Phy 504 hw 5 solution

I. STATISTICAL MECHANICS OF A GAS OF PHOTONS

1, Suppose $n_s(k)$ is the number of photons with k and polarization s, then the partition function is

$$\begin{aligned} Z &= \sum_{\{n_s(k)\}} \exp\left(-\beta \sum_{k,s} \hbar\omega(k) n_s(k) + E_0\right) \\ &= \exp(-\beta E_0) \prod_{k,s} \frac{1}{1 - e^{-\beta\hbar\omega(k)}} \end{aligned} \tag{1}$$

So the free energy is

$$\begin{aligned} F &= -kT \ln Z \\ &= E_0 + k_B T \sum_s \frac{V}{(2\pi)^3} \int dk \ln(1 - e^{-\beta\hbar\omega(k)}) \end{aligned} \tag{2}$$

the free energy per unit volume is

$$\frac{F}{V} = \frac{E_0}{V} + \frac{2kT}{(2\pi)^3} \int dk \ln(1 - e^{-\beta\hbar c k}) \tag{3}$$

2) The internal energy density can be calculated by the equation

$$\begin{aligned} \frac{U}{V} &= \frac{8\pi}{(hc)^3} \int_0^\infty \frac{\epsilon^3}{e^{\epsilon/(kT)} - 1} d\epsilon \\ &= \frac{8\pi(kT)^4}{(hc)^3} \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{8\pi(kT)^4}{(hc)^3} \frac{\pi^4}{15} \end{aligned} \tag{4}$$

Therefore

$$\frac{U}{V} = \sigma T^4 \tag{5}$$

where $\sigma = \frac{8\pi^5 k^4}{15(hc)^3} = 7.566 \times 10^{-16} J/m^3 k^4$

3) The magnon dispersion relation is $\omega(k) = A|k|^2$. Because there is only one polarization

$$\epsilon = \hbar\omega(k) = \hbar A |\vec{k}|^2 \tag{6}$$

Therefore

$$U = \int \epsilon(\vec{k}) n dk = 4\pi\hbar A \int_0^\infty \frac{k^4}{e^{\hbar A k^2/kT} - 1} dk \tag{7}$$

Change of variables and get

$$U = \frac{2\pi(kT)^{5/2}}{(\hbar A)^{3/2}} \int_0^\infty \frac{x^3}{e^x - 1} dx \tag{8}$$

Therefore

$$U \sim T^{5/2} \tag{9}$$

so $p = 5/2$ for magnons. For a general dispersional relation $\omega(k) \sim |\vec{k}|^r$. The exponential would be $p = \frac{3}{r} + 1$

4) Planck Distribution: convert integrand of $\frac{U}{V}$ from ϵ to frequency.

$$\begin{aligned} \frac{U}{V} &= \frac{2\pi}{(hc)^3} \int_0^\infty \frac{\epsilon^3}{e^{\epsilon/kr} - 1} d\epsilon \\ &= \frac{8\pi}{(hc)^3} \int_0^3 \frac{(h\nu)^3}{e^{h\nu/kT} - 1} h d\nu \end{aligned} \tag{10}$$

Therefore

$$u(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{11}$$

II. THE PHASE TRANSITION IN THE IDEAL BOSE GAS

1) In the low density limit, we have $z << 1$,

$$\begin{aligned} \frac{P}{kT} &= \frac{1}{\lambda_T^3} g_{5/2}(z) \\ &= \frac{1}{\lambda_T^3} \left(z + \frac{z^2}{2^{3/2}} \right) + \cdots \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{N}{V} &= \frac{1}{\lambda_T^3} g_{3/2}(z) \\ &= \frac{1}{\lambda_T^3} \left(z + \frac{z^2}{2^{3/2}} \right) + \cdots \end{aligned} \tag{13}$$

Therefore expand them

$$\frac{P}{kT} = \frac{B_1(T)}{\lambda_T^3} z + \left(\frac{B_1(T)}{2^{3/2}\lambda_T^3} + \frac{B_2(T)}{\lambda_T^6} \right) + \cdots \tag{14}$$

The coeffcients are

$$B_1(T) = 1 \tag{15}$$

$$B_2(T) = \lambda_T^3(\frac{1}{2^{5/2}} - \frac{1}{2^{3/2}}) \tag{16}$$

$$= -0.18\lambda_T^3 \tag{17}$$

2) Start at the Grand Canonical Potential

$$\Omega = -kT \sum_\alpha \ln\left(1 - e^{-\beta(\epsilon_\alpha - \mu)}\right) \tag{18}$$

Therefore the total number of particles are

$$\langle N \rangle = \sum_\alpha \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1} \tag{19}$$

where

$$\langle N \rangle = \langle N_0 \rangle + \langle N_{excited} \rangle \tag{20}$$

The condition for the critical temperature is

$$\langle N_{excited} \rangle = \langle N \rangle \tag{21}$$

where

$$\langle N \rangle = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \sqrt{\epsilon} \frac{d\epsilon}{e^{\epsilon/kT} - 1} \tag{22}$$

we get the critical density n_C

$$n_c \equiv \langle N \rangle / V = \frac{1}{4\pi^2} \left(\frac{2mkT_c}{\hbar^2}\right)^{3/2} \zeta(3/2)\Gamma(3/2)$$

$$= \zeta(3/2) \left(\frac{mkT_C}{2\pi\hbar^2}\right)^{3/2} \tag{23}$$

Therefore

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{n_c}{\zeta(3/2)}\right)^{2/3} \tag{24}$$

Define $\lambda_{T_c}^3 = \left(\frac{2\pi\hbar^2}{mkT_C}\right)^{3/2}$ at T_c and $\lambda_T^3 = \left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}$ below T_C

At T_C we have $\langle N \rangle = \frac{V}{\lambda_{T_C}^3} g_{3/2}(1)$, where $g_{3/2}(1)$ is a Bose function for $z = 1$.

Therefore

$$\frac{\langle N\rangle}{\langle N\rangle_T} = \frac{T^{3/2}}{T_c^{3/2}} \tag{25}$$

Thus, the condensate fraction is

$$\frac{\langle N_0\rangle}{\langle N\rangle} = 1 - \left(\frac{T}{T_C}\right)^{3/2} \tag{26}$$

3) The internal energy of the system is

$$U = \frac{3V}{2\beta\lambda_T^3} g_{5/2}(z) \tag{27}$$

Therefore the specific heat is

$$\frac{c_V}{k} = \frac{\partial(U/V)}{\partial(kT)} \tag{28}$$

$$= \frac{\partial}{\partial(kT)}\left(\frac{3kT}{2\lambda_T^3} g_{5/2}(z)\right) \tag{29}$$

$$= \frac{15}{4\lambda_T^3} g_{5/2}(z) + \frac{3}{2\beta\lambda_T^3}\left(\frac{\partial z}{\partial(kT)}\right)\frac{g_{3/2}(z)}{z} \tag{30}$$

In the regime of $z < 1$, the particle number density is

$$\frac{1}{\nu} = \frac{1}{\lambda_T^3} g_{3/2}(z) \tag{31}$$

Differentiating by kT at constant ν gives

$$-\frac{\lambda_T}{2kT}\frac{3\lambda_T^2}{\nu} = \frac{\partial z}{\partial(kT)}\frac{1}{z} g_{1/2}(z) \tag{32}$$

Eliminating the partial derivative of z in c_V/k_B with this equation, we obtain

$$\frac{c_V}{k_B} = \frac{15}{4\lambda_T^3} g_{5/2}(z) - \frac{9 g_{3/2}(z)}{g_{1/2}(z)\nu} \tag{33}$$

In the regime $z = 1$, the partial derivative $\partial z/\partial T$ vanishes, and so the specific heat is

$$\frac{1}{k_B}\left(\frac{\partial c_V}{\partial T}\right)_{T\to T_c^+} - \frac{1}{k_B}\left(\frac{\partial c_V}{\partial T}\right)_{T\to T_c^-} = -\frac{9}{4\nu}\left(\frac{\partial z}{\partial T}\right)_{T\to T_c^+}\left(\frac{\partial g_{3/2}(z)}{\partial z g_{1/2}(z)}\right)_{z\to 1^-} \tag{34}$$

The asymptotic behavior of g_α gives us the discontinuity in the temperature derivative of the specific heat

$$\frac{1}{k_B}\left(\frac{\partial c_V}{\partial T}\right)_{T\to T_c^+} - \frac{1}{k_B}\left(\frac{\partial c_V}{\partial T}\right)_{T\to T_c^-} = -3\frac{a_2}{a_0^2}\frac{1}{T_C}\frac{g_{3/2}(1)}{\nu} \tag{35}$$

Therefore

$$\left(\frac{\partial c_V}{\partial T}\right)_{T\to T_c^+} - \left(\frac{\partial c_V}{\partial T}\right)_{T\to T_c^-} = -3.66\frac{Nk_B}{T_c} \tag{36}$$

4)

$$\rho(x) = a^\dagger(x)a(x) = \frac{1}{V}\sum_{k,q} e^{i(k-q)\cdot x}a_q^\dagger a_k = \frac{1}{V}\sum_{k\neq q} e^{i(k-q)\cdot x}a_q^\dagger a_k + \frac{N}{V} \tag{37}$$

$$\rho(0) = \frac{1}{V}\sum_{k\neq q} a_q^\dagger a_k + \frac{N}{V} \tag{38}$$

Therefore

$$\begin{aligned}\Gamma(x) &= \langle\rho(x)\rho(0)\rangle - \frac{N^2}{V^2} \\ &= \frac{1}{V^2}\sum_{k\neq q} e^{i(k-q)\cdot x}\langle a_q^\dagger a_k a_k^\dagger a_q\rangle \qquad (39)\\ &= \frac{1}{V^2}\sum_{k\neq q} e^{i(k-q)\cdot x}\langle n_q(n_k+1)\rangle \qquad (40)\end{aligned}$$

So

$$\begin{aligned}\Gamma(x) &= \frac{1}{V^2}\langle\sum_{k,q} e^{i(k-q)\cdot x}n(q)n(k) + \sum_{k,q} e^{i(k-q)\cdot x}n(q) \\ &= \frac{1}{V^2}\langle\sum_{q} e^{-iqx}n(q)\sum_{k} e^{ikx}n(k) + \sigma(x)n(x)\end{aligned} \tag{41}$$

In the limit $V\to\infty$, the correlation becomes

$$\Gamma(x) = |\frac{1}{V}\sum_k e^{ik\cdot x}n(k)|^2 \tag{42}$$

III. THERMODYNAMICS OF THE IDEAL FERMI GAS

1) The grand potential is

$$\Omega = -kT\sum_\alpha \ln(1+e^{-\beta(\epsilon_\alpha-\mu)})$$

Therefore

$$\begin{aligned}\Omega &= -kTV\int\frac{d^3p}{h^3}\ln(1+e^{-\beta(\epsilon-\mu)}) \\ &= -\frac{2}{\sqrt{\pi}}kTV\frac{1}{\lambda_T^3}\int_0^\infty dx\sqrt{x}\ln(1+ze^{-x})\end{aligned} \tag{43}$$

Consdering that $\Omega = -PV$ we get that

$$\frac{1}{V} = \frac{4\pi}{h^3} mkT\sqrt{2mkT} \int_0^\infty \frac{\sqrt{x}}{1+z^{-1}e^x} dx = \frac{2}{\sqrt{\pi}} \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \int_0^\infty \frac{\sqrt{x}}{1+z^{-1}e^x}$$
$$= \frac{1}{\lambda_T^3} f_{3/2}(z) \tag{44}$$

2)Expand the $f_{3/2}$ to the first order

$$\frac{P}{kT} = \frac{1}{\lambda_T^3}(\lambda_T^3 \left(\frac{N}{V}\right) - 2^{-5/2}\lambda_T^6 (N/V)^2 + \cdots)$$
$$= N(1 - 2^{-5/2}\lambda_T^3 (N/V)^2) + \cdots \tag{45}$$

So the second virial coefficient is $a_2 = 2^{-5/2}$ for the ideal Fermi gas. This is the opposite of what we found for the Bose. So it is repulsive.

3)

$$\rho = \frac{1}{V} = \frac{4\pi}{h^3} \int_0^\infty \frac{p^2 dp}{e^{\beta(\epsilon-\mu)}+1} \tag{46}$$
$$= \frac{1}{\hbar^3}(2m)^{3/2} \int_0^\infty \frac{\sqrt{\epsilon} d\epsilon}{e^{\beta(\epsilon-\mu)}+1} \tag{47}$$

In the same way

$$u = \frac{U}{V} = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1} \tag{48}$$

4) Writing the U in expansion

$$U = \int_0^\infty \epsilon g(\epsilon) n_{FD} d\epsilon$$
$$\approx \int_0^\mu g(\epsilon) d\epsilon + \frac{\pi^2}{6\beta^2} g'(\mu) + \cdots$$
$$= \frac{3}{5} N \frac{\mu^{5/2}}{\epsilon_F^{3/2}} + \frac{3\hbar^2}{8} N \frac{kT^2}{\epsilon_F} + \cdots$$
$$= \frac{3}{5} N\epsilon_F + \frac{\pi^2}{4} N \frac{(kT)^2}{\epsilon_F} \tag{49}$$

The heat capacity is therefore

$$\frac{c_V}{N} = \frac{\pi^2}{2}\frac{k^2 T}{\epsilon_F} \tag{50}$$

5)

$$\begin{aligned} P = \frac{2}{3}\frac{U}{V} &= \frac{2}{3}\left(\frac{3}{5}\frac{N\epsilon_F}{V} + \frac{\pi^2}{4}N\frac{(kT)^2}{\epsilon_F V} + \cdots\right) \\ &= \frac{2}{5}\frac{N\epsilon_F}{V} + \frac{\pi^2}{6}N\frac{(kT)^2}{\epsilon_F V} \end{aligned} \tag{51}$$

At $T = 0$, the pressure takes on the value $P_0 = \frac{2}{5}\frac{N\epsilon_F}{V}$ which is known as the degeneracy pressure.