Physics 504: Statistical Mechanics
Department of Physics, UIUC
Spring Semester 2013
Professor Eduardo Fradkin

Problem Set No. 5:
Quantum Statistical Mechanics
Due Date: April 7, 2013

1 Statistical Mechanics of a Gas of Photons

Consider a gas of photons in equilibrium at temperature T. Since photons are not conserved it is not possible to fix the density of this gas and hence the chemical potential is zero. The Hamiltonian is

$$\hat{H} = \sum_{\vec{k}, s=\pm} \hbar\omega(\vec{k})\ \hat{a}_s^\dagger(\vec{k})\hat{a}_s(\vec{k}) + E_0$$

where $s = \pm$ labels the two polarizations of the photons, the dispersion relation is $\omega(\vec{k}) = c|\vec{k}|$, and E_0 is the ground state energy.

1. Calculate the free energy per unit volume of this gas at temperature T. You may leave your result in the form of a momentum (or rather wave vector) integral.

2. Use the result you just derive to calculate the internal energy density and the specific heat of this gas at temperature T. Show that the internal energy density has the form

   $$\frac{U}{V} = \sigma T^p$$

   and compute the coefficient σ and the exponent p. Use the following integral

   $$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

3. The low energy excitations of a quantum ferromagnet are waves called magnons. Magnons have a dispersion relation

   $$\omega(\vec{k}) = A|\vec{k}|^2$$

   where A is a constant proportional to the exchange energy. Unlike photons, magnons have only one polarization state and like photons magnons are not conserved either. Using this information, calculate the exponent

p for the case of magnons. What would the exponent p be for some hypothetical particles which obeyed instead a general power law dispersion of the form

$$\omega(\vec{k}) = \text{const.}|\vec{k}|^r$$

where r is some exponent?

4. Use these results to derive the Planck distribution, $U(\nu, T)$, for the number of photons per unit volume of either polarization with frequency between ν and $\nu + d\nu$, per unit frequency, at temperature T.

2 The Phase Transition in the Ideal Bose Gas

In class we discussed the non-relativistic ideal Bose gas of particles of mass M in some detail. In this problem you will work with these results and use them to find the thermodynamic properties of this gas at temperature T and specific volume v.

1. Consider first the low density, $v \gg v_c$, or equivalently high temperature limit $T \gg T_c$, of the ideal Bose gas. Show that the equation of state of the ideal Bose gas in the low density lim it has the form of the virial expansion. Calculate the second virial coefficient for this gas and compare your result with the corresponding one for a dilute interacting classical gas. What does the sign of this result tell you?

2. Rederive the formula for the condensate fraction $\langle n_0 \rangle / \langle N \rangle$ for all $T \leq T_c$, as well as the expression for T_c.

3. We showed in class that the thermodynamic properties of the ideal Bose gas can be written in terms of the function $g_{5/2}(z)$

$$g_{5/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}}$$

where z is the fugacity. We gave detailed expressions for small values of z. Here you will be doing calculations close to the phase transition where $z \to 1^-$. In this regime the asymptotic behavior of this function is (F. London)

$$g_{5/2}(z) = a_0(|\log z|)^{3/2} + a_1 - a_2|\log z| + O(|\log z|^2)$$

where $a_0 = 2.363$, $a_1 = 1.342$ and $2.612\ldots = \zeta(3/2)$ approximately. Use this expansion to show that the the slope of the specific heat of an ideal Bose gas has a discontinuity at T_c given by

$$\left(\frac{\partial C_v}{\partial T}\right)_{T \to T_c^+} - \left(\frac{\partial C_v}{\partial T}\right)_{T \to T_c^-} = 3.66 \frac{N k_B}{T_c}$$

4. In the lectures we worked out in detail the behavior of the one-particle density matrix or correlation function of the order parameter. Here you will consider the density-density correlation function:

$$\Gamma(\vec{x}) = \langle \rho(\vec{x})\rho(0)\rangle - \left(\frac{N}{V}\right)^2$$

where $\rho(\vec{x}) = \hat{a}^\dagger(\vec{x})\hat{a}(\vec{x})$ is the density operator. Show that

$$\Gamma(\vec{x}) = \frac{1}{V^2}\sum_{\vec{k}\neq\vec{q}} e^{i(\vec{k}-\vec{q})\cdot\vec{x}}\langle \hat{n}(\vec{q})(\hat{n}(\vec{k})+1)\rangle \xrightarrow[V\to\infty]{} \left|\frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\langle \hat{n}(\vec{k})\rangle\right|^2$$

Use these results to determine the long distance behavior of the density-density correlation function both above T_c and below T_c.

3 Thermodynamics of the Ideal Fermi Gas

In this problem you will work out the thermodynamic properties of an ideal spinless non-relativistic *Fermi* gas at finite temperature T and density $1/v$, where v is the specific volume.

1. Use the Grand Canonical Ensemble to show that the equation of state is given by

$$\frac{P}{k_BT} = \frac{1}{\lambda_T^3}\, f_{5/2}(z)$$
$$\frac{1}{v} = \frac{1}{\lambda_T^3}\, f_{3/2}(z)$$

where

$$f_{5/2}(z) = \frac{2}{\sqrt{\pi}}\int_0^\infty dx\, \sqrt{x}\, \log(1+z\, e^{-x}) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{5/2}}$$
$$f_{3/2}(z) = \frac{2}{\sqrt{\pi}}\int_0^\infty dx\, \frac{\sqrt{x}}{1+z\, e^{-x}} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{3/2}}$$

2. Show that in the low density (or high temperature) limit the equation of state free Fermi gas also has the form of a virial expansion. Compute the second virial coefficient. Compare your result with what you found in problem 2 for the Bose gas. Give a physical explanation for whatever differences you find.

3. Show that the density $\rho = 1/v$ and the energy density $u = U/V$ can be written in the form

$$\rho = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}+1}$$
$$u = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1}$$

4. You will now use the integrals you just derived to investigate the low temperature limit of this gas. You will need to use the Sommerfeld expansion of these integrals:

$$
\begin{aligned}
I &= \int_0^\infty d\epsilon \, \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \\
&= \int_0^\mu g(\epsilon)\, d\epsilon + \int_0^\infty \frac{g(\mu + x/\beta)}{e^x+1}\frac{dx}{\beta} - \int_0^{\beta\mu} \frac{g(\mu - x/\beta)}{e^x+1}\frac{dx}{\beta} \\
&\approx \int_0^\mu g(\epsilon)\, d\epsilon + \frac{\pi^2}{6\beta^2}\, g'(\mu) + \ldots
\end{aligned}
\tag{1}
$$

Use this approximation to compute the specific heat of a Fermi gas at low temperatures. Write your answers in terms of the Fermi energy ϵ_F, the limiting value of the chemical potential at $T = 0$. Find the relation between ϵ_F and the specific volume v.

5. Use the same approximation to calculate the pressure in a Fermi gas at very low temperatures. What is the limiting value P_0 of the pressure at $T = 0$?