Conformal Quantum Criticality Order and Deconfinement in Quantum Dimer Models

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Outline

- Quantum Dimer Models and Generalizations
- The Quantum Lifshitz Model and Conformal Quantum Criticality
- Quantum Eight Vertex Model: Phase Diagram, Ordered (Confined) and Topological (Deconfined) Phases; Quantum Criticality
- Generic Critical Behavior of Perturbed Quantum Dimer Models: Honeycomb and Square lattices
- Phase Diagram: Tilted Phases, Incommensurate States and Devil Staircases
- Conclusions

Quantum Dimer Models

- Simple local models describing strongly frustrated and ring exchange quantum spin systems with a large spin gap and no long range spin order
- They typically exhibit spin gap phases with different types of valence bond crystal orders
- QDM have special solvable points, the Rokhsar-Kivelson (RK) point, where the exact ground state wave function has the short range RVB form

$$|\Psi_{\rm RVB}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

- On bipartite lattices, the RK points are quantum (multi) critical points described by an effective field theory with $z=2$, exponents that depend continuously on the coupling constant of a strictly marginal operator (a "Luttinger parameter"), and have massless deconfined spinons
- On non-bipartite lattices QDMs have topological $\mathbb{Z}_2$ deconfined phases with massive spinons and a topological 4-fold ground state degeneracy on a torus

Questions

- Is the connection with 2D classical systems peculiar to the QDM? Conformal invariance in 2D? Central charge?
- Is this quantum phase transition generic?, *i.e.* is it robust to local short range perturbations of the QDM?
- Why this is not a first order transition (as naively expected)?
- QDM models are known to have columnar, plaquette and “staggered" phases. These are simple commensurate phases. Are there also incommensurate phases? Devil staircases? How are they related to this QCP? Global phase diagram?
- What is the effective field theory of these phases and of these quantum phase transitions? Is this a new universality class?
- Are there quantum disordered phases?. How are they related to these QCPs?
- Models with ordered, disordered phases and quantum critical points? Quantum vertex models! Phase diagram and quantum critical behavior?
- What’s next? Other universality classes? Loops? $c < 1$? Non-Abelian Phases? See Paul Fendley’s talk!

The Quantum Dimer Model

$$H_{\rm RK} = \sum_i (vV_i - tF_i), \qquad \text{Rokhsar and Kivelson (1988)}$$

$$V_i = |\,=\,\rangle\langle\,=\,| + |\,\|\,\rangle\langle\,\|\,| \qquad F_i = |\,=\,\rangle\langle\,\|\,| + |\,\|\,\rangle\langle\,=\,|$$

Here each bar represents a spin singlet bond.

For $t = v$ $\qquad \Rightarrow H_{\rm RK} = \sum_i Q_i^\dagger Q_i$, with $Q_i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

The ground state wave function $|\Psi_0\rangle$ has $E = 0$

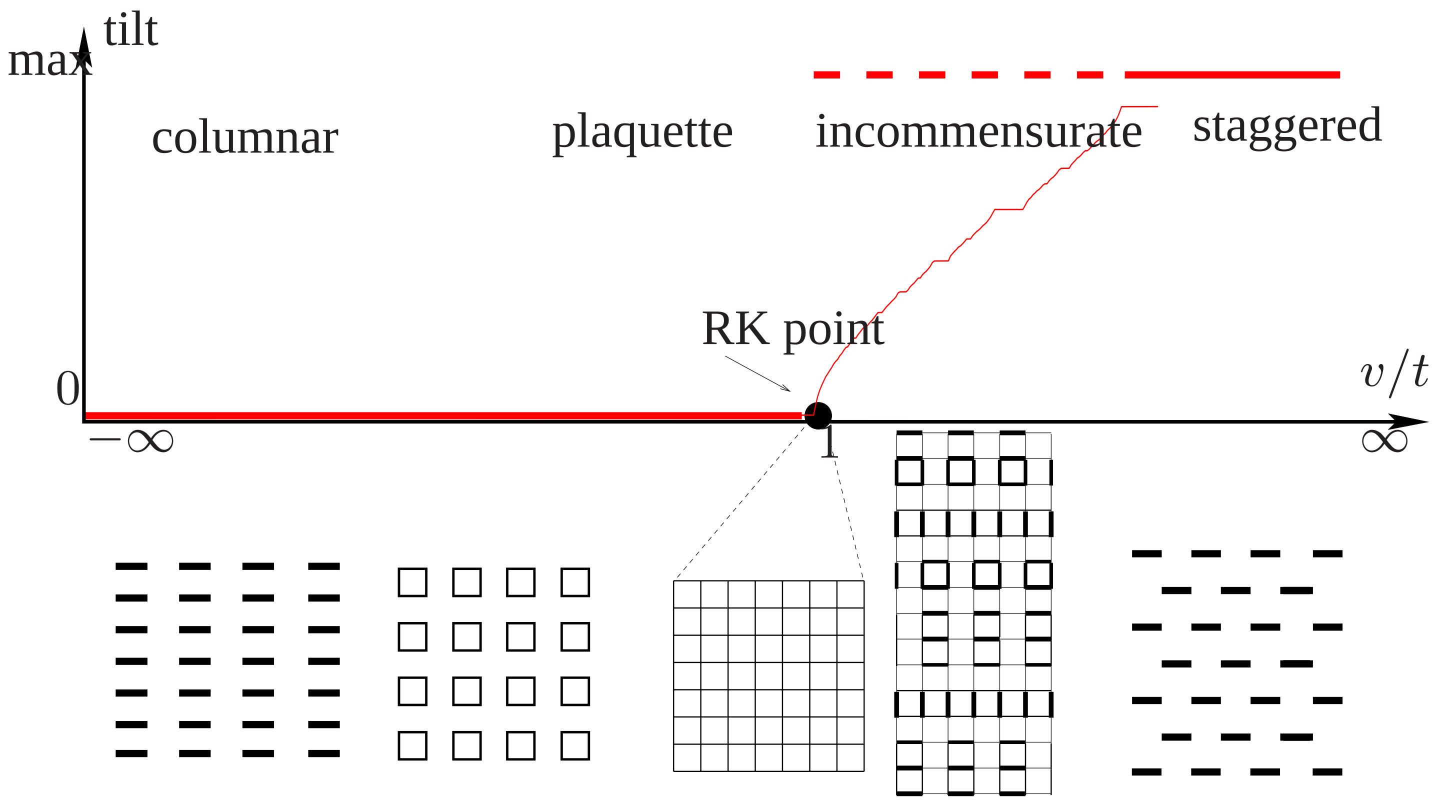
$$|\Psi_0\rangle = \frac{1}{\sqrt{Z_{\rm cl}}} \sum_C |C\rangle \ ,$$

where $Z_{\rm cl}$ is the sum over all dimer configurations

Equal-*time* correlators in the quantum dimer model at the RK point are given by correlators of the classical dimer model.

This is a critical system on a square lattice (Kivelson and Rokhsar; Fradkin and Kivelson), but non-critical, deconfined, on a triangular lattice (Moessner and Sondhi, 1998).

Schematic Phase Diagram of Generalized QDMs



Square lattice: Sachdev and Jalabert (1990)

Effective Field Theories of QDMs

QDMs are gauge theories and are dual to height models, (Fradkin and Kivelson, 1988)

Gauge theory picture:

- an electric field $\gamma - 1$ is assigned to a link occupied by a dimer and an electric field -1 if unoccupied ($\gamma =$ coordination number of the lattice).
- Electric field configurations satisfy a lattice version of Gauss' Law:

$$\forall \text{ lattice sites } \mathbf{r} \quad \Rightarrow \quad \nabla \cdot \mathbf{E}(\mathbf{r}) = 0$$

  This the hard dimer constraint: every site belongs to one and only one dimer

- Holons and spinons are violations of the hard dimer constraint, and correspond to gauge charges
- The coarse grained effective continuum Hamiltonian at the RK point

$$H = \int d^2x \left[\frac{\kappa^2}{2} (\nabla \times \mathbf{E})^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 \right]$$

$$\text{Physical states obey Gauss' Law}: \quad \nabla \cdot \mathbf{E}(\mathbf{r}) \, |\text{Phys}\rangle = 0$$

$$[E_j(x), A_k(y)] = i \, \delta_{jk} \delta^2(x - y)$$

Dimers, heights and continuum limit

- The QDM can be mapped to a height model
- The heights live on the dual lattice, and going around a vertex of the even sublattice clockwise, the height changes by $+3$ if a dimer is present, and by -1 if there is no dimer.

$h = 3/2$ $\quad$ $h = 1/2$ $\quad$ $h = -1/2$ $\quad$ $h = -3/2$

- Plaquette flip changes the height of that plaquette by ± 4, and the average height of the surrounding sites by ± 1.
- Equivalent configurations: $h \cong h + 4$.
- Columnar phase, $\langle h \rangle \neq 0$
  Staggered phase, $\langle \partial h \rangle \neq 0$
- Continuum limit: $h \cong 4\varphi(x)$
  Compactification Radius: $\varphi(x) \cong \varphi(x) + 1$.

The Quantum Lifshitz model

- Duality $\Leftrightarrow$ Solving the Gauss Law constraint

$$\nabla \cdot \mathbf{E} = 0 \;\Rightarrow\; \mathbf{E} = \nabla \times \varphi$$

- Canonical Conjugate Momentum

$$\Pi(\mathbf{r}) = B(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})\;, \qquad [\varphi(\mathbf{r}), \Pi(\mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}')$$

- Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{\kappa^2}{2}\left(\nabla^2 \varphi\right)^2 \right]$$

This is the Quantum Lifshitz Model. (Henley; Moessner, Sondhi and Fradkin)

- Action in imaginary time τ:

$$S = \int d^2x \int d\tau \left[\frac{1}{2}\left(\partial_\tau \varphi\right)^2 + \frac{\kappa^2}{2}\left(\nabla^2 \varphi\right)^2 \right]$$

Same as the free energy of smectic layers in 3D at the Lifshitz transition. (Grintsein 1982).

Ground State Wave Function and 2D Classical Critical Phenomena

- Schrödinger picture $\Rightarrow \Pi = -i\delta/\delta\varphi$.

$$\int d^2\vec{x}\left[-\frac{1}{2}\left(\frac{\delta}{\delta\varphi}\right)^2 + \frac{\kappa^2}{2}(\nabla^2\varphi)^2\right]\Psi[\varphi] = E\Psi[\varphi]$$

- The Hamiltonian has the form

$$H = \int d^2x\; Q^\dagger(\mathbf{x})Q(\mathbf{x})$$

$$Q(\mathbf{x}) \equiv \frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\varphi} + \kappa\nabla^2\varphi\right) \qquad Q^\dagger(\mathbf{x}) \equiv \frac{1}{\sqrt{2}}\left(-\frac{\delta}{\delta\varphi} + \kappa\nabla^2\varphi\right)$$

- Ground state wave-function, $\Psi_0[\varphi]$

$$Q(\vec{x})\Psi_0[\varphi] = 0 \qquad \Rightarrow \qquad \Psi_0[\varphi] \propto e^{-\frac{\kappa}{2}\int d^2x\; (\nabla\varphi(\mathbf{x}))^2}$$

$$\|\Psi_0\|^2 = \int \mathcal{D}\varphi\; e^{-\kappa\int d^2x\; (\nabla\varphi(\mathbf{x}))^2}$$

Mapping to a 2D $c = 1$ Euclidean CFT

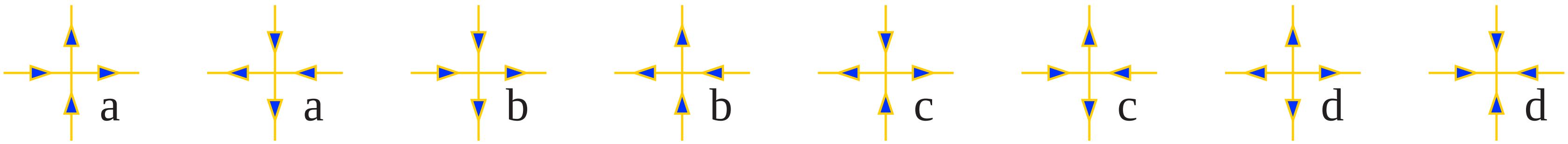
- The probability for a configuration $|\varphi\rangle$ is the Gibbs weight of a 2D classical Gaussian model, a Euclidean 2D free massless scalar field.

- The equal-time expectation value for operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$. Time-dependent correlators exhibit powerlaw behavior with dynamical exponent $z = 2$.

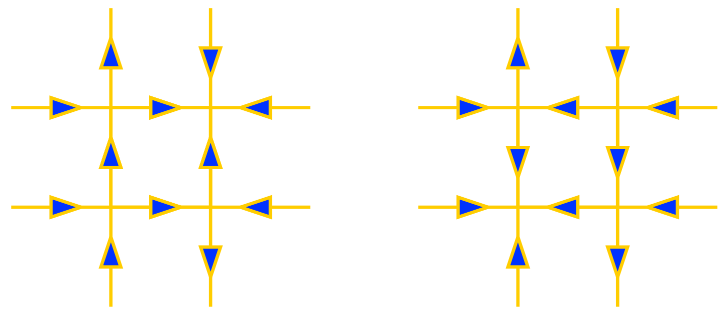
- Matching the correlation functions of the RK and Lifshitz models, one finds $\kappa = 1/2\pi$.

- Adding the term $(\nabla\varphi)^2$ with positive (negative) coefficient is a relevant perturbation which drives the system into the columnar (staggered) phase. (More on this below.)

The quantum eight-vertex model



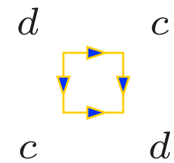
- Quantum dynamics: plaquettes are allowed to be flipped.

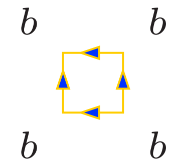


- The potential energy of a plaquette depends on the four vertices at its corners.
- At the RK point, H is of the form (with $Q_i = Q_i^\dagger \propto Q_i^2$)

$$H_{\mathrm{q8v}} = \sum_i Q_i \qquad \text{sum over plaquettes } \{i\}$$

$$Q_i = \begin{pmatrix} c^{\tilde{n}_c - n_c} d^{\tilde{n}_d - n_d} & -1 \\ -1 & c^{\tilde{n}_c - n_c} d^{\tilde{n}_d - n_d} \end{pmatrix}, \qquad \begin{cases} n_c \text{ \# of c-vertices in } i \\ \tilde{n}_c \text{ \# of c-vertices in } i \textit{ after} \text{ the flip} \end{cases}$$

- Example, consider the plaquettes $\mathbb{P}_1 =$ 

$\longleftrightarrow$ 

$= \mathbb{P}_2$.

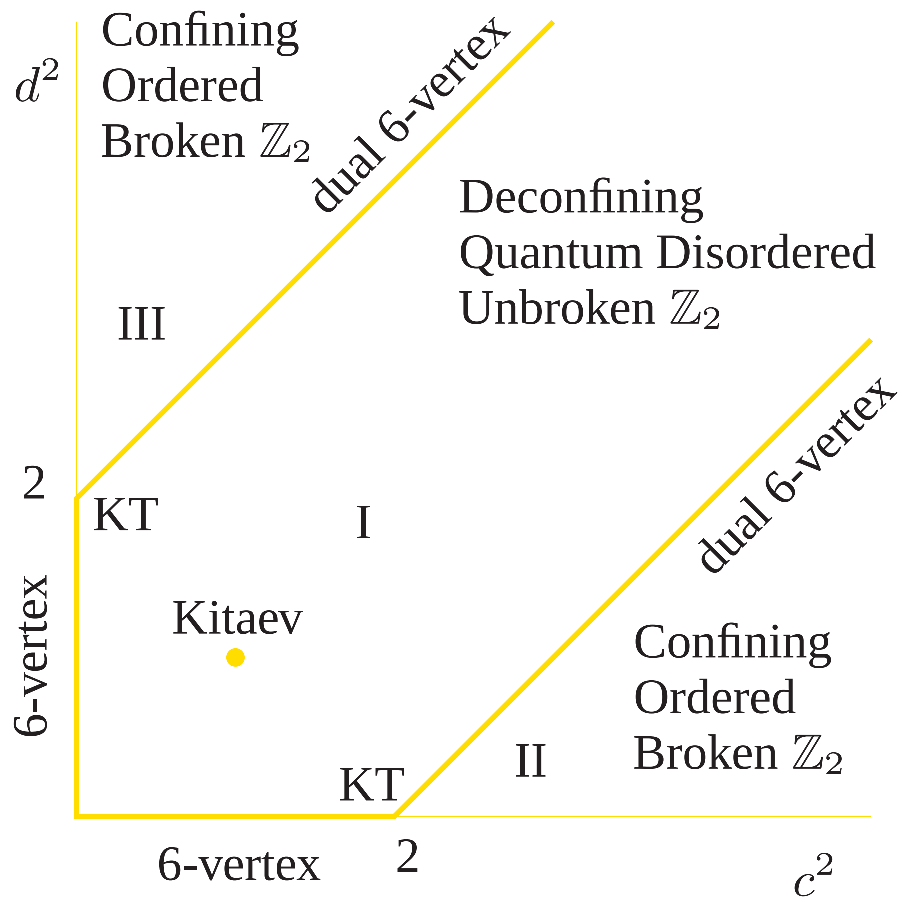
$$\Rightarrow \qquad \begin{pmatrix} \frac{1}{c^2 d^2} & -1 \\ -1 & c^2 d^2 \end{pmatrix} \begin{pmatrix} c^2 d^2 \mathbb{P}_1 \\ \mathbb{P}_2 \end{pmatrix} = 0$$

- H_{q8v} has an exact $E = 0$ ground state, $|\Psi_0\rangle$, the Baxter wave function. Its norm is the partition function of the 2D classical Baxter model.

- Amplitude for a state with N_c c-vertices and N_d d-vertices:

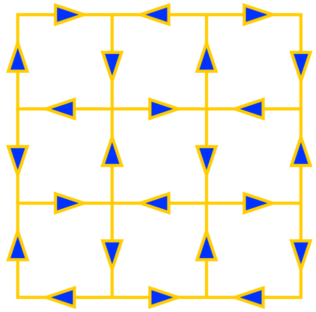
$$\Psi_0[N_c, N_d] = \frac{c^{N_c} d^{N_d}}{\sqrt{Z_{\mathrm{cl}}(c^2, d^2)}}$$

Phase diagram



Phases and Critical Behavior

- Regions II & III are ordered phases ("DDW") with broken $\mathbb{Z}_2$ symmetry. The (local) polarization operator $\langle \tau(A)\tau(B) \rangle$ is non-vanishing.
  The energy for violating the constraint at two points at a distance R costs an energy $\sigma R \quad \Rightarrow \quad$ confinement.



- Region I: Quantum disordered phase. No long-range order. The dual order parameters have a non-zero expectation value. The state exhibits topological order and deconfinement.

- Quantum critical behavior: We can use the quantum Lifshitz model at the critical lines. Using Baxter's result for the correlation length and the scaling exponent for the energy operator $x = 1/2\pi\kappa$

$$\kappa^{-1} = 8 \cot^{-1}(cd) \qquad \text{for } |c^2 - d^2| = 2$$

$$\kappa^{-1} = 8 \cot^{-1}\left(\sqrt{\frac{4}{c^4} - 1}\right) \qquad \text{for } 0 < c^2 \leq 2\,,\, d = 0$$

Is the effective field theory near the RK QCP sufficient?

$$\mathcal{L} = \frac{1}{2}(\partial_\tau h)^2 + \frac{1}{2}\rho_2(\nabla h)^2 + \frac{1}{2}\rho_4(\nabla^2 h)^2 + \lambda\cos(2\pi h)$$

Henley 1997, Moessner, Sondhi, Fradkin 2002

- It is rotationally invariant and does not depend on the symmetries of the underlying lattice.
- ρ_2 changes sign at the RK point where it vanishes.
- λ keeps track of the discreteness of the microscopic heights
- $\rho_4 = \kappa^2$ is non universal. Square lattice, $\rho_4 = (\pi/32)^2$; Honeycomb lattice, $\rho_4 = (\pi/18)^2$.
- Compactification Radius: minimum shift of heights describing the same dimer configuration. $R = 3, 4$ (honeycomb ,square)

Life Outside the RK Point

- The QDM shows continuous quantum phase transition from zero tilt states discontinuously to maximally tilted states, *e.g.* plaquette to staggered.
- Are there less commensurate or even incommensurate phases in generalized QDMs? Are they pinned or sliding?
- Why are these transitions not first order?
- The transition between the plaquette and staggered phases is unusual.
  - The plaquette order parameter vanishes continuously but the staggered order parameter appears in full strength.
  - Two diverging length scales on the plaquette side: $\chi \sim |v-t|^{-1/2}$ and $\chi_c \sim \lambda^{-1/2}\chi_c^{\theta}$, $\theta = 6, 5/2$ (square and honeycomb).
- On bipartite lattices only highly commensurate ordered states (and critical points) are allowed in the QDM.
- On non-bipartite lattices, and in the eight vertex model, quantum disordered phases appear, and are continuously connected to the RK QCP.

Mapping for Dimer Density Operators

- Dimer density operators for the honeycomb lattice:

$$n_1 - \frac{1}{3} = \frac{1}{3}\partial_x h + \frac{1}{2}[\exp(2\pi i h/3)\exp(4\pi i x/3) + \text{h.c.}]$$

$$n_2 - \frac{1}{3} = \frac{1}{3}(-\frac{1}{2}\partial_x + \frac{\sqrt{3}}{2}\partial_y)h + \frac{1}{2}[\exp(2\pi i h/3)\exp(4\pi i x/3 + 4\pi i/3) + \text{h.c.}]$$

$$n_3 - \frac{1}{3} = \frac{1}{3}(-\frac{1}{2}\partial_x - \frac{\sqrt{3}}{2}\partial_y)h + \frac{1}{2}[\exp(2\pi i h/3)\exp(4\pi i x/3 - 4\pi i/3) + \text{h.c.}]$$

- Square Lattice:

$$n_x - \frac{1}{4} = \frac{1}{4}(-1)^{x+y}\partial_y h + \frac{1}{2}[(-1)^x \exp(\pi i h/2) + \text{h.c.}]$$

$$n_y - \frac{1}{4} = \frac{1}{4}(-1)^{x+y+1}\partial_x h + \frac{1}{2}[(-1)^y \exp(\pi i h/2 + \pi i/2) + \text{h.c.}]$$

Effective Field Theory a Honeycomb Lattice

- For the honeycomb lattice, the heights live on the (dual) triangular lattice, and take the values 0, 1 and 2 modulo 3 on the three sublattices of the triangular lattice.
- To be a symmetry, rotations by $\pi/3$ and inversion require $h \to -h$.
- The effective Lagrangian must include the relevant cubic perturbation

$$\mathcal{L}_3 = g_3(\partial_x h)(\frac{1}{2}\partial_x h - \frac{\sqrt{3}}{2}\partial_y h)(\frac{1}{2}\partial_x h + \frac{\sqrt{3}}{2}\partial_y h)$$

and the marginal quartic perturbation,

$$\mathcal{L}_4 = g_4 \left[\nabla h \cdot \nabla h\right]^2 .$$

- If $g_3 \neq 0$, the relevant perturbation takes over drives the transition first order and the state has maximal tilt.

Stability of the RK Point at $g_3 = 0$

- RG flow for $g_3 = 0$:

$$\frac{d\lambda}{dt} = -\left(\frac{\pi}{2\rho_4^{1/2}} - 2\right)\lambda$$

$$\frac{dg_4}{dt} = -\frac{9}{4\pi\rho_4^{3/2}} g_4^2 \, ,$$

- $g_4 > 0 \quad \Rightarrow \quad \lambda$ irrelevant and g_4 marginally irrelevant
- logarithmic corrections corrections to scaling
- finite renormalizations of ρ_4
- The multicritical RK point is stable on a surface of codimension 2 ($\rho_4^{\text{eff}} \leq \pi^2/16$).

The Titled Phase

- For g_4 large enough, the tilt is not maximal. For $g_3 \neq 0$ the direction of the tilt depends on the sign of g_3.

- Small fluctuations in the weakly tilted state ($g_3 < 0$, tilt along the x axis, $|g_3|$ small)

$$h(\mathbf{r}, \tau) = \mathbf{C} \cdot \mathbf{r} + \delta h(\mathbf{r}, \tau), \qquad \mathbf{C} = C\mathbf{e}_x$$

$$\delta \mathcal{L} = \frac{1}{2}(\partial_\tau \delta h)^2 + \frac{\rho_4}{2}(\nabla^2 \delta h)^2 + \frac{v_l^2}{2}(\partial_x \delta h)^2 + \frac{v_t^2}{2}(\partial_y \delta h)^2$$

- $|\rho_2| < g_3^2/g_4 \Rightarrow v_t \approx v_l$; there is a single correlation length $\xi^{-1} = v_l/\sqrt{\rho_4}$.

- New Bragg peaks in the dimer density structure factor
  - *commensurate* Bragg peak at the wavevector of the maximally “staggered" state located at the origin
  - *incommensurate* peaks displaced from the wavevector of the columnar/plaquette pattern, $(4\pi/3, 0)$, by an amount proportional to C.

- The existence of gapless modes has important implications: test monomers (spinons) interact through a logarithmic potential.

Incommensurate Locked Crystals: The Devil's Staircase

- We ignored the possibility of the height locking into tilted configurations.
- The height and the lattice points together define a 3D lattice $\Gamma = \{(h, \mathbf{x})\}$, the simple cubic lattice with the $[111]$ direction measuring height.
- General lock-in potential

$$V_{\text{lock}}(h, \mathbf{x}) = \sum_{\{\mathbf{G}\}} V_{\mathbf{G}} \; e^{iG_h h + \mathbf{G}_{\mathbf{x}} \cdot \mathbf{x}}, \quad \mathbf{G} \in \Gamma^*$$

- Tilt commensurate with any of the $\{\mathbf{G}\}$: Gaussian fluctuations about a given tilted state lead to locking at any point in the tilted phase. The ground state is a VBC with a gapped spectrum.
- Incommensurate tilts: it depends upon the strength of V_{lock} relative to the remaining quantum fluctuations.

Strong fluctuations or Weak Locking

Near the RK point, $\rho_2 = 0$ and g_3 small,

- Higher order commensuration to lock-in $|\mathbf{G}| \sim 1/C$
- Locking potentials get weak
- Locking operators are more irrelevant
- $V_{\text{lock}}^{\text{eff}} \to 0$, the gap $\Delta \to 0$ (and range of the commensurate phases) as $C \to 0$

$$\Delta \sim C^{a/(\rho_4 C^2}, \qquad \xi \sim \frac{1}{C}\ , \qquad \xi_c \sim \xi^{a' \xi^2/\rho_4}$$

- Close to the RK point even the commensurate phases are essentially gapless!
- Close to the RK point, the ground state is an incommensurate VBC with a Bragg peak at the incommensurate wavevector but with a gapless (phason) spectrum, a“photon" $\Rightarrow$ Deconfinement

Weak fluctuations or Strong Locking

- Ground states‘ form a Devil’s Staircase
- Aubry’s “breaking of analyticity” transition beyond which
  - the incommensurate ground states are pinned,
  - their low lying excitations are localized and gapless,
  - the incommensurate ground states occupy a set of measure zero of the phase diagram.
- Cantor deconfinement!

Effective Theory of the Square Lattice

Generic case: (a) allow ρ_4 to vary, (b) include the quadratic and strictly marginal term,

$$\mathcal{L}_m = \frac{1}{2}\tilde{\rho}_4 \left[(\partial_x^2 h)^2 + (\partial_y^2 h)^2\right] \ ,$$

and (c) include the interactions

$$\mathcal{L}_{\text{int}} = g_4 \left[\nabla h \cdot \nabla h]\right)^2 + \tilde{g}_4 \left[(\partial_x h)^4 + (\partial_y h)^4\right]^2 \ .$$

- Now we have a 2D surface of fixed points
- No cubic invariant is allowed; continuous transition in mean field theory.
- RG flow around the line with $\widetilde{\rho}_4 = 0$:

$$\begin{aligned}
\frac{d\lambda}{dt} &= -\left(\frac{\pi}{2\rho_4^{1/2}} - 2\right)\lambda \\
\frac{dg_4}{dt} &= -\frac{9}{4\pi(\rho_4)^{3/2}}(g_4^2 + g_4\tilde{g}_4 + \frac{1}{4}\tilde{g}_4^2) \\
\frac{d\tilde{g}_4}{dt} &= -\frac{9}{4\pi(\rho_4)^{3/2}}(\frac{2}{3}g_4\tilde{g}_4 + \frac{1}{2}\tilde{g}_4^2)
\end{aligned}$$

- The flows in the $(g_4, \tilde{g}_4)$ plane are attracted to the origin only along the positive g_4 axis
- Generically the flows run away to the region where the action is unstable at quartic order in gradients.
- Fluctuation-driven first order transition!
- The RK point behavior requires a further fine tuning, $\tilde{g}_4 = 0$ multicritical surface of codimension two.
- Runaway flows: the pattern of symmetry breaking is indicated by the initial sign of $\tilde{g}_4$ and depending on that there are four states with the tilt either aligned or at angle $\pi/4$ to the lattice axes. All of them exhibit a modulation of the dimer density at wavevector (π, π).
- Tilted Phase: we need the reciprocal lattice vectors of the diamond lattice, with the direction $[100]$ measuring the height. Bragg peaks at (π, π), $(\pi, 0)$ and $(0, \pi)$. Devil's staircase.

Conclusions

- We showed that the Quantum Lifshitz Model describes the universality class of Quantum Dimer Models, the Quantum Eight Vertex Model and their generalizations
- These are multicritical points with dynamic critical exponent $z = 2$ and codimension 2, and are ultra-deconfined.
- We showed that these (multi)critical points describe phase transitions between confining valence bond crystalline states with different tilts, and between these ordered phases and topological (spin liquid) deconfined phases.
- We showed on the effective field theory in the vicinity of the (multi)critical point admits additional operators (not included in the Quantum Lifshitz Model) which either make the transition first order or give rise to a complex sequence of commensurate and incommensurate phases.