Solution of the homework 1 for PHY 504 Statistical Physics

1, The methematical knowledge tells us the number of different configurations for $N_\uparrow$ up spin and $N_\downarrow$ down spin is

$$g(N, E) = \frac{N!}{N_\uparrow!\, N_\downarrow!}$$

The task reduces to calculate what $N_\uparrow$ and $N_\downarrow$ are. It is not hard if you still remember $N = N_\uparrow + N_\downarrow$

$$N_\uparrow = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)$$

$$N_\downarrow = \frac{1}{2}\left(N + \frac{E}{\mu B}\right)$$

So

$$g(N, E) = \frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu B}\right)!\left(\frac{N}{2} + \frac{E}{2\mu B}\right)!}$$

2, The entropy can be approximated by Log[N!] = N Log N - N. The third term in the question is of different order of magnitude

$$S(N, E, B) = \mathrm{Log}\left[\frac{N!}{N_\uparrow!\, N_\downarrow!}\right] = N\,\mathrm{Log}[N] - N_\uparrow \mathrm{Log}[N_\uparrow] - N_\downarrow \mathrm{Log}[N_\downarrow]$$

$$= N\,\mathrm{Log}[N] - \left(\frac{N}{2} - \frac{E}{2\mu B}\right)\mathrm{Log}\left[\frac{N}{2} - \frac{E}{2\mu B}\right] - \left(\frac{N}{2} + \frac{E}{2\mu B}\right)\mathrm{Log}\left[\frac{N}{2} + \frac{E}{2\mu B}\right]$$

The entropy is extensive, to show this use the first line equation, for any coefficient α

$$S(\alpha N, \alpha E, B) = \alpha N\,\mathrm{Log}[\alpha N] - \alpha N_\uparrow \mathrm{Log}[\alpha N_\uparrow] - \alpha N_\downarrow \mathrm{Log}[\alpha N_\downarrow]$$

$$= \alpha N\,\mathrm{Log}\,N - \alpha N_\uparrow \mathrm{Log}[N_\uparrow] - \alpha N_\downarrow \mathrm{Log}[N_\downarrow] + (\alpha N\,\mathrm{Log}[\alpha] - \alpha (N_\uparrow + N_\downarrow)\,\mathrm{Log}[\alpha])$$

$$= \alpha S(N, E, B)$$

Therefore it is extensive. Note that B is an intensive quantity.

3.

a. The minimum energy is the state with all spin upward, therefore

$$E_{min} = -\mu B N$$

There is only one configuration for this energy, so the entropy must be zero.

b. The entropy reaches maximum when its derivative with E is zero

$$\frac{\partial S}{\partial E} = \frac{k}{2 B\mu}\,\mathrm{Log}\left[\frac{N - \frac{E}{B\mu}}{N + \frac{E}{B\mu}}\right] = 0$$

The is just the inverse of temperature. The solution of energy $E^* = 0$. The corresponding entropy is the case of $N_\uparrow = N_\downarrow$

$$S(N, 0, B) = kN\,\mathrm{Log}[2]$$

The explanation of this result is natural. The zero energy means the upward spin and downward spin are equal, i.e. E = 0. For this macrostate the system has

maximum ways of configuration. The entropy is therefore maximum.

The temperature for this situation is infinity because $\frac{\partial S}{\partial E} = 0$.

4. In the last question, we have calculated the $\frac{\partial S}{\partial E}$, **by definition it is just** $\frac{1}{T}$ **So**

$$T = \frac{2\,\mu B}{K} \frac{1}{\text{Log}\left[\frac{N-\frac{E}{B\mu}}{N+\frac{E}{B\mu}}\right]}$$

5, The expression above is nothing but

$$T = \frac{2\,\mu b}{K} \frac{1}{\text{Log}\left[\frac{N_\uparrow}{N_\downarrow}\right]}$$

where $N_\uparrow = N - N_\downarrow$

Therefore $N_\downarrow = \dfrac{N}{1 + \text{Exp}\left[\frac{2\,B\mu}{kT}\right]}$

There are two limits to be considered

a. For $T \to 0$, $N_\downarrow \to 0$ **because** $\text{Exp}\left[\frac{2\,B\mu}{kT}\right] \to \infty$

b. For $T \to \infty$, $\text{Exp}\left[\frac{2\,B\mu}{kT}\right] \to 1$, **so** $N_\downarrow = N/2$.

The energy scale E_0 **is just the difference of the two case, which is obviously** $NB\mu$

6, The calculation specific heat c is a bit lengthy. First,

start from $T = \dfrac{2\,\mu b}{K} \dfrac{1}{\text{Log}\left[\frac{N-\frac{E}{B\mu}}{N+\frac{E}{B\mu}}\right]}$, **you can inverse the function and get**

$$E = BN\mu\, \text{Tanh}\left[\frac{B\mu}{kT}\right]$$

Second, take the derivative of E with T

$$c = \frac{\partial E}{\partial T} = \frac{B^2 N\mu^2}{kT^2} \text{Sech}^2\left[\frac{B\mu}{kT}\right]$$

The specific heat has some special property by the Taylor expansion $\text{Sech}[x]^2 = 1 - x^2 + \ldots$

This shows the $T \to \infty$ **behavior is** $c \to \dfrac{B^2 N\mu^2}{kT^2}$ **which tends to 0 for** $T \to \infty$ **On the other hand,**

for $T \to 0$, Sech^2 **fastly tends to zero,**

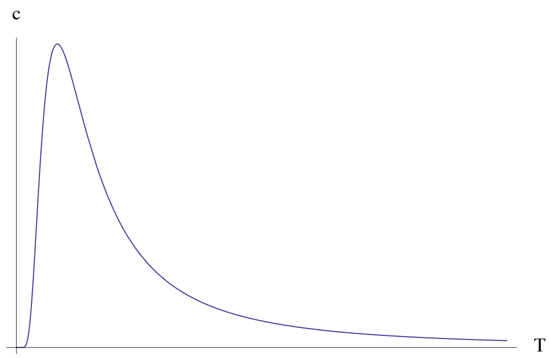
this is much faster than $\frac{1}{T^2}$. **Therefore** $c \to 0$ **in this limit. If you feel insecure,**

expand Sech[x] and see the limit value.

$\mathrm{Lim}_{T\to 0}\, c =$

$$\mathrm{Lim}_{T\to 0} \frac{B^2 N\mu^2}{kT^2} \mathrm{Sec}^2\left[\frac{B\mu}{kT}\right] = \mathrm{Lim}_{T\to 0} \frac{B^2 N\mu^2}{kT^2} \frac{4}{\left(\mathrm{Exp}\left[\frac{B\mu}{kT}\right] + \mathrm{Exp}\left[-\frac{B\mu}{kT}\right]\right)^2} = \mathrm{Lim}_{T\to 0} \frac{4 B^2 N\mu^2}{kT^2} \frac{1}{\mathrm{Exp}\left[\frac{2 B\mu}{kT}\right]}$$

The limit above is zero due to the mathematical fact that $\mathrm{Lim}_{x\to 0} \frac{1}{x^2} \mathrm{Exp}\left[-\frac{1}{x}\right] =$ 0. The exponential behavior is due to the gap in the energy spectrum $2 \mu B$
The figure is drawn here. Clearly the specific heat is 0 at both limits.



7, The $T = \frac{2 \mu b}{K} \frac{1}{\mathrm{Log}\left[\frac{N-\frac{E}{B\mu}}{N+\frac{E}{B\mu}}\right]}$ has a special property. For $E > E^* = 0$,

then $\mathrm{Log}\left[\frac{N - \frac{E}{B\mu}}{N + \frac{E}{B\mu}}\right] < \mathrm{Log}[1] = 0$. That means $T < 0$. The negative temperature is caused by the decrease of entropy with increasing energy. The system tends to lower the energy in order to increase the entropy, although this is counterintuitive.

8, For this equation,
remember two things : the temperature has to be same for two parts of system before the system reaches equilibrium. Second, the energy is conserved.
a. The initial temperature has been discussed above. For I,
then spins are all pointing up. The energy is $-B\mu N$,
the temprature is 0. For the system II, the temperature is ∞.

What is the temperature for final state? The energy for the two system is $E_{total} = -\mu BN$. So

$$T_{final} = \frac{2 B\mu}{k \log\left[\frac{3N}{N}\right]} = \frac{2 B\mu}{k \log[3]}$$

b. The initial entropy comes from system II only, it is $S = kN \, Log[2]$. The final entropy is easily calculated by the equation derived in problem 2.

$S_{total} =$

$$k\left[2\,N\,Log[2\,N] - \frac{1}{2}\left(2\,N - \frac{E}{B\mu}\right)Log\left[N - \frac{E}{2\,B\mu}\right] - \frac{1}{2}\left(2\,N + \frac{E}{B\mu}\right)Log\left[N + \frac{E}{2\,B\mu}\right]\right] = kN\,Log\left[\frac{16}{3\sqrt{3}}\right] > kN\,Log[2]$$

$$\Delta S = kN\,Log\left[\frac{16}{3\sqrt{3}}\right] - kN\,Log[2] > 0$$

This is not surprising because the system is isolated and the process is spontaneous (irreversible), the second thermodynamics law tells us that the entropy for the system will increase.