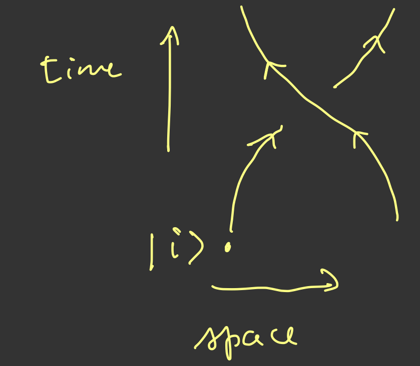
odd denominators

* The excitations of FQH fluids are vortices ("quasiholes") that

ⓐ carry fractional charge $q = \frac{1}{2sp\pm1}$ ←

ⓑ fractional braiding statistics (anyons) (Halperin '84, Arovas, Schrieffer and Wilczek '84)

ⓒ m degenerate ground states on a torus (topological protection)



amplitude ~ $e^{i\varphi}$

✓ $\varphi = \frac{\pi}{2sp\pm1}$ ⇒ anyons labelled by one-dimensional representations of the Braid Group

✓ $e^{i\varphi_1} e^{i\varphi_2} \to e^{i(\varphi_1+\varphi_2)}$ ("fusion")

Wen: Effective Field Theory
~1990
(Laughlin states)

$$\mathcal{L} = \frac{m}{4\pi}\varepsilon_{\mu\nu\lambda}\underset{\uparrow}{a^\mu}\partial^\nu a^\lambda + \underbrace{\frac{e}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\nu a^\lambda}_{\text{current}}\underset{\uparrow}{A^\mu} + \underset{\uparrow}{j^{\nu}_{\mu}}\, a^\mu \equiv$$

Chern-Simons term

a_μ: hydrodynamic gauge field

e.m. field

vortex world lines

Duality at the FQH Plateau Transition

(Hart Goldman & EF, 2019)

* Limiting value of the Jain sequence

$$\lim_{p\to\infty} \frac{p}{2np \pm 1} = \boxed{\frac{1}{2n}}$$

* In this limit the average CS field cancels A_μ

* Halperin - Lee - Read: this is a "Fermi Liquid"

* Good phenomenology but...

* singular forward scattering interactions and violation of particle-hole symmetry at $\nu = 1/2$ $(\nu \leftrightarrow 1-\nu)$

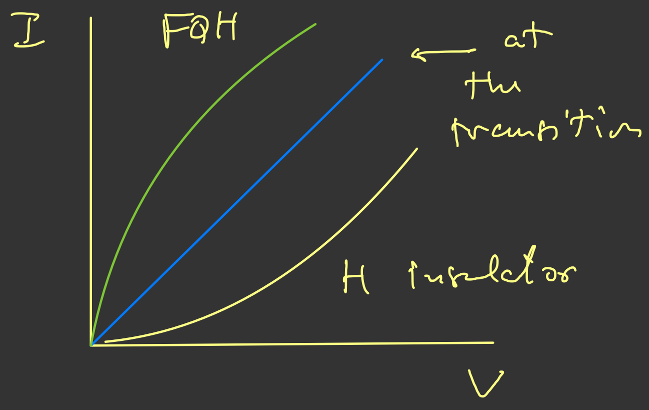
Symmetry of the I-V curves at the transition

- The I-V curves show a "mirror" symmetry at all transitions

- For general Jain states

$$\nu = \frac{p}{2np+1} \leftrightarrow \nu' = \frac{1+p}{2n(1+p)-1}$$

- For $\nu = 1/2 \Longleftrightarrow$ PH symmetry



I-V curves at the $0 \leftrightarrow \frac{1}{3}$ transition

$(\nu \cong \frac{1}{4})$

Hall insulator ↔ FQH

$\nu = \frac{1}{2}$: Son's Conjecture

General case : $\nu = \frac{1}{2n}$

$$\mathcal{L}_{1/2n} = i \underbrace{\bar{\Psi} \underset{\smile}{\not{D}_a}}_{} \Psi - \frac{1}{4\pi} \underbrace{\left(\frac{1}{2} - \frac{1}{2n} \right)}_{} a\, d a + \frac{1}{2\pi} \frac{1}{2n} a\, dA + \underset{\smile}{\frac{1}{2n}} \frac{1}{4\pi} A\, dA$$

$\longrightarrow$

$\partial \times A = B$

a : flux attachment

Electron filling $\quad \nu = \frac{2\pi}{B} \left\langle \frac{\delta \mathcal{L}_{\nu = 1/2n}}{\delta A_0} \right\rangle = \frac{1}{2n} \left(1 + \frac{b_*}{B} \right)$

$$b^* = \partial \wedge a = 0 \Rightarrow \nu = \frac{1}{2n}$$

Composite fermion ψ Fermi Surface set by a_0

$$\rho_\psi = \frac{1}{2\pi} \left(\frac{1}{2} - \frac{1}{2n} \right) b_* - \frac{1}{2\pi} \frac{B}{2n}$$

$\uparrow$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{filling} \\ \text{fraction} \\ \text{of } \psi}}{\nu_\psi} = 2\pi \frac{\rho_\psi}{b_*} = \frac{1}{2} + \frac{\nu}{1-2n\nu}$$

$$\Rightarrow \quad \text{If} \quad \nu_\psi = \underset{\substack{\uparrow \\ (\text{Dirac})}}{p + \frac{1}{2}} \Rightarrow \nu = \frac{p}{2np+1}$$

$$\text{But, if} \quad \underset{(\text{PH transf.})}{\nu_\psi \to -\nu_\psi} \Rightarrow \nu = \frac{p}{2np+1} \longrightarrow \frac{1+p}{2n(1+p)-1} \quad \checkmark \quad !$$

$\Rightarrow$ PH transf. of the Dirac composite fermion is equivalent to the reflection symmetry!

Self-Duality at the Transition

* Use fermion - boson duality

$$\mathcal{L}_{1/2n} \longleftrightarrow |D_{g-A}\Phi|^2 - |\Phi|^4 + \frac{1}{4\pi}\frac{1}{2n-1}\, g\,dg \longleftarrow \boxed{\nu_\Phi}$$

* Followed by a (boson) particle - vortex duality

$$\mathcal{L}_{1/2n} \longleftrightarrow |D_h\varphi|^2 - |\varphi|^4 - \frac{2n-1}{4\pi}\, h\,dh + \frac{1}{2\pi}\, h\,dA$$

$$\uparrow \boxed{-\frac{1}{\nu_\varphi}}$$

* $\nu = \frac{1}{2n} \longleftrightarrow \nu_\Phi = -\nu_\varphi = 1$

* Reflection symmetry $\nu_\Phi(\nu) = -\nu_\varphi(\nu')$

* Reflection $\Longleftrightarrow$ boson - vortex symmetry !

* Reflection symmetry <u>at</u> $\nu = \frac{1}{2n} \longleftrightarrow$ boson self-duality !

Non-Abelian States: Moore-Read (1991)

$$\Psi_{MR}(\{z_i\}) \sim \overbrace{\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)}^{\text{Pfaffian}} \underbrace{\prod_{i<j}(z_i-z_j)^n}_{\uparrow} \; e^{-\frac{1}{4\ell_0^2}\sum_{i=1}^{N}|z_i|^2}$$

Pfaffian: expectation value of chiral Majorana fermions $\underline{\chi(z)=\chi^\dagger(z)}$

Propagator: $\langle \chi(z)\,\chi(w)\rangle = \frac{1}{z-w}$

$\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right) = \langle \chi(z_1)\dots\chi(z_N)\rangle \longleftarrow$ "paired states" (p_x+ip_y superconductor)

$\varphi(z)$: chiral boson $\quad \varphi(z)\sim\varphi(z)+2\pi\sqrt{n}$

$$\Psi_{MR}\sim\langle\chi(z_1)\dots\chi(z_N)\rangle\left\langle\left(\prod_{i=1}^{N}e^{i\sqrt{n}\,\varphi(z)}\right)e^{-\int d^2z'\sqrt{n}\,\rho_0\,\varphi(z')}\right\rangle$$

Filling fraction: $\nu=\frac{1}{n}$

n even → fermions ; n odd ↔ bosons ; e.g. $\nu=\frac{1}{2}$ fermions

$\nu=1$ bosons

Generalization: Read-Rezayi states (RR) (1998)

Based on $\mathbb{Z}_k$ parafermions (and $SU(2)_k$)

$$\psi_n(z) * \psi_m(z') \sim \frac{1}{(z-z')^{\Delta_n+\Delta_m-\Delta_{n+m}}}\, \psi_{n,m}(z') + \dots$$

Fradkin & Kadanoff (1980) (!)

$$\Delta_n = \frac{n(k-n)}{k}, \quad n,m = 1,\dots,k-1$$

RR states use the parafermion CFT (Zamolodchikov & Fateev, 1985
Gepner & Qiu, 1987)

$$\Psi_{RR}(\{z_i\}) \sim \overbrace{\langle \psi_1(z_1)\dots\psi_1(z_N)\rangle}^{\downarrow} \prod_{i<j} \overbrace{(z_i-z_j)}^{}{}^{M+\frac{2}{k}} \times \text{gaussians}$$

|| vanishes when k+1 particles come together.
clustering

$M \in \mathbb{Z}$ divisible by k; M even: bosons, M odd: fermions; $\nu = \frac{k}{Mk+2}$

The most interesting case is $k=3$ ($\mathbb{Z}_3$) ($\nu = \frac{3}{2}$ (B), $\frac{3}{5}$ (F))

In addition to the $\mathbb{Z}_3$ parafermion, it has a Fibonacci anyon τ

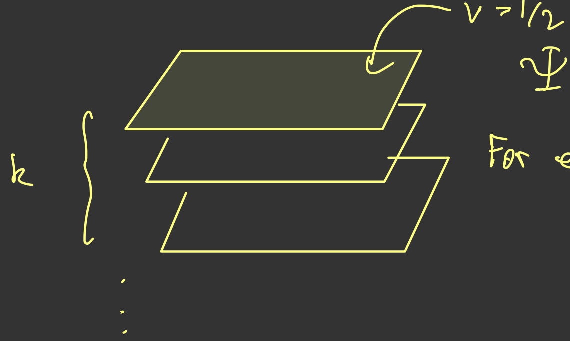
Fusion rule: $\boxed{\tau * \tau = I + \tau}$ ⇒ its unitary braiding matrices cover SU(2)

(Fibonacci sequence) ⇒ universal quantum computer

Effective Field Theory Approaches (Fradkin, Nayak, Schoutens, 1999) (Goldman, Sohal, EF, 2019, 2020)

We will discuss bosons for simplicity, $\nu = \frac{k}{2}$

Consider $\boxed{k}$ layers of bosons in a $\boxed{\nu = \frac{1}{2}}$ Laughlin state



$$\Psi_{1/2} \sim \prod_{i<j} (z_i - z_j)^{\boxed{2}} \times \text{gaussians} \longleftarrow U(1)_2$$

For each layer $\mathcal{L} = \frac{\boxed{2}}{4\pi} \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + \dots$

$$\equiv \frac{\boxed{2}}{4\pi} a\, d a + \frac{1}{2\pi} A\, d a + \dots$$

Symmetry $\underbrace{U(1)_2 \times \dots \times U(1)_2}_{k \text{ factors}}$

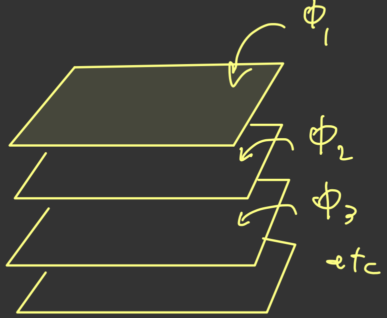
Chern-Simons level-rank duality $\quad U(1)_2 \longleftrightarrow SU(2)_1 \quad$ group is non-abelian

$I, \; e^{i\varphi/\sqrt{2}} \qquad j = 0, \frac{1}{2} \qquad$ the braids are abelian

Q: how to get to a state with non-abelian statistics?

Hint: somehow we need a theory on $\boxed{SU(2)_k}$

you need $U(1)_2 \times \dots \times U(1)_2 \longrightarrow \boxed{SU(2)_k}$



$\langle \phi_j \, \phi_{j+1} \rangle \neq 0$

Ⓐ ① use the Chern-Simons level-rank duality

$SU(2)_1 \times \dots \times SU(2)_1$

② construct a condensate $\rightarrow$ $SU(2)_k$

The 1999 paper did this by condensing pairs of excitations on two layers at a time

$\Rightarrow$ Higgs (Meissner) mechanism projects onto a state with symmetry $SU(2)_k$ (clustering)

1999 was basically right (but not completely)

$\Rightarrow$ Dualities solve the problem

Construction of a Fibonacci FQH state (Goldman, Sohal, EF, 2021)

*Want a FQH state with <u>only</u> Fibonacci anyons

$\tau * \tau = \mathbb{1} + \tau$ (and no other anyons)

$\Rightarrow$ universal quantum computing (3 τ's form a qubit)

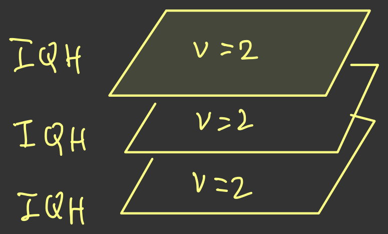
* Topological QFT?

$$(G_2)_1 \leftrightarrow U(2)_{3,1} = \frac{SU(2)_3 \times U(1)_2}{\mathbb{Z}_2}$$

$$\mathcal{L}_{Fib} = \frac{3}{4\pi} \mathrm{Tr}\left[\underset{\substack{\uparrow \\ SU(2)\text{ gauge field}}}{a}\, d a - \frac{2}{3} i a^3\right] - \frac{1}{4\pi} \mathrm{Tr}[a] \underset{\substack{\uparrow \\ U(1)_2}}{d}\, \mathrm{Tr}[a] + \frac{1}{4\pi} \underset{\substack{\uparrow \\ \text{background}}}{A}\, d\, \mathrm{Tr}[a]$$

$$\Rightarrow \quad \nu = 2 \qquad \left(\sigma_{xy} = 2 \frac{e^2}{h}\right)$$

* Start with 3 layers of Diracs at $\nu = 2 \to 1$ transition (IQH)



$$\mathcal{L} = \sum_{n=1}^{3} \left[\bar{\Psi}_n (i \not{D}_A - M) \Psi_n - \frac{3}{2} \frac{1}{4\pi} A d A \right]$$

$$D_A = \partial - i A$$

$\uparrow$ Parity anomaly

Duality: Free Dirac $\Psi \longleftrightarrow$ Wilson-Fisher boson ϕ + $U(N)_1$

OK since $U(N)_1 \leftrightarrow \mathcal{L}_{eff} = -\frac{N}{4\pi} A d A$ (trivial)

* Set $N=2$

$$\mathcal{L} = \sum_n \left[|D a_n \phi_n|^2 - r|\phi_n|^2 - |\phi|^4 + \mathcal{L}_{CS}[a_n] \right] + \frac{1}{2\pi} A d \, \mathrm{Tr}[a_1 - a_2 + a_3]$$

* Clustering: $\langle \Gamma_{mn} \rangle = \langle \phi_m^\dagger \phi_n \rangle \neq 0 \ (m \neq n), \ \langle \phi_n \rangle = 0$

$\Rightarrow$ pins $a_1 = a_2 = a_3 \equiv a \Rightarrow \frac{1}{2\pi} A d \, \mathrm{Tr}[a_1 - a_2 + a_3] \equiv \frac{1}{2\pi} A d \, \mathrm{Tr}[a]$

* The physical densities are pinned $\rho_1 = -\rho_2 = \rho_3$

$\Rightarrow$ layer exchange symmetry is broken

$$\Rightarrow \mathcal{L}_{U(2)_3} = 3\,\mathcal{L}_{CS}[a] + \frac{1}{2\pi} A\, d\, \mathrm{Tr}\,[a]$$

* To get Fibonacci $\Leftrightarrow$ attach a unit of flux to the fermions

$\Rightarrow$ fermions $\rightarrow$ bosons

flux attachment: $$3\,\mathcal{L}_{CS}[a] + \frac{1}{2\pi}\, \underset{\substack{\uparrow \\ \text{fluctuating} \\ U(1)\ \text{gauge field}}}{b}\, d\, \mathrm{Tr}\,[a] + \frac{1}{4\pi} (b+A)\, d\, (b+A)$$

* Integrating out b_μ $\Rightarrow$ obtain $\mathcal{L}_{Fib}$!

$\Rightarrow$ interpret $\underline{\phi^\dagger t^a \phi}$ as the Fibonacci anyon τ

* Alternatively we can attach (+) flux to layers 1, 3
and (-) to layer 2

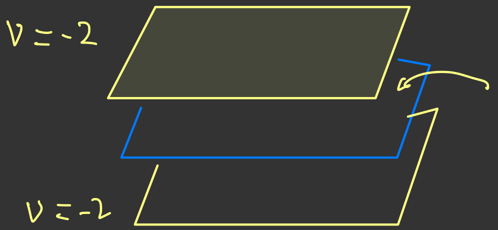
<u>before</u> clustering

$\Rightarrow$ layers 1, 3 become $|D_A \Phi|^2 + \frac{2}{4\pi} A dA$ (trivial)

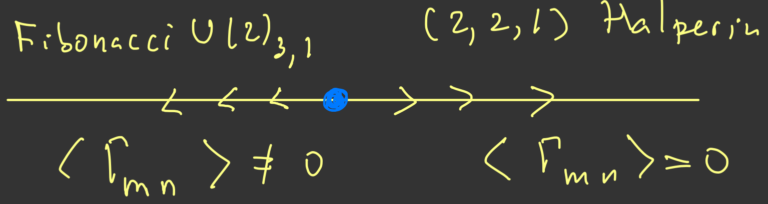
layer 2 : $|D_\alpha \Phi|^2 + \frac{2}{4\pi} \alpha d \alpha + \frac{2}{4\pi} \beta d \beta + \frac{1}{2\pi} \alpha d\beta + \frac{1}{2\pi} \beta d A$

layer 2 $\Rightarrow$ Halperin (2,2,1) state

Transition (2,2,1) $\leftrightarrow$ Fibonacci



(2,2,1) $\Rightarrow$



One can use this construction to derive the Fibonacci wave function!

- * Non-Abelian dualities can be used to understand the landscape of non-abelian FQH states
- * define physical parent states
- * construct ideal wave functions using CFT methods
- * hopefully to find simple Hamiltonians!
- * Opens a window to universal TQC!

References: Goldman, Sohal, EF PRB 100, 115111 (2019)
102, 195151 (2020)
103, 235118 (2021)