Realizing non-Abelian statistics in time-reversal-invariant condensed matter systems

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Outline

- Motivation
- Braiding Statistics in $2+1$ Dimensions and Algebras; $SU(2)$ and $SO(3)$
- Quantum Loop Gases, S-matrices and Braid Matrices
- Wave Function Engineering in $2+1$ Dimensions from Field Theories in $1+1$ Dimensions
- Quantum Loop Lattice Models with Non Abelian Statistics, $SU(2)$ and $SO(3)$ and Generalized Potts Models
- Topological Phases and Phase Transitions
- Conclusions

Spin Liquids and Topological States of Matter

- Liquid phases of electron fluids and spin systems without long range order, with or without time reversal symmetry breaking
- Quasiparticles: vortices with fractional charge and fractional statistics (Abelian and non-Abelian)
- Hidden Topological Order and Topological Vacuum Degeneracy
- Finite-dimensional quasiparticle Hilbert spaces $\Rightarrow$ universal topological quantum computer

“Known” Topological Quantum Liquids

- 2DEG Fractional Quantum Hall Liquids
  - Abelian FQH states (Laughlin and Jain): fractional charge and Abelian fractional statistics
  - Non-Abelian FQH states: Is $\nu = 5/2$ a Pfaffian (Moore-Read) FQH state? Parafermion states?
- Rapidly rotating Bose gases: possible non-Abelian (Pfaffian) FQH state of bosons at $\nu = 1$
- Time-Reversal Breaking Superconductors: Is Sr_2RuO_4 a $p + ip$ superconductor?

Challenges

- To develop a consistent theory of topological phases and to understand the mechanisms
- What are the generic phases of models of topological liquids
- Is the gap necessary? Can a topological liquid be gapless?
- Concrete examples of lattice models with local interactions with topological phases
- Fractional Statistics: Abelian and non-Abelian
- There has been some progress in constructing models with Abelian statistics
- To find experimentally realizable models

Statistics and Quantum Mechanics

In Quantum Mechanics the wave-function depends on the positions of the particles and their quantum numbers $i_1 i_2 \ldots$. To make the notation simpler, we just denote the labels $i_1; i_2 \ldots$ by a single one a:

$$\Psi_a(x_1, x_2, \ldots)$$

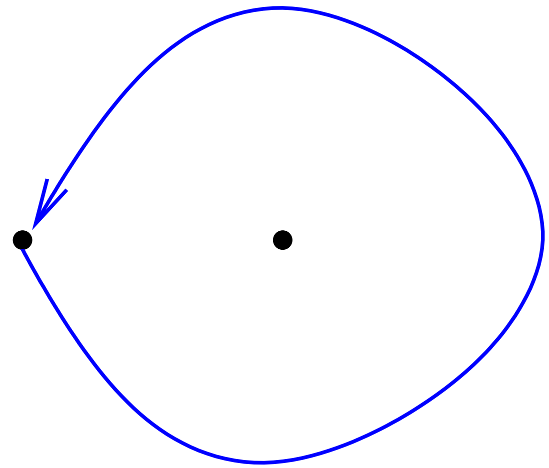
The statistics of the particles comes from the behavior of Ψ under the interchange $x_1 \leftrightarrow x_2$.

In $3+1$ dimensions the only allowed symmetry of the wave function under exchange requires that the particles are either fermions and bosons

$$\Psi_a(x_1, x_2, \ldots) = \pm\Psi_a(x_2, x_1, \ldots)$$

Statistics and Adiabatic Evolution

In $2+1$ dimensions there are more possibilities. We will regard the identical particles as having a hard core and we will consider an adiabatic time evolution which corresponds to an exchange process:



- $3+1$ dimensions: this path is topologically trivial
- $2+1$ dimensions: this path is topologically non- trivial $\Rightarrow$ Braids!

For Laughlin (and Jain) states

$$\Psi_a(x_1, x_2, \ldots) = e^{i\theta}\Psi_a(x_2, x_1, \ldots), \qquad \theta = \frac{\pi}{m}$$

Anyons with Abelian (braid) fractional statistics!

Non-Abelian Fractional Statistics

However, life can be more complicated .... and more interesting

- Consider an N-particle state with quantum numbers $i_1, \ldots, i_N$.

$$\Psi_{i_1 \ldots i_N, r}(z_1, \ldots, z_N)$$

$r = 1, \ldots, p$ labels p linearly independent states with the *same quantum* numbers

- Upon an "exchange" process these states mix as follows:

$$\Psi_{i_1 i_2 \ldots i_N, r}(z_1, z_2, \ldots, z_N) \rightarrow \sum_{s=1}^{p} B_{rs} \; \Psi_{i_2 i_1 \ldots i_N, s}(z_2, z_1, \ldots, z_N)$$

where B_{rs} is a $p \times p$ *(braid) matrix*

- non-Abelian fractional statistics!

Fractional Statistics in Laughlin FQH States

$$\Psi_m(z_1, \ldots, z_N) = \prod_{i>j=1}^{N} (z_i - z_j)^m \; e^{-\frac{1}{2\ell^2}\sum_{i=1}^{N}|z_i|^2}$$

$$z = x + iy, \qquad \ell = \sqrt{\frac{\hbar c}{eB}}$$

- The Laughlin FQH states are uniform fluid phases with filling factor $\nu = 1/m$, where m is an odd integer
- The topological degeneracy of these ground states is m^g where g is the genus of a closed surface ($g = 0$ for a sphere, $g = 1$ for a torus, etc.)
- Their charged excitations are quasiholes which are anyons with (Abelian) fractional statistics $\theta = \pi/m$ and fractional charge $e^* = e/m$

Non-Abelian Pfaffian (Moore-Read) FQH States

$$\Psi_{\rm Pf} \;=\; {\rm Pf}\left(\frac{1}{z_i - z_j}\right)\;\prod_{i>j}(z_i - z_j)^q\;\; e^{-\frac{1}{4\ell^2}\sum |z_i|^2}$$

${\rm Pf}\left(\frac{1}{z_i-z_j}\right) = \mathcal{A}\left(\frac{1}{z_1-z_2}\,\frac{1}{z_3-z_4}\,\ldots\right)$ is the antisymmetrized product

- Filling fraction: $\nu = \frac{1}{q}$, with $q \in \mathbb{Z}$
- Pfaffian factor: also appears in $p + ip$ superconductors and in $\nu = 1$ rotating BECs
- Generalization: Read-Rezayi states $\Rightarrow$ parafermions

Non-Abelian Braiding Statistics of Quasiholes

- There are 2^{n-1} linearly independent states of $2n$ quasiholes at fixed positions $\Rightarrow$ non-Abelian statistics

- The braiding matrices are associated with the group $SO(2n)$ (Wilczek and Nayak)

- $SO(2n)$ has a 2^{n-1} dimensional (Majorana) spinor representation, constructed from 2n real γ-matrices satisfying $\{\gamma_i, \gamma_j\} = \delta_{i,j}$

- The braiding of two quasiholes leads to a rotation in the two-dimensional Hilbert space of four-quasihole states

$$\text{Braiding matrix} \;\Rightarrow\; \frac{e^{i\pi\left(\frac{1}{8}+\frac{1}{4q}\right)}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Time Reversal Invariant Spin Liquids: Quantum Dimer Models

- Simple local models describing strongly frustrated and ring exchange quantum spin systems with a large spin gap and no long range spin order
- They typically exhibit spin gap phases with different types of valence bond crystal orders
- QDM have special solvable points, the Rokhsar-Kivelson (RK) point, where the exact ground state wave function has the short range RVB form

$$|\Psi_{\text{RVB}}\rangle = \sum_{\{C\}} |C\rangle, \qquad \{C\} = \text{ all dimer coverings of the lattice}$$

-
  - Bipartite lattices: the RK points are quantum (multi) critical points, described by an effective field theory with $z = 2$ and massless deconfined spinons, or first order transitions
  - Non-bipartite lattices: QDMs have topological $\mathbb{Z}_2$ deconfined phases with massive spinons and a topological 4-fold ground state degeneracy on a torus (Moessner and Sondhi, 1998)

The Quantum Dimer Model

$$H_{\rm RK} = \sum_i (vV_i - tF_i), \qquad \text{Rokhsar and Kivelson (1988)}$$

Here each bar represents a spin singlet bond.

For $t = v$ $\Rightarrow H_{\rm RK} = \sum_i Q_i^\dagger Q_i$, with $Q_i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

- The ground state wave function $|\Psi_0\rangle$ has $E = 0$

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z_{\rm cl}}} \sum_C |C\rangle \ ,$$

  where $Z_{\rm cl}$ is the sum over all dimer configurations

- Equal-*time* correlators in the quantum dimer model at the RK point are given by correlators of the classical dimer model.

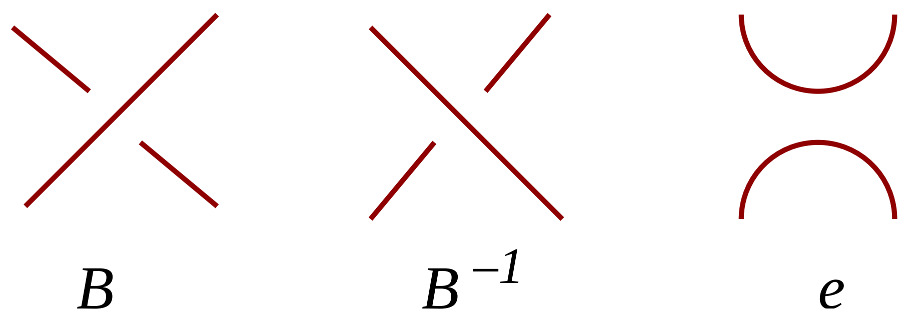
- This is actually a loop model: loops are the dimer moves from a reference state. This is the simplest loop model: the $SU(2)_1$ fully packed loop model.

Strategy for a Generalization

- Each basis state in the Hilbert space is a loop configuration in 2D
- We start with the statistics we wish to have, and work backward
- Algebraic characterization of braiding in both $SU(2)_k$ and $SO(3)_k$ Chern-Simons theories.
- Braid matrix of a 2+1-dimensional theory as a limit of the S-matrix of an associated relativistic 1+1 dimensional model
- We construct quantum 2D models with these braid relations by utilizing the structure of the factorizable S-matrices of integrable 1D models.
- We embed the 1D model in 2D Euclidean space, and find a (Rokhsar-Kivelson-type) quantum Hamiltonian whose ground state has the properties expected of a model with non-Abelian statistics.
- Loop gases:
  - $SU(2)_k$ case: $O(n)$ lattice model with $n = 2\cos(\pi/(k+2))$ (self-avoiding and mutually-avoiding loops)
  - $SO(3)_k$ case: domain walls of a Q-state Potts model with $Q = 4\cos^2(\pi/(k+2))$ (loops intersect and branch: nets)

Braids and Algebras

- Consider the worldlines of a set of particles with “hard cores” in $2+1$ dimensions
- Project each configuration on the 2D plane: N world lines $\Leftrightarrow$ N planar strands
- Braiding $\Leftrightarrow$ crossing and undercrossing of strands
- Amplitude for a configuration of strands $\Leftrightarrow$ Scattering process in a equivalent $1+1$ theory whose “time” is one of the directions of the plane
- The Braid Group acts on a N-particle space of states $V(N) = V_1 \otimes V_2 \otimes \ldots \otimes V_N$
- The elements of the braid group corresponding to overcrossings and undercrossings are denoted B_i and B_i^{-1}, acting on strands i and $i+1$

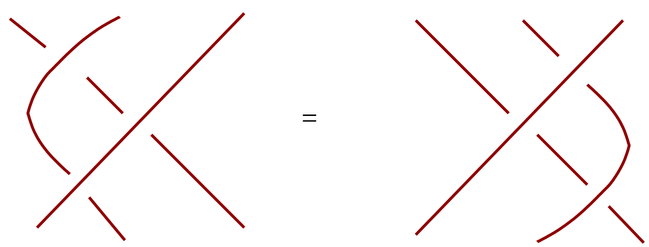


braid, reverse braid, and Temperley-Lieb generator

The Braid Group

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1},$$
$$B_i B_j = B_j B_i \qquad |i-j| \geq 2.$$

- If the matrices B_i are diagonal, then the particles have Abelian statistics
- For bosons the B_i matrices are all the identity
- for anyons their entries are phases.
- Non-Abelian representations of the braid group, so that particles obey non-Abelian statistics: the wave function changes form depending on the order in which the particles are braided.



Consistency relation for braiding

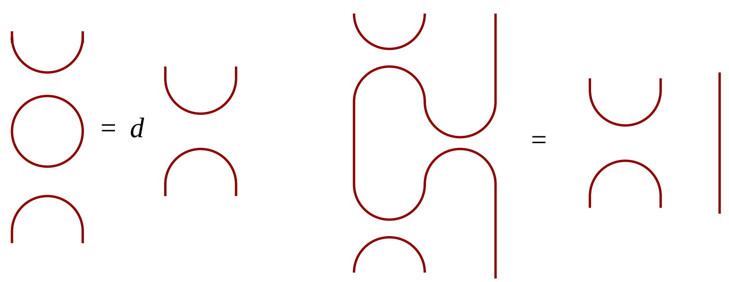
$SU(2)$ Chern-Simons and Temperley-Lieb Algebra

Braiding of Wilson and Polyakov loops in the $S = 1/2$ representation of $SU(2)_k$ CS

$$S = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} \left(A^a_\mu \partial_\nu A^a_\lambda + \frac{2}{3} f_{abc} A^a_\mu A^b_\nu A^c_\lambda \right)$$

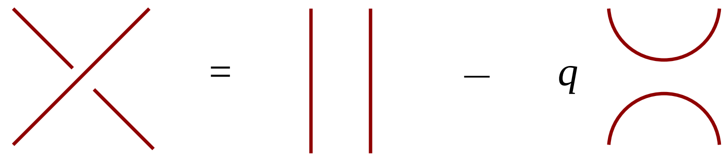
$$\begin{aligned} e_i^2 &= d e_i, \\ e_i \, e_{i\pm1} \, e_i &= e_i, \\ e_i \, e_j &= e_j \, e_i \quad (|j-i| \geq 2). \end{aligned}$$

e_i acts non-trivially on the ith and $(i+1)$th particles



d is the weight of a closed loop

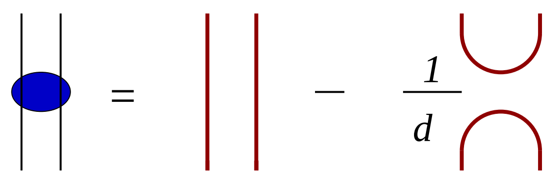
$$B_i = I - q e_i, \qquad B_i^{-1} = I - q^{-1} e_i$$



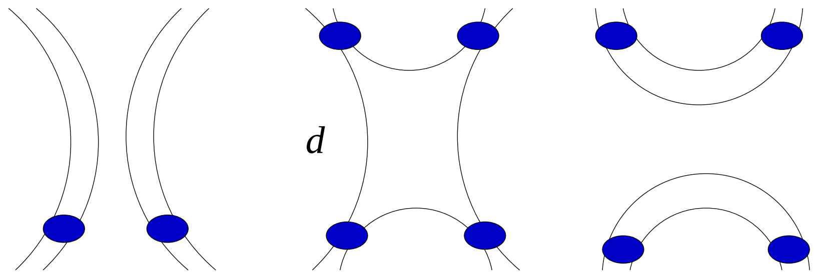
- It obeys the Braid Group if $d = q + q^{-1}$
- Braid relations are those of Wilson loops in the $S = 1/2$ representation of $SU(2)_k$ Chern-Simons theory when $d = 2\cos\left(\frac{\pi}{k+2}\right)$ or $q = e^{i\pi/(k+2)}$ (Witten)
- A topological ground state must satisfy a suitable projection operator, known as the Jones-Wenzl projector which acts on $k+1$ strands (Freedman, Nayak, Shtengel, Walker and Wang)

$SO(3)$ Chern-Simons: the BMW Algebra

- Braiding of Wilson and Polyakov loops in the $S = 1$ representations of $SO(3)_k$
- Representations of the Braid Group are given in terms of the $SO(3)$ Birman-Murakami-Wenzl (BMW) Algebra
- Projector onto $S = 1$: $P_i = I - \frac{1}{d} e_i, \quad P_i e_i = 0$
- The single-particle space of states W_j in the $so(3)$ BMW algebra is comprised of two "fused" Temperley-Lieb strands, projected onto the spin-1 representation, $W_j = P_{2j-1}[V_{2j-1} \otimes V_{2j}]$.



- Two non-trivial generators: X_j and E_j acting on pairs of adjacent strands:



$$E_j \quad = \quad P_{2j-1}\, P_{2j+1}\, e_{2j}\, e_{2j-1}\, e_{2j+1}\, e_{2j}\, P_{2j-1}\, P_{2j+1},$$

$$X_j \quad = \quad d\, P_{2j-1}\, P_{2j+1}\, e_{2j}\, P_{2j-1}\, P_{2j+1}.$$

- These generators (with $Q = d^2$) act on the two-particle states in $W_j \otimes W_{j+1}$.

$$\begin{aligned} (E_i)^2 &= (Q-1)E_i, \\ (X_i)^2 &= (Q-2)X_i + E_i. \\ E_iX_i = X_iE_i &= (Q-1)E_i. \end{aligned}$$

- The E's satisfy a Temperley-Lieb Algebra with $d = Q - 1$
- closed loops of "spin-1" particles get a weight $Q - 1 = d^2 - 1 = 1 + q^2 + q^{-2}$

The Braid Group for $SO(3)$

- $SO(3)$ representation of the Braid Group

$$B_j^{SO(3)} = q^2\mathcal{I} - X_j + q^{-2}E_j, \qquad \left(B_j^{SO(3)}\right)^{-1} = q^{-2}\mathcal{I} - X_j + q^2 E_j$$

- the particles are integer spin representations of the quantum group $U_q(sl_2)$

- Projector onto spin-s representation of $U_q(sl_2)$: $\mathcal{P}^{(s)}$
  Jones-Wenzl Projector (acts on $k+1$ strands): $\mathcal{P}_j^{([k+1]/2)} = 0$, $\forall j$

- $\mathcal{P}_j^{(1)} = P_j = I - \frac{e_j}{d}$ projects onto the spin-1 representation.
  $k = 1$: the JW projector is $\mathcal{P}_j^{(1)} = P_j = 0$ which is automatic
  $\Rightarrow SO(3)_1$ is trivial.

$SO(3)_3$: the "Yang-Lee Model"

- $SO(3)_3$: the "Yang-Lee Model", is the simplest model of non-Abelian statistics
- $k = 3$: two strands

$$\mathcal{P}_j^{(2)} = P_{2j-1}P_{2j+1} - \frac{1}{d^2-2}X_j + \frac{1}{(d^2-2)(d^2-1)}E_j$$

- For $k = 3$ the Jones-Wenzl projector sets $\mathcal{P}_j^{(2)} = 0$
  in $SU(2)_3$ this is a relation with 4 strands
  in $SO(3)_3$ it is a relation for 2 fused strands
- $SO(3)_3$ BMW generators: $X_j = (Q-2)(\mathcal{I} + E_j)$
- $Q - 2 = q^2 + q^{-2} \Rightarrow B_j^{SO(3)_3} = -q^2\mathcal{I} - q^{-2}E_j$
- For $k = 3$ the generators E_j obey the same Temperley-Lieb algebra as the e_i,
  because $d = 2\cos(\pi/5) = d^2 - 1$
- $q^5 = -1 \Rightarrow B_j^{SO(3)_3} \equiv B_j^{TL} \Rightarrow SO(3)_3$ is *almost* equivalent to $SU(2)_3$ except
  that $SO(3)_3$ is projected and $SU(2)_3$ still needs projection.

Wave Function Engineering in $2+1$ Dimensions from Field Theories in $1+1$ Dimensions

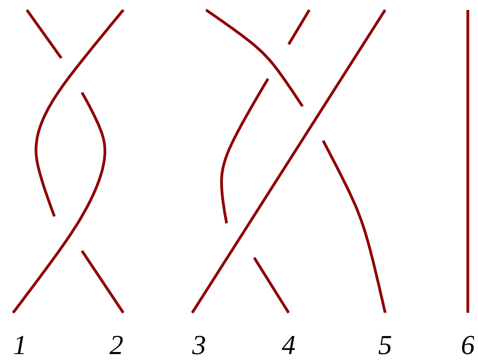
- Project the world lines of the particles down to the plane $\Rightarrow$ loops
- The basis states $|s\rangle$ of the Hilbert space are configurations of 2D loops
- This assumes that the $2+1$-dimensional theory is holographic
- The wave function Ψ of this ground state can be written as

$$\langle s|\Psi\rangle = \frac{e^{-\mathcal{S}(s)}}{\sqrt{Z}}$$

- $\mathcal{S}(s)$: action of the *classical* 2D loop model for the configuration s.
- Z is the 2D partition function with weight $|\langle s|\Psi\rangle|^2$, which is the functional integral over all configurations s with weight $e^{-\mathcal{S}(s)-\mathcal{S}^*(s)}$.

2D wave functions and 1D S-matrices

- The plane is a $1+1$-dimensional Euclidean space time
- A strand configuration is the "time" evolution of a system of particles in 1D



- A 2D wave function is given by an evolution in 1+1 dimensions
- it is the evolution of a vector in $V^{\otimes N}$, specified at the boundary
  a 1D wave function specified in terms of a set of coordinates and momenta of the particles, $x_1, p_1, \ldots, x_N, p_N$ at the boundary
- The evolution is specified by the 1D S-matrix which is a matching condition for $x_i < x_{i+1}$ and $x_i > x_{i+1}$

- $\psi_{V_i \otimes V_{i+1}}(x_i \gg x_{i+1}) = S_i(\theta_{i,i+1}) \psi_{V_i \otimes V_{i+1}}(x_i \ll x_{i+1})$
  where $\theta_{i,i+1}$ is the relative *rapidity* of the two particles

- For integrable systems the S-matrix obeys the Yang-Baxter Equation (same for Boltzmann weights in lattice models)

- Correspondence between the S-matrix and the braid generators

$$B = \lim_{\theta \to \infty} \widetilde{S}(\theta), \qquad B^{-1} = \lim_{\theta \to \infty} \widetilde{S}(-\theta)$$

- $\widetilde{S}$ obeys Yang-Baxter $\Rightarrow$ B obeys the Braid Group algebra

- There is a natural representation of the Braid Group associated to a given integrable theory with a factorizable S-matrix

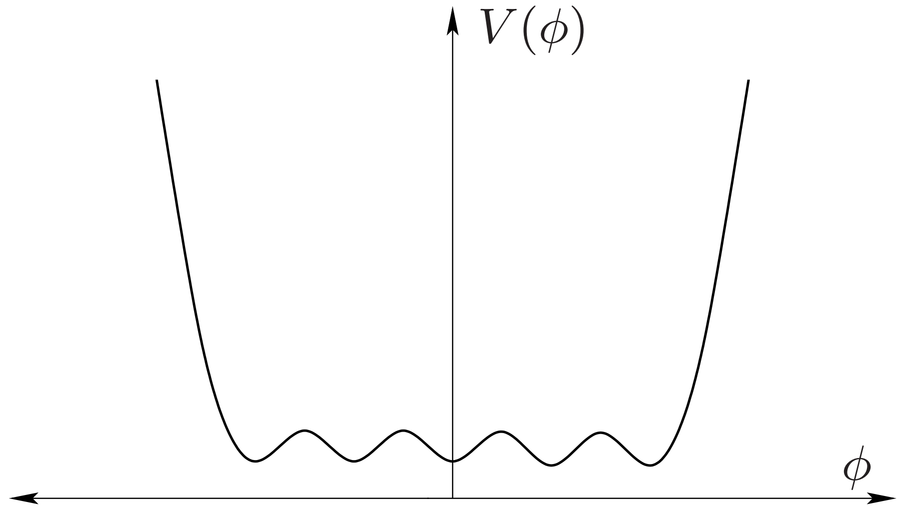
- This connection gives a prescription for constructing wave functions with excitations with non-Abelian braid statistics

- Topological theory: Unitarity requires that braiding be compatible with the Jones-Wenzl projection

- A 2D Hamiltonian can be found à la Rokhsar-Kivelson

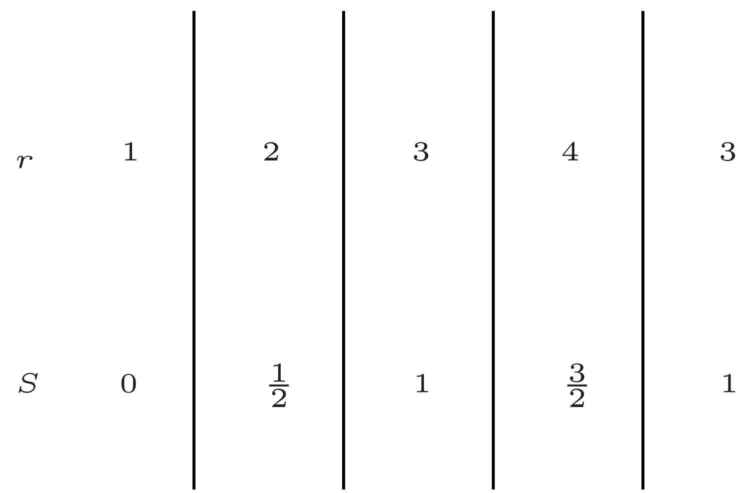
Quantum Loop Lattice Models with Non Abelian Statistics, and Generalized Potts Models

Consider a 2D classical problem with $k+1$ states; *e.g.* an RSOS model with “dual spins” (or “heights”) taking values $1, \ldots, k+1$ with a Landau-Ginzburg potential



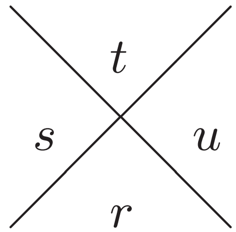
Strands and Domain Walls

- The heights can only change by ± 1 across a domain wall
- We can regard the domain walls of this model as the strands carrying a $S = 1/2$ representation of $U_q(sl_2)$
- We can also associate a spin $S = \frac{h-1}{2}$ representation to a height $h = 1, \ldots, k+1$
- Crossing a strand $\Leftrightarrow$ tensor products of spin of the region on the left (S_L) and the spin $(1/2)$ of the strand: $S_L \otimes 1/2 = (S_L + 1/2) \oplus (S_L - 1/2) \Leftrightarrow h_R = h_L \pm 1$
- The Jones-Wenzl projector is satisfied: $k+1$ consecutive strands cannot have spin $(k+1)/2$



RSOS and Braiding Matrices for $SU(2)_k$

- RSOS representation of the Temperley-Lieb generator e_i for k integer:
  - label it by four dual heights $r, s, t, u = 1, \ldots, k+1$, with $|r-s| = |s-t| = |t-u| = |r-u| = 1$
  - the matrix elements of e_i are

$e_i =$ 

$$= \delta_{su} \frac{\sqrt{[r]_q [t]_q}}{[u]_q}$$

$[h]_q \equiv (q^h - q^{-h})/(q - q^{-1})$, $q = e^{i\pi/(k+2)}$

- S-matrix:

$$\widetilde{S}_i(\theta) = I - \frac{e^{\lambda\theta} - e^{-\lambda\theta}}{q^{-1}e^{\lambda\theta} - q e^{-\lambda\theta}}\, e_i$$

obeys the Yang-Baxter equation $\forall \lambda \Rightarrow \lim_{\theta \to \pm\infty} \widetilde{S}(\theta)$ are $SU(2)_k$ braid matrices

- Jones-Wenzl is satisfied

- Restrictions on the allowed domain walls
  $\Rightarrow$ Dimension of the space with N particles: $d^N = 2^N \cos^N \left(\frac{\pi}{k+2}\right)$, for large N
  $\Rightarrow$ d is the weight of an isolated loop

The $SO(3)_k$ case

- The S-matrix now is

$$\widetilde{S}_j = \frac{q^2 e^{\lambda\theta} - q^{-2} e^{-\lambda\theta}}{e^{\lambda\theta} - e^{\lambda\theta}} \mathcal{I} - \frac{q e^{\lambda\theta} + q^{-1} e^{-\lambda\theta}}{q^3 e^{\lambda\theta} + q^{-3} e^{-\lambda\theta}} E_j + X_j$$

  Taking the $\theta \to \infty$ limit yields the braiding matrix $B^{SO(3)}$

- We can build a representation of the E_j and X_j from the e_i of a Temperley-Lieb representation by projecting into spin 1 double strands

- The representation in terms of dual heights works but the rules for adjacent dual heights are different since the strands are spin 1 representations of $U_q(sl_2)$
  - For dual heights $1, \ldots, k-1$ the rules are the same as for spin 1 reps of sl_2: *e.g.* $(h-1)/2 \otimes 1 = (h-3)/2 \oplus (h-1)/2 \oplus (h+1)/2$ for $3 \leq h \leq k-1$.
  - For dual spins k and $k+1$ we have respectively $(k-1)/2 \otimes 1 = (k-3)/2 \oplus (k-1)/2$, $k/2 \otimes 1 = (k-2)/2$
  - Strands can split $\Rightarrow$ trivalent vertices

- Restrictions on the allowed domain walls
  $\Rightarrow$ the number of states for N spin-1 strands grows as $(d^2-1)^N$ at large N
  $\Rightarrow$ $d^2 - 1$ is the weight of an isolated loop

Dimers, heights and continuum limit

Moessner, Sondhi and Fradkin; Ardonne, Fendley and Fradkin

- The QDM can be mapped to a height model
- The heights live on the dual lattice, and going around a vertex of the even sublattice clockwise, the height changes by $+3$ if a dimer is present, and by -1 if there is no dimer.

$h = 3/2$ $\quad$ $h = 1/2$ $\quad$ $h = -1/2$ $\quad$ $h = -3/2$

- Plaquette flip changes the height of that plaquette by ± 4, and the average height of the surrounding sites by ± 1.
- Equivalent configurations: $h \cong h + 4$.
- Continuum limit: $h \cong 4\varphi(x)$
  Compactification Radius: $\varphi(x) \cong \varphi(x) + 1$.

The Quantum Lifshitz model

- Hamiltonian:

$$H = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{\kappa^2}{2} \left(\nabla^2 \varphi\right)^2 \right]$$

This is the Quantum Lifshitz Model. (Henley; Moessner, Sondhi and Fradkin)

- Action in imaginary time τ:

$$S = \int d^2x \int d\tau \left[\frac{1}{2} \left(\partial_\tau \varphi\right)^2 + \frac{\kappa^2}{2} \left(\nabla^2 \varphi\right)^2 \right]$$

Same as the free energy of smectic layers in 3D at the Lifshitz transition.

Ground State Wave Function and 2D Classical Critical Phenomena

- $$\int d^2\vec{x}\left[-\frac{1}{2}\left(\frac{\delta}{\delta\varphi}\right)^2+\frac{\kappa^2}{2}(\nabla^2\varphi)^2\right]\Psi[\varphi]=E\Psi[\varphi]$$

$$Q(\mathbf{x})\equiv\frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\varphi}+\kappa\nabla^2\varphi\right)\qquad Q^\dagger(\mathbf{x})\equiv\frac{1}{\sqrt{2}}\left(-\frac{\delta}{\delta\varphi}+\kappa\nabla^2\varphi\right)$$

- Ground state wave-function, $\Psi_0[\varphi]$

$$Q(\vec{x})\Psi_0[\varphi]=0\qquad\Rightarrow\qquad\Psi_0[\varphi]\propto e^{-\frac{\kappa}{2}\int d^2x\ (\nabla\varphi(\mathbf{x}))^2}$$

$$\|\Psi_0\|^2=\int\mathcal{D}\varphi\ e^{-\kappa\int d^2x\ (\nabla\varphi(\mathbf{x}))^2}$$

Mapping to a 2D $c = 1$ Euclidean CFT

- The probability for a configuration $|\varphi\rangle$ is the Gibbs weight of a 2D classical Gaussian model, a Euclidean 2D free massless scalar field.

- The equal-time expectation value for operators in the quantum Lifshitz model are given by correlators of the massless free boson conformal field theory with central charge $c = 1$. Time-dependent correlators exhibit power-law behavior with dynamical exponent $z = 2$.

- Matching the correlation functions of the RK and Lifshitz models, one finds $\kappa = 1/2\pi$.

Quantum Loop Models on a Honeycomb Lattice

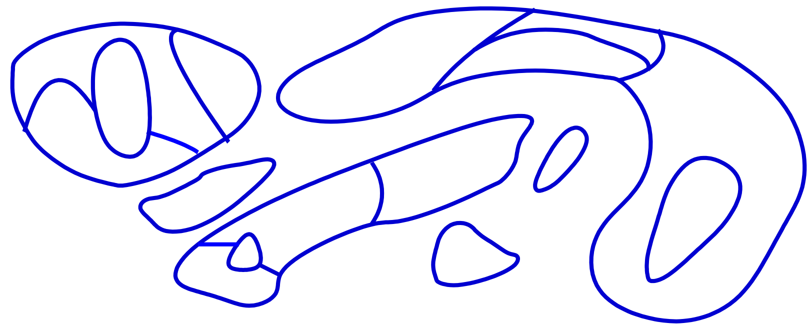
- We want to define local Hamiltonians on a honeycomb lattice whose ground state wave functions are given by the weights of loop models
- The Hamiltonians have the RK form: they are a sum of projection operators
- The ground state is annihilated by all projection operators and has zero energy
- The off-diagonal terms in the Hamiltonians are ergodic in the configuration space
  - Every link of the lattice is either occupied by a strand or empty
  - $SU(2)_k$: occupied links are assigned a spin $1/2$ representation of $U_q(sl_2)$
  - $SO(3)_k$: occupied links are assigned a spin 1 representation of $U_q(sl_2)$
  - An empty site corresponds to the identity
  - At each vertex the configurations that appear in the ground state obey the fusion rules of $U_q(sl_2)$

The $SU(2)_k$ lattice loop model

- The ground state must consist of a superposition of configurations where the strands form self- and mutually-avoiding loops which are not fully packed
- Each loop should have a weight d, and to be a purely topological ground state, there should be no weight per unit length.
- The strands form closed non-intersecting loops, *i.e.* each vertex has either 0 or 2 links with occupied links touching it.
- topologically identical configurations must have the same weight
- d-isotopy: If two configurations are identical except for one having a closed loop around a single plaquette then the weight of the configuration without the single-plaquette loop is d times that of the one with it.
- Freedman, Nayak and Shtengel constructed local Hamiltonians on a honeycomb lattice satisfying these rules for general d (without Jones-Wenzl projection)
- For $d \leq 2$ the ground states are critical and correspond to the CFT of the $O(n)$ loop models with $n = d = 2\cos(\pi/(k+2))$

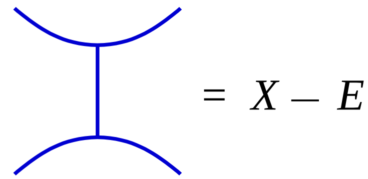
The $SO(3)_k$ loop model on the honeycomb lattice

- The strands are assigned a spin 1 representation of $U_q(sl_2)$
- At a vertex we either have 0, 2 or 3 strands
- A typical configuration in the spin-1 loop model

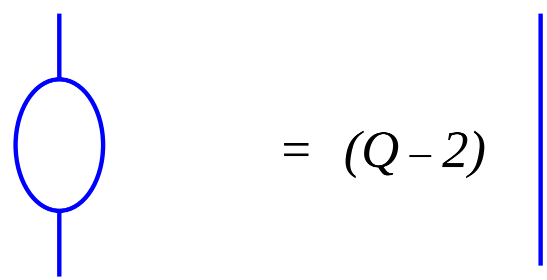


- The lines in this figure represent “spin-1” particles
- We must now allow for trivalent vertices, *i.e.* the loops are now allowed to branch and merge $\Rightarrow$ the spin-1 loop model has branching loops
- the reason for the trivalent vertices is that spin 1 appears in the tensor product of two spin-1 representations
- The BMW relation $E_j^2 = (Q-1)E_j$ implies that isolated loops in the spin-1 model receive a weight of $Q - 1 = d^2 - 1$

- Because trivalent vertices occur here, however, all loops need not be isolated. The projector onto spin-1 is proportional to $X - E$, so we associate this with two neighboring trivalent vertices,



- The relation $(X_j - E_j)^2 = (Q-2)(X_j - E_j)$ means that a configuration with a loop with just two lines emanating from it has a weight $Q - 2$ times the configuration with the loop removed.



- Because $(X_j - E_j)E_j = 0$, no graph can contain any loop with just one external line attached to it $\Rightarrow$ no "tadpoles" are allowed.

The Chromatic Loop Model

- For the spin-1 loop model we will use the weights of the classical 2D Q state Potts model whose S-matrix yields the $SO(3)$ braid matrix
- The Potts "spins" reside on the dual (triangular) lattice and its domain walls occupy the links of the honeycomb lattice
- The domain walls can intersect but do not have tadpoles
- If we shade each region of like "dual spins" with some color $\Rightarrow \chi_Q$ is the number of ways this shading can be done with Q colors so that no two adjacent regions have the same color
- The number of "dual spin" configurations which have the same loop configuration $\mathcal{L}$ is the number of Q-colorings $\chi_Q(\mathcal{L})$.
- $\chi_Q(\mathcal{L})$ is the *chromatic polynomial* of the graph dual to $\mathcal{L}$.
- $\chi_Q(\mathcal{L})$ vanishes for any configuration with a tadpole, or a strands with dangling end

The Chromatic Loop Model: the ground state of the $SO(3)$ quantum loop gas

- Strands form closed loops, but now we allow trivalent vertices
- topologically identical configurations have the same weight
- Each loop configuration $\mathcal{L}$ receives a weight $\chi_Q(\mathcal{L})$. For example, if two configurations are identical except for one having a closed loop around a single plaquette (a loop of length 6 on the honeycomb lattice, length 4 on the square), then the weight of the configuration without the single-plaquette loop is $Q-1$ times that of the one with it.
- We are only interested in the regime in which the domain walls proliferate: this is the disordered phase
- We have given an explicit construction on the honeycomb lattice

Topological Phases and Phase Transitions

- To determine the phase diagram, remember that a configuration s is weighted by $|\Psi(s)|^2$ in the quantum theory.
- Thus each weight is squared: each loop gets a weight $(Q-1)^2$.
- This suggests that the phase diagram is that of the Q_{eff}-state Potts model, where

$$Q_{\text{eff}} - 1 = (Q-1)^2 = (d^2-1)^2 = 1 + 2\cos[2\pi/(k+2)]$$

(Can be proven using Tutte's Theorem and assigning a weight to trivalent vertices)

- There is a critical point when $Q_{\text{eff}} \leq 4$: $k = 1, 2, 3$. $k = 1$ is trivial, $k = 2$ is abelian.
- $k = 3$ is the Yang-Lee theory, the braiding rules are those of the Yang-Lee CFT.
- The critical point with $d = (1+\sqrt{5})/2$ and

$$Q_{\text{eff}} = 1 + \left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1\right]^2 = \frac{5+\sqrt{5}}{2}$$

is the conformal field theory with $c = 14/15$ (before Jones-Wenzl projection)

- Determines the equal-time correlators in the ground state of the quantum loop gas.

Conclusions

- There are lattice models and field theories which exhibit topological order and conformal quantum critical points. For $SO(3)_k$, Potts; for $SU(2)_k$, $O(n)$ model.
- Equal-time correlators at the critical points can be computed exactly.
- There is a gapped field theory with Chern-Simons topological field theory describing the ground state.
- The excitations of this theory obey non-abelian statistics.
- The Jones-Wenzl projected $SO(3)_3$ theory is the simplest universal quantum computer! (using Freedman's rules)
- Lots of work to be done, *e.g.* determine the basin of stability and find a more realistic microscopic model