Phys 504 hw 6 solution

I. THE ONE DIMENSIONAL CLASSICAL ISING MODEL

1,

$$
\begin{aligned}
Z &= \sum_{\sigma} e^{-\beta E[\sigma]} \\
&= \sum_{\sigma} \prod_{i=1}^{N} \exp(\beta J \sigma_i \sigma_{i+1} + \beta H \sigma_i) \\
&= tr(\prod_{i=1}^{N} \exp(\beta \sigma_i \sigma_{i+1} + \frac{\beta H}{2}(\sigma_i + \sigma_{i+1})))
\end{aligned}
$$

Let $\langle \sigma_i | T | \sigma_j \rangle = \exp(\beta J \sigma_i \sigma_j + \frac{\beta H}{2}(\sigma_i + \sigma_{i+1}))$

Then

$$
\begin{aligned}
Z &= tr[\langle \sigma_1 | T | \sigma_2 \rangle \langle \sigma_2 | T | \sigma_3 \rangle \cdots \langle \sigma_N | T \sigma_1 \rangle] \\
&= tr[T^N] \qquad (1)
\end{aligned}
$$

The transfer matrix is therefore

$$
T = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix} \qquad (2)
$$

2, The transfer matrix has eigenvalues

$$
\begin{aligned}
t_{\pm} &= e^{\beta J} \cosh \beta H \pm \sqrt{e^{2\beta J} \cosh^2 \beta H - 2 \sinh 2\beta J} \qquad (3) \\
&= e^{\beta J} \left(\cosh(\beta H) \pm \sqrt{\sinh^2 \beta H + e^{-4\beta J}} \right) \qquad (4)
\end{aligned}
$$

3, The partition function can be written as

$$
Z = t_+^N + t_-^N \qquad (5)
$$

The free energy is therefore

$$
\begin{aligned}
F_N &= -kT \ln Z \\
&= -kT \ln(t_+^N + t_-^N) \qquad (6)
\end{aligned}
$$

4, In the thermodynamic limit $N \to \infty$ the free energy becomes

$$\begin{aligned} F_N &= -kT\{N\ln t_+ + \ln(1+\left(\frac{t_-}{t_+}\right)^N)\} \\ &\to -NkT\ln t_+ \end{aligned} \tag{7}$$

which is determined by the behavior at large t_+. Since F_N is proportional to N, it is extensive. In the no-field limit $H \to 0$, the free energy beomes

$$F_N \to -NkT\ln(2\cosh\beta J) \tag{8}$$

By direct differentiation

$$\frac{1}{Nk}\frac{\partial F_\infty}{\partial T} = -\ln(2\cosh\beta J) + \beta J\tanh(\beta J) \tag{9}$$

$$\frac{1}{Nk^2}\frac{\partial^2 F_\infty}{\partial T^2} = -\beta^3 J^2 \mathrm{sech}^2\beta J \tag{10}$$

which is differentiable for the 2nd order for $T > 0$. No phase transition for it.

5,

In the low temperature limit with $\beta J \geq 1$ and $\beta H \geq 1$

$$\begin{aligned} t_+ &= e^{\beta J}\left(\cosh\beta H + \sqrt{\sinh^2\beta H + e^{-4\beta J}}\right) \\ &\to e^{\beta(J+H)}\left(1+e^{-\beta(4J+2H)}\right) \end{aligned} \tag{11}$$

On the other hand, expand it in the high temperature limit with small β

$$t_+ \approx 2 + (\beta H)^2 + (\beta J)^2 \tag{12}$$

So

$$\ln t_+ \approx \ln 2 + \frac{\beta^2}{2}(H^2+J^2) \tag{13}$$

a) From $F_\infty = -NkT\ln t_+$, the average magnetization per site is

$$\begin{aligned} m(T,J,H) &= \frac{1}{N}\frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial H} \\ &= -\frac{1}{N}\frac{\partial F_\infty}{\partial H} \\ &= kT\frac{1}{t_+}\left(e^{\beta J}\beta\sinh\beta H + \frac{e^{2\beta J}\beta\sinh\beta H\cosh\beta H}{\sqrt{e^{2\beta J}\sinh^2\beta H + e^{-2\beta J}}}\right) \\ &= \frac{e^{\beta J\sinh\beta H}}{t_+}\left(1+\frac{e^{\beta J}\cosh\beta H}{\sqrt{e^{2\beta J}\sinh^2\beta H + e^{-2\beta J}}}\right) \end{aligned} \tag{14}$$

In the low temperature,

$$m(T,J,H) \to kT\frac{\partial}{\partial H}(\beta(J+H)+e^{-\beta(4J+2H)})$$
$$= 1 - 2e^{-\beta(4J+2H)} \tag{15}$$

The system approaches the configuration with all spins aligned with the magnetic field as the temperature decrease. In the high temperature limit

$$m(T,J,H) \to kT\frac{\partial}{\partial H}(\ln 2 + \frac{\beta^2}{2}(H^2+J^2))$$
$$= \beta H \tag{16}$$

So the system approaches the completely disordered configuration as the temperature increases.

b) The magnetic susceptibility is

$$\chi(T,J,H) = \frac{\partial m}{\partial H} \tag{17}$$

In the low temperature limit,

$$\chi(T,J,H) \to 4\beta e^{-\beta(4J+2H)} \tag{18}$$

and in the high temperature limit

$$\chi(T,J,H) \to \beta \tag{19}$$

c) The entropy is

$$S = -\frac{\partial F}{\partial T} \tag{20}$$

In the low temperature limit,

$$S \to N\frac{\partial}{\partial T}(kT\beta(J+H) + kTe^{-\beta(4J+2H)})$$
$$\approx \frac{N}{T}(4J+2H)e^{-\beta(4J+2H)} \tag{21}$$

In the high temperature limit

$$S \to N\frac{\partial}{\partial T}(kT\ln 2 + kT\frac{\beta^2}{2}(H^2+J^2))$$
$$= N(k\ln 2 - \frac{1}{2kT^2}(H^2+J^2)) \tag{22}$$

d) The specific heat

$$c = \frac{T}{N}\frac{\partial S}{\partial T} \tag{23}$$

In the low temperature limit

$$\begin{aligned} c &\to (4J+2H)\frac{\partial}{\partial T}e^{-\beta(4J+2H)} \\ &= (4J+2H)^2\frac{1}{kT^2}e^{-\beta(4J+2H)} \end{aligned} \tag{24}$$

while in the high temperature limit

$$\begin{aligned} c &\to T\frac{\partial}{\partial T}(-\frac{1}{2kT^2}(H^2+J^2)) \\ &= \frac{1}{kT^2}(H^2+J^2) \end{aligned} \tag{25}$$

II. MEAN FIELD THEORY

1) Under $\sigma \to -\sigma$

$$H[-\sigma] = -J\sum_{r_A,r_B}\sigma(r_A)\sigma(r_B) + h\sum_{r_A}\sigma(r_A) - h\sum_{r_B}\sigma(r_B)$$

This Hamiltonian will give the same free energy as $H[\sigma]$ since the A and B sublattices are equivalent.

2. The antiferromagnetic state $\sigma(r_A) = \sigma(r_B) = 1$ has energy

$$H_{af} = -2NJ \tag{26}$$

while the ferromagnetic state $\sigma(r_A) = -\sigma(r_B) = 1$

$$H_f = 2NJ - Nh \tag{27}$$

Therefore, there is a ground state phase transition when $H_{af} = H_f$ at the critical field $h_c = 4J$

3.

$$F \le \tilde{F} = F_0 + \langle H - H_0 \rangle \tag{28}$$

$$H - H_0 = [-J \sum_{r_A, r_B} \sigma(r_A)\sigma(r_B) - h \sum_{r_A} \sigma(r_A) + h \sum_{r_B} \sigma(r_B)] - [-h_A \sum_{r_A} \sigma(r_A) - h_B \sigma_{r_B} \sigma(r_B)] \tag{29}$$

$$= -J \sum_{r_A, r_B} \sigma(r_A)\sigma(r_B) - (h - h_A) \sum_{r_A} \sigma(r_A) + (h + h_B) \sum_{r_B} \sigma(r_B) \tag{30}$$

In addition

$$\langle H - H_0 \rangle_0 = \langle H \rangle_0 - \langle H_0 \rangle_0 \tag{31}$$

$$\langle H_0 \rangle_0 = -2NJ \langle \sigma_A \rangle \langle \sigma_B \rangle - \frac{h}{2} N \langle \sigma_A \rangle + \frac{h}{2} N \langle \sigma_B \rangle \tag{32}$$

$$\langle H_0 \rangle_0 = -\frac{N}{2} [h_A \langle \sigma_A \rangle + h_B \langle \sigma_B \rangle] \tag{33}$$

Therefore

$$\langle H - H_0 \rangle_0 = -2NJ \langle \sigma_a \rangle \langle \sigma_B \rangle + \frac{N}{2} \langle \sigma_A \rangle (h_A - h) + \frac{N}{2} \langle \sigma_B \rangle (h_0 - h) \tag{34}$$

Therefore

$$\tilde{F} = F_0(\beta h_A, \beta h_B) - 2NJ \langle \sigma_A \rangle_0 \langle \sigma_B \rangle_0 - \frac{Nh}{2} (\langle \sigma_A \rangle_0 - \langle \sigma_b \rangle_0) + \frac{N}{2} (h_A \langle \sigma_A \rangle_0 + h_B \langle \sigma_B \rangle_0) \tag{35}$$

Note that the two sublattices in H_0 are non interacting, we can write

$$\begin{aligned} \langle \sigma(r_A)\sigma(r_B) \rangle_0 &= \langle \sigma_H \rangle_0 \langle \sigma_B \rangle_0 \\ &= m_A m_B \end{aligned} \tag{36}$$

where m_A and m_B are the magnetization of the two sublattices.

$$H - H_0 = -J \sum_{\langle r_A, r_B \rangle} \sigma(r_A)\sigma(r_B) + (h_A - h) \sum_{r_A} \sigma(r_A) + (h_B + h) \sum_{r_B} \sigma(r_B) \tag{37}$$

Hence the variational free energy is

$$\tilde{F} = F_0(\beta h_A, \beta h_B) - 2NJ m_A m_B - \frac{1}{2} Nh (m_A - m_B) + \frac{1}{2} N (h_A m_A + h_B m_B) \tag{38}$$

4. With the partition function of H_0 as

$$\begin{aligned} Z_0 &= \sum_{\sigma} e^{-\beta H_0[\sigma]} \\ &= (2 \cosh \beta h_A)^{N/2} (2 \cosh \beta h_B)^{N/2} \end{aligned} \tag{39}$$

we compute the sublattice magnetization

$$
\begin{aligned}
m_A &= \frac{1}{Z_0} \sum_{\sigma} \sigma(r_A) e^{-\beta H_0[\sigma]} \\
&= \frac{\sum_{\sigma(r_A)} \sigma(r_A) e^{\beta h_A \sigma(r_A)}}{\sum_{\sigma(r_A)} e^{\beta h_A \sigma(r_A)}} \\
&= \tanh(\beta h_A)
\end{aligned} \tag{40}
$$

$$
m_B = \tanh(\beta h_B) \tag{41}
$$

It follows that

$$
\begin{aligned}
F_0 &= -kT \ln Z_0 \\
&= -\frac{1}{2} NkT (\ln(2\cosh \beta h_A) + \ln(2\cosh \beta h_B)) \\
&= -\frac{1}{2} NkT \left(\ln \frac{2}{\sqrt{1-m_A^2}} + \ln \frac{2}{\sqrt{1-m_B^2}} \right)
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\tilde{F} = &-\frac{1}{2} NkT [2\ln 2 - \frac{1}{2}\ln(1-m_A^2) - \frac{1}{2}\ln(1-m_B^2)] - 2NJm_A m_B \\
&-\frac{1}{2} Nh(m_A - m_B) + \frac{1}{2} NkT(m_A \tanh^{-1} m_A + m_B \tanh^{-1} m_B)
\end{aligned}
$$

Then differentiating with respect to m_A and m_B

$$
\begin{aligned}
\frac{\partial \tilde{F}}{\partial m_A} &= \frac{1}{2} NkT \tanh^{-1} m_A - 2NJm_B - \frac{1}{2} Nh \\
\frac{\partial \tilde{F}}{\partial m_B} &= \frac{1}{2} NkT \tanh^{-1} m_A - 2NJm_B + \frac{1}{2} Nh
\end{aligned} \tag{43}
$$

we find that $\tilde{F}$ is minimized when

$$
\begin{aligned}
m_A &= \tanh(4\beta J m_B + \beta h) \\
m_B &= \tanh(4\beta J m_B - \beta h)
\end{aligned}
$$

5.

Notice that $m_A = -m_B$ is always a solution to the set of equations, this correspond to the state with $m = 0$. Summing the two equations gives

$$m = \tanh(\beta h + 4\beta J(m-n)) + \tanh(4\beta J(m+n) - \beta h) \tag{44}$$

We cannot solve this transcendental equation but we do find some useful things. There will be another solution when the slope of the right handside is bigger than 1 In other words, the critical point is given by

$$4\beta_c J[\mathrm{sech}^2(\beta_c h - 4\beta_c J_n) + \mathrm{sech}^2(4\beta_c J_n - \beta_c h)] \tag{45}$$

Therefore

$$8\beta_c J \mathrm{sech}^2(\beta_c h - 4\beta_c J_n) = 1 \tag{46}$$

6. Substitute m_A and m_B for m and n, and expand about small m

$$\begin{aligned}
\tilde{F} &= -\frac{1}{2}NkT(2\ln 2 - \frac{1}{2}\ln(1-m_A^2) - \frac{1}{2}\ln(1-m_B^2)) \\
&\quad - 2NJm_Am_B - \frac{1}{2}Nh(m_A - m_B)\frac{1}{2}NkT(m_A \tanh^{-1} m_A + m_B \tanh^{-1} m_B) && (47) \\
&\approx -\frac{1}{2}NkT[2\ln 2 - \frac{1}{2}\ln(1-2n^2+n^4) + \frac{1+n^2}{(1-n^2)^2}m^2 + \frac{1+6n^2+n^4}{2(1-n^2)^4}m^4] && (48)
\end{aligned}$$

where

$$\begin{aligned}
A &= -\frac{1}{2}NkT[2\ln 2 - \frac{1}{2}\ln(1-2n^2+n^4) - 2n\tanh^{-1} n] + 2NJn^2 - Nhn \\
B &= -2NJ + \frac{1}{2}NkT\frac{1}{1-n^2} \\
C &= \frac{1}{12}NkT\frac{1+2n^2-3n^4}{(1-n^2)^4} && (49)
\end{aligned}$$