Physics 504: Statistical Mechanics
Department of Physics, UIUC
Spring Semester 2013
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Problem Set No. 1:
The Microcanonical Ensemble
Due Date: February 3, 2013

In this problem set we will consider the statistical mechanics of a set of N atoms . The atoms are at fixed postions. Each atom has a spin-1/2 magnetic moment. A magnetic field $\vec{B} = B\hat{z}$ is present. We will assume that the spins are sufficiently far away from each other that we can ignore spin-spin interactions. Thus, the only energy involved in this system is the Zeeman energy.

Let $N_\uparrow$ be the number of atoms with spin up and $N_\downarrow$ be the number of atoms wit spin down. The total (Zeeman) energy of the system with $N = N_\uparrow + N_\downarrow$ spins is given by the Hamiltonian

$$H = -\mu B(N_\uparrow - N_\downarrow) \tag{1}$$

where we have absorbed in the quantity μ both the Bohr magneton and the magnitude of the spin magnetic moment.

1. Derive a formula for $g(N, E)$ the number of configurations of this system with N spins and total energy E, in terms of N and E alone.

2. Derive an expression for the entropy $S(N, E, B)$ of this system in the limit $N_\uparrow \gg 1$ and $N_\downarrow \gg 1$.
   Note: use the Stirling approximation

   $$\log N! = N \log N - N + \frac{1}{2}\log(2\pi N) + O(\frac{1}{N}) \tag{2}$$

   where we have used the notation $\log N = \ln N$. Show that the entropy is *extensive*.

3. Find an asymptotic expresssion for the energy dependence of the entropy $S(N, E, B)$ in the limits:

   (a) Near E_{min}, the minumum of the total energy E. What is the value of E_{min} ? What is the entropy at E_{min} ? Explain your result.

   (b) Near the energy E^* where the entropy is at a maximum. For what value of the E^* ? What is the value of the entropy at E^* ? Give an intuitive explanation for this value of the entropy.

4. Use the results you derived above to find an expression for the temperature T of this system of spins when the total energy is E.

5. Use the results you derived above to find an expression for the temperature dependence of the expected number of downspins in the limits: (a) $T \to 0$, and (b) $T \to \infty$. What is the energy scale E_0 separating these regimes? Write E_0 in terms of the parameters of the Hamiltonian.

6. Use your results to find a general expression for the specific heat of this system (at constant B) as a function of temperature, and find the asymptotic behavior of the specific heat in the low and high temperature regimes.

7. How does the temperature of the system behave for $E > E^*$? Explain your answer.

8. Imagine now that we have two spin systems, each with N spins and both interacting with the same magnetic field. Let us call these systems I and II. Suppose that system I is in a state with all its spins pointing up, and that system II is in a state wit half of its spins pointing up and half of its spins pointing down (assume that N is even). If we were to allow these systems to become in thermal contact with each other, find

   (a) The temperature of the final common equilibrium state. What were their initial temperatures?

   (b) The change in the total entropy. Has it increased or decreased? Explain your result.