Quantum Statistical Mechanics

We will now turn our attention to the problem of the equilibrium properties of quantum mechanical systems at non-zero temperature. We have already discussed quantum systems with just a few degrees of freedom (say a particle) at non-zero temperature. Here we will be interested in many-particle systems. Many particle systems have a distinct feature brought about by the fact that quantum mechanics imposes constraints on the way the states of identical particles ought to behave.

Let us begin by considering an N-particle system of identical particles, i.e. the particles only differ by their label. If the Hamiltonian $\hat{H}$ is invariant under the permutation of any pair of particles, i.e. if there is no way to distinguish them

$$\Rightarrow |\Psi(1,\ldots,i,\ldots,j,\ldots,N)|^2 = |\Psi(1,\ldots,j,\ldots,i,\ldots,N)|^2$$

In general, the only freedom this leaves us is the requirement that

$$\hat{P}_{ij}\,\Psi(\cdots i\cdots j\cdots N) = \Psi(\cdots j\cdots i\cdots) = z\,\Psi(\cdots i\cdots j\cdots)$$

with $|z|=1$. $[\hat{P}_{ij}, \hat{H}] = 0$ (symmetry)

If $d \geqslant 3 \Rightarrow z \in \mathbb{R}$ and $z = \pm 1$

$$\Psi_{\pm}(\cdots i\cdots j\cdots) = \pm\,\Psi_{\pm}(\cdots j\cdots i\cdots)$$

(+) = bosons

(−) = fermions

If the state satisfies $\Psi_- \Rightarrow$

$$\Psi_-(\cdots x\cdots y\cdots) = -\,\Psi_-(\cdots y;\cdots x\cdots)$$

$$\Rightarrow \Psi_-(\cdots x\cdots x\cdots) = 0 \qquad \text{Pauli principle.}$$

$\Rightarrow$ Fermions have antisymmetric wave functions and Bosons have symmetric wave functions.

Note: For $d \leq 1,2$, z can be a complex phase, $e^{i\theta}$ when θ is the statistical angle $\Rightarrow$ these particles are anyons (neither fermions nor bosons) $\Rightarrow$ fractional statistics.

Examples: e^-, p^+, He_3, quarks, neutrinos are fermions.

He_4, photons, gluons, phonons are bosons.

We will consider a few of these systems: the electron gas, weakly interacting Bose gases, phonons in a crystal and black body radiation.

In addition to their <u>statistics</u> (i.e. bosons vs fermions) these systems differ and major and important ways: in some cases the number of these particles is conserved (e.g. electron gas or in He_4) but <u>not</u> in others (e.g. phonons, photons etc.)

If a certain quantity $\hat{Q}$ is conserved, i.e. if $[\hat{Q}, \hat{H}] = 0 \Rightarrow$ we can defined a Legendre transform of $\hat{H}$: $\hat{H} - \mu \hat{Q}$

we recognize that if $\hat{Q} = \hat{N}$, the # of particles, $\Rightarrow$ μ is the chemical potential.

These considerations motivate the introduction of the notion of systems with an undefined number of particles, i.e. the Grand Canonical Ensemble.

L31 The space of states of a quantum mechanical system with a fixed number of particles is the Hilbert space $\mathcal{H}_N$. Let $|\psi_1\rangle, \ldots, |\psi_N\rangle$ be a set of one-particle states, i.e. each state $|\psi_i\rangle \in \mathcal{H}_1$ $(i=1,\ldots,N)$

$\Rightarrow$ The properly symmetrized (antisymmetrized) N-particle state is obtained in terms of a tensor product of N, 1-particle states

$$|\psi_1 \cdots \psi_N\rangle \equiv |\psi_1\rangle \times \cdots \times |\psi_N\rangle = \frac{1}{\sqrt{N!}} \sum_P z^P |\psi_{P(1)}\rangle \times \cdots \times |\psi_{P(N)}\rangle$$

P: permutations of N objects, and $z = \pm 1$.

The one-particle states $|\psi_i\rangle$ are themselves linear combination of a set of basis states of $\mathcal{H}_1$

$\{|\alpha_i\rangle\}$ / $\sum_{\alpha_i} |\alpha_i\rangle\langle\alpha_i| = \hat{I}$ (identity in $\mathcal{H}_1$)

$$\langle\alpha_i|\beta_j\rangle = \delta_{ij} \quad \text{(orthonormality)}$$

with $|\psi_i\rangle = \sum_i \psi_{\alpha_i} |\alpha_i\rangle$

Inner Product:

$$\langle \psi_1 \cdots \psi_N | \chi_1 \cdots \chi_N \rangle =$$

$$= \frac{1}{N!} \sum_{P,Q} z^{P+Q} \langle \psi_{P(1)} | \chi_{Q(1)} \rangle \cdots \langle \psi_{P(N)} | \chi_{Q(N)} \rangle$$

Since $PQ = P'$ / $$= \sum_{P'} z^{P'} \langle \psi_1 | \chi_{P'(1)} \rangle \cdots \langle \psi_N | \chi_{P'(N)} \rangle$$

$$= \begin{vmatrix} \langle \psi_1 | \chi_1 \rangle & \cdots & \langle \psi_1 | \chi_N \rangle \\ \vdots & & \\ \langle \psi_N | \chi_1 \rangle & \cdots & \langle \psi_N | \chi_N \rangle \end{vmatrix}_z$$

For $z = -1$ this is a (Slater) <u>determinant</u> (which is antisymmetric) ~~and~~ while for $z = +1$ it is a <u>permanent</u> (which is symmetric).

If we consider now states obtained by tensor products of the orthonormal basis ~~state~~ one-particle states $\Rightarrow$ the labels α_j ($j = 1, \ldots, N$) can be arranged in a monotonic sequence $\alpha_1 \leq \alpha_2 \leq \cdots$ for bosons, and in terms of a strict monotonic sequence $\alpha_1 < \alpha_2 < \cdots$

for fermions. ~~For~~ Let n_j be an integer which counts how many particles there are in the j-th single particle state $\Rightarrow$ the boson states are normalized as follows

$$\frac{1}{\sqrt{n_1! \dots n_N!}} \, |\alpha_1 \dots \alpha_N\rangle \qquad (\alpha_1 \le \alpha_2 \le \dots \le \alpha_N)$$

and $n_j = 0, 1, 2, \dots$ For fermions $n_j = 0, 1$ only and $|\alpha_1 \dots \alpha_N\rangle$ is correctly normalized. These N-particle states are complete and orthonormal.

Fock Space and "Second Quantization"

Consider a quantum mechanical system in which the # of particles is not fixed. The Hilbert space of such a system is called a Fock space.

$$\mathcal{H} = \underbrace{\mathcal{H}_0}_{\substack{\text{space}\\ \text{without}\\ \text{particles}}} \oplus \underbrace{\mathcal{H}_1}_{\substack{\text{one-particle}\\ \text{states}}} \oplus \mathcal{H}_2 \underbrace{\oplus}_{\text{direct sum}} \dots \oplus \underbrace{\mathcal{H}_N}_{\substack{N\text{-particle}\\ \text{states.}}} \oplus \dots$$

$\Rightarrow$ an arbitrary state $|\psi\rangle \in \mathcal{H}$ can be expanded

$$|\Psi\rangle = |\Psi^{(0)}\rangle + |\Psi^{(1)}\rangle + \cdots |\Psi^{(N)}\rangle + \cdots$$

$\mathcal{H}_0$ is a subspace with no particles. It is a one-dimensional Hilbert space spanned by a single state $|0\rangle$, the "vacuum".

Inner Product in Fock space:

$$\langle \chi | \Psi \rangle = \sum_{j=0}^{\infty} \underbrace{\langle \chi^{(j)} | \Psi^{(j)} \rangle}_{\text{inner product in } \mathcal{H}_j}$$

It vanishes if $|\Psi\rangle$ and $|\chi\rangle$ belong to $\neq$ subspaces.

<u>Creation and Annihilation Operators</u>

Let $|\phi\rangle$ be an arbitrary one-particle state, $|\phi\rangle \in \mathcal{H}_1$. Let us define the operator $\hat{a}^{+}(\phi)$ ~~which~~ by it's action in Fock space

$$\hat{a}^{+}(\phi) : \mathcal{H}_N \to \mathcal{H}_{N+1}$$ (i.e. it adds a particle in state $|\phi\rangle$)

<u>Creation Operator</u>

$$\hat{a}^{+}(\phi)|\Psi_1 \ldots \Psi_N\rangle = |\phi, \Psi_1 \ldots \Psi_N\rangle$$ (properly symmetrized)

<u>Destruction Operator</u> : $\hat{a}(\phi) : \mathcal{H}_{N+1} \to \mathcal{H}_N$, <u>removes</u> a particle from the single particle state $|\phi\rangle$.

$\hat{a}(\phi)$ is <u>defined</u> as the <u>adjoint</u> of $\hat{a}^+(\phi)$

i.e.

$$\langle x_1 \cdots x_{N-1} | \hat{a}(\phi) | \psi_1 \cdots \psi_N \rangle =$$

$$\equiv \langle \psi_1 \cdots \psi_N | \hat{a}(\phi)^+ | x_1 \cdots x_{N-1} \rangle^*$$

$$= \langle \psi_1 \cdots \psi_N | \phi, x_1 \cdots x_{N-1} \rangle^*$$

$$= \begin{vmatrix} \langle \psi_1 | \phi \rangle & \langle \psi_1 | x_1 \rangle \cdots & \langle \psi_1 | x_{N-1} \rangle \\ \vdots & & \\ \langle \psi_N | \phi \rangle & \langle \psi_N | x_1 \rangle \cdots & \langle \psi_N | x_{N-1} \rangle \end{vmatrix}_z^*$$

Expand the determinant (permanent):

$$\langle x_1 \cdots x_{N-1} | \hat{a}(\phi) | \psi_1 \cdots \psi_N \rangle =$$

$$= \sum_{k=1}^{N} z^{k-1} \langle \psi_k | \phi \rangle^* \begin{vmatrix} \langle \psi_1 | x_1 \rangle \cdots & \langle \psi_1 | x_{N-1} \rangle \\ \text{--- no } \psi_k \text{ ---} & \\ \langle \psi_N | x_1 \rangle \cdots & \langle \psi_N | x_{N-1} \rangle \end{vmatrix}_z^*$$

$$= \sum_{k=1}^{N} z^{k-1} \langle \phi | \psi_k \rangle \langle x_1 \cdots x_{N-1} | \psi_1 \cdots \overset{\text{removed}}{\hat{\psi}_k} \cdots \psi_N \rangle$$

$$\Rightarrow \boxed{\hat{a}(\phi) | \psi_1 \cdots \psi_N \rangle = \sum_{k=1}^{N} z^{k-1} \langle \phi | \psi_k \rangle | \psi_1 \cdots \hat{\psi}_k \cdots \psi_N \rangle}$$

$\Rightarrow$ (check!) $\hat{a}(\phi_1)^\dagger \hat{a}(\phi_2)^\dagger = z\, \hat{a}(\phi_2)^\dagger \hat{a}(\phi_1)^\dagger$

Let us introduce the notation

$$[\hat{A}, \hat{B}]_{-z} \equiv \hat{A}\hat{B} - z\,\hat{B}\hat{A}$$

i.e. $[\hat{A}, \hat{B}]_+ \equiv \{\hat{A}, \hat{B}\}$

$$[\hat{A}, \hat{B}]_- \equiv [\hat{A}, \hat{B}]$$

$\Rightarrow$ $[\hat{a}(\phi_1)^\dagger, \hat{a}(\phi_2)^\dagger]_{-z} = 0$ (commute for bosons and anticommute for fermions)

Also:

$$[\hat{a}(\phi_1), \hat{a}(\phi_2)]_{-z} = 0$$

and $[\hat{a}(\phi_1), \hat{a}(\phi_2)^\dagger]_{-z} = \langle \phi_1 | \phi_2 \rangle\, \hat{I}$

inner product of the one-particle states.

Occupation Number Representation:

$$|n_1 \dots n_k \dots\rangle = \frac{1}{\sqrt{n_1!\, n_2! \dots n_k! \dots}}\, |\overbrace{1\dots 1}^{n_1}\ \overbrace{2\dots 2}^{n_2} \dots\rangle$$

Let $|\alpha\rangle$ be the α-th one-particle state $\Rightarrow$

$$\hat{a}^\dagger_\alpha |n_1, \dots, n_\alpha, \dots\rangle = \sqrt{n_\alpha + 1}\, |n_1, \dots, n_\alpha + 1, \dots\rangle$$

and $\hat{a}_\alpha |n_1, \dots, n_\alpha, \dots\rangle = \sqrt{n_\alpha}\, |n_1, \dots, n_\alpha - 1, \dots\rangle$

$\Rightarrow$ For fermions and bosons $\hat{a}_\alpha$ annihilate states with $n_\alpha = 0$ and only for fermions $\hat{a}_\alpha^+$ annihilates the states with $n_\alpha = 1$ (Pauli)

$\Rightarrow$ For bosons: $[\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}_\alpha^+, \hat{a}_\beta^+] = 0$

and $[\hat{a}_\alpha, \hat{a}_\beta^+] = \delta_{\alpha\beta}$

For fermions: $\{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}_\alpha^+, \hat{a}_\beta^+\} = 0$

and $\{\hat{a}_\alpha, \hat{a}_\beta^+\} = \delta_{\alpha\beta}$

If a unitary transformation is performed among one-particle states $\Rightarrow$ it induces an unitary transformation among creation and annihilation operators, if $|\chi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$

$$\Rightarrow \hat{a}(\chi) = \alpha^* \hat{a}(\psi) + \beta^* \hat{a}(\phi)$$

$$\hat{a}^+(\chi) = \alpha\, \hat{a}^+(\psi) + \beta\, \hat{a}^+(\phi)$$

i.e. $\hat{a}^+(\chi)$ transforms like $|\chi\rangle$ and $\hat{a}^*(\chi)$ transforms like $\langle\chi|$.

e.g. if we choose momentum state $|\vec{p}\rangle$ as the (complete) set of one-particle states, $\{|\vec{p}\rangle\} \Rightarrow$

$$[\hat{a}^+(\vec{p}), \hat{a}^+(\vec{q})]_{-z} = [\hat{a}(\vec{p}), \hat{a}(\vec{q})]_{-z} = 0$$

$$[\hat{a}(\vec{p}), \hat{a}^+(\vec{q})]_{-z} = (2\pi)^d \, \delta^d(\vec{p}-\vec{q})$$

since $\langle\vec{p}|\vec{q}\rangle = (2\pi)^d \, \delta^d(\vec{p}-\vec{q})$

N-particle states:

$$|\vec{p}_1, \cdots, \vec{p}_N\rangle = \hat{a}^+(\vec{p}_1)\cdots\hat{a}^+(\vec{p}_N)|0\rangle$$

This is the momentum space representation. In coordinate space we have instead

$$|\vec{x}_1, \cdots, \vec{x}_N\rangle = \hat{a}^+(\vec{x}_1)\cdots\hat{a}^+(\vec{x}_N)|0\rangle$$

where $[\hat{a}(x_1), \hat{a}(x_2)]_{-z} = [\hat{a}^+(\vec{x}_1), \hat{a}^+(\vec{x}_2)]_{-z} = 0$

and $[\hat{a}(\vec{x}_1), \hat{a}^+(\vec{x}_2)]_{-z} = \delta^d(\vec{x}_1-\vec{x}_2)$

Since $|\vec{p}\rangle = \int d\vec{x} \; |\vec{x}\rangle\langle\vec{x}|\vec{p}\rangle = \int d\vec{x}\, e^{i\vec{p}\cdot\vec{x}}|\vec{x}\rangle$

$$\Rightarrow \hat{a}^+(\vec{p}) = \int d\vec{x} \; e^{i\vec{p}\cdot\vec{x}}\,\hat{a}^+(\vec{x})$$

and $\hat{a}^+(\vec{x}) = \int \frac{d\vec{p}}{(2\pi)^d}\, e^{-i\vec{p}\cdot\vec{x}}\,\hat{a}^+(\vec{p})$

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General Operators in Fock Space

Let $A^{(1)}$ be some operator acting on the one-particle states of $\mathcal{H}_1 \Rightarrow$ define an extension to Fock space (or to $\mathcal{H}_N$) as follows

$$\hat{A}|\psi\rangle \equiv \sum_{j=1}^{N} |\psi_1\rangle \times \cdots \times \hat{A}^{(1)}|\psi_j\rangle \times \cdots \times |\psi_N\rangle$$

If $\{|\psi_j\rangle\}$ are (one-particle) eigenstates of $A^{(1)}$ with e.v.'s $\{a_j\} \Rightarrow$

$$\hat{A}|\psi\rangle = \left(\sum_{j=1}^{N} a_j\right)|\psi\rangle$$

We wish to write $\hat{A}$ in terms of creation and annihilation operators. Let us consider $A^{(1)}_{\alpha\beta} = |\alpha\rangle\langle\beta|$

$$\Rightarrow \hat{A}_{\alpha\beta}|\psi\rangle = \sum_{j=1}^{N} |\psi_1\rangle \times \cdots \times |\alpha\rangle \times \cdots \times |\psi_N\rangle \langle\beta|\psi_j\rangle$$

$$\equiv \sum_{j=1}^{N} |\psi_1, \ldots, \overset{j}{\alpha}, \ldots, \psi_N) \langle\beta|\psi_j\rangle$$

But

$$\hat{a}^{+}(\alpha)\hat{a}(\beta)|\psi\rangle = \sum_{k=1}^{N} |\alpha, \psi_1 \cdots \psi_{k-1}\, \psi_{k+1} \cdots \psi_N\rangle\, z^{k-1} \langle\beta|\psi_k\rangle$$

$$= \sum_{k=1}^{N} |\psi_1 \ldots \overset{k}{\alpha} \ldots \psi_N) \langle\beta|\psi_k\rangle$$

$$\Rightarrow \hat{A}_{\alpha\beta}|\psi\rangle \equiv \hat{a}^{+}(\alpha)\hat{a}(\beta)|\psi\rangle$$

$\Rightarrow$ a general one-particle operator

$$\hat{A}^{(1)} = \sum_{\alpha,\beta} A_{\alpha\beta} |\alpha\rangle\langle\beta|$$

is represented by

$$\hat{A} = \sum_{\alpha,\beta} A_{\alpha\beta}\, \hat{a}^{+}(\alpha)\, \hat{a}(\beta)$$

where $A_{\alpha\beta} = \langle\alpha| A^{(1)} |\beta\rangle$

Examples

(1) Identity operator: $\hat{I} = \sum_{\alpha} |\alpha\rangle\langle\alpha|$

$$\Rightarrow \hat{I}_{\alpha\beta} = \delta_{\alpha\beta}$$

$$\Rightarrow \hat{I} \equiv \sum_{\alpha,\beta} \delta_{\alpha\beta}\, \hat{a}^{+}(\alpha)\, \hat{a}(\beta)$$

$$\Rightarrow \hat{N} = \sum_{\alpha} \hat{a}^{+}(\alpha)\, \hat{a}(\alpha)$$ which is the number operator.

$$\hat{N} = \int d\vec{x}\; \hat{a}^{+}(\vec{x})\, \hat{a}(\vec{x}) = \int \frac{d^d\vec{p}}{(2\pi)^d}\, \hat{a}^{+}(\vec{p})\, \hat{a}(\vec{p})$$

(2) Linear Momentum:

$$\hat{P}_j^{(1)} = \int \frac{d^d p}{(2\pi)^d}\, p_j\, |\vec{p}\rangle\langle\vec{p}| = \int d^d x\; |\vec{x}\rangle \frac{\hbar}{i}\vec{\nabla} \langle\vec{x}|$$

$\Rightarrow$ Total Linear Momentum $\vec{P}$

$$\hat{P}_j = \int \frac{d^d\vec{p}}{(2\pi)^d}\, p_j\, \hat{a}^{+}(\vec{p})\, \hat{a}(\vec{p}) = \int d^d x\; \hat{a}^{+}(\vec{x}) \frac{\hbar}{i}\vec{\nabla}_x \hat{a}(\vec{x})$$

(3) ~~3~~ **Hamiltonian**

$$\hat{H}^{(1)} = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

$$\Rightarrow \langle \vec{x} | \hat{H}^{(1)} | \vec{y} \rangle = -\frac{\hbar^2}{2m} \nabla^2 \delta^d(\vec{x}-\vec{y}) + V(\vec{x})\, \delta^d(\vec{x}-\vec{y})$$

$\Rightarrow$ in Fock space we get

$$\hat{H} = \int d^d x \; \hat{a}^+(\vec{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 + \hat{V}(\vec{x}) \right] \hat{a}(\vec{x})$$

$$\tilde{V}(\vec{q}) = \int dx^d \; e^{-i \vec{q} \cdot \vec{x}} \; \hat{V}(\vec{x})$$

$$\hat{H} = \int \frac{d^d p}{(2\pi)^d} \frac{\vec{p}^2}{2m} \hat{a}^+(\vec{p})\, \hat{a}(\vec{p}) +$$
$$+ \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \; \tilde{V}(\vec{q})\, \hat{a}^+(\vec{p}+\vec{q})\, \hat{a}(\vec{p})$$

(4) **Two-body interactions**

$\hat{V}^{(2)}$ acts on $\mathcal{H}_2$

$$\hat{V}^{(2)} = \frac{1}{2} \sum_{\alpha,\beta} |\alpha,\beta\rangle V^{(2)}_{(\alpha,\beta)} \langle \alpha,\beta| \qquad \text{(diagonal op.)}$$

$$\Rightarrow \hat{V} = \frac{1}{2} \sum_{\alpha,\beta} V^{(2)}_{(\alpha,\beta)} \hat{a}^+(\alpha)\, \hat{a}^+(\beta)\, \hat{a}(\beta)\, \hat{a}(\alpha)$$

$$\Rightarrow \quad \hat{V} = \frac{1}{2}\int d^d x \int d^d y \; V^{(2)}(\vec{x},\vec{y})\, \hat{a}^\dagger(\vec{x})\, \hat{a}^\dagger(\vec{y})\, \hat{a}(\vec{y})\, \hat{a}(\vec{x})$$

$\hat{\rho}(\vec{x}) = \hat{a}^\dagger(\vec{x})\,\hat{a}(\vec{x})$ density operator

$$\hat{V} \equiv \frac{1}{2}\int d^d x \int d^d y \; V^{(2)}(\vec{x},\vec{y})\, \hat{\rho}(x)\, \hat{\rho}(y)$$
$$+ \frac{1}{2}\int d^d x \; V^{(2)}(\vec{x},\vec{x})\, \hat{\rho}(\vec{x})$$

or if $V^{(2)}(\vec{x},\vec{y}) \equiv V^{(2)}(\vec{x}-\vec{y}) \Longrightarrow$

$$\hat{V} = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\int \frac{d^d q}{(2\pi)^d}\int \frac{d^d k}{(2\pi)^d} \; \tilde{V}(\vec{k})\, \hat{a}^\dagger(\vec{p}+\vec{k})\, \hat{a}^\dagger(\vec{q}-\vec{k})\, \hat{a}(\vec{q})\, \hat{a}(\vec{p})$$

Coulomb: $\quad V^{(2)}(\vec{x},\vec{y}) = \dfrac{e^2}{|\vec{x}-\vec{y}|}$

and $\quad \tilde{V}^{(2)}(\vec{k}) = \dfrac{e^2}{4\pi \vec{k}^2}$

Ground State of a System of Free Fermions

Let $\hat{H} = \sum_\alpha E_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha$

for $E_1 \leq E_2 \leq E_3 \leq \ldots$

We want a system with N particles $\Rightarrow$ define

$$\hat{H} - \mu\hat{N} = \sum_\alpha (E_\alpha - \mu)\, \hat{a}^\dagger_\alpha \hat{a}_\alpha$$

what is the ground state? $|G\rangle$

$$\hat{H}\,|G\rangle = E_G\,|G\rangle$$

$$\hat{N}\,|G\rangle = N\,|G\rangle$$

let $\quad |G\rangle = \prod_{\alpha=1}^{N} \hat{a}_\alpha^{+}\,|0\rangle = |\overbrace{1\ldots 1}^{N}0\ldots 0\ldots\rangle$

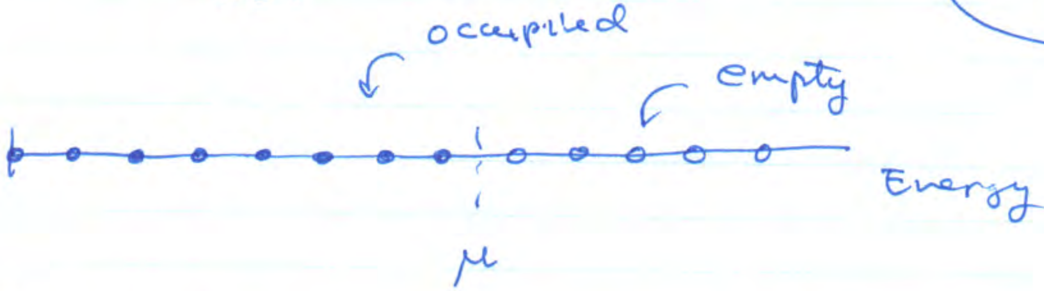
$|G\rangle$ is the <u>Fermi Sea</u> (in RQM it is the <u>Dirac Sea</u>)

$$E_G|G\rangle = \hat{H}\,|G\rangle = \left(\sum_\alpha E_\alpha\, \hat{a}_\alpha^{+}\hat{a}_\alpha\right)\prod_{\beta=1}^{N}\hat{a}_\beta^{+}\,|0\rangle$$

using $\quad \{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}_\alpha^{+}, \hat{a}_\beta^{+}\} = 0$ and

$$\{\hat{a}_\alpha, \hat{a}_\beta^{+}\} = \delta_{\alpha\beta}$$

$$\Rightarrow E_G = \sum_{\alpha=1}^{N} E_\alpha = E_1 + \cdots + E_N \quad ; \quad E_N = E_{Fermi}$$



$$(\hat{H} - \mu\hat{N})|G\rangle = (E_G - \mu N)|G\rangle$$

why is this the ground state at fixed <u>N</u>?

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Consider now a state like

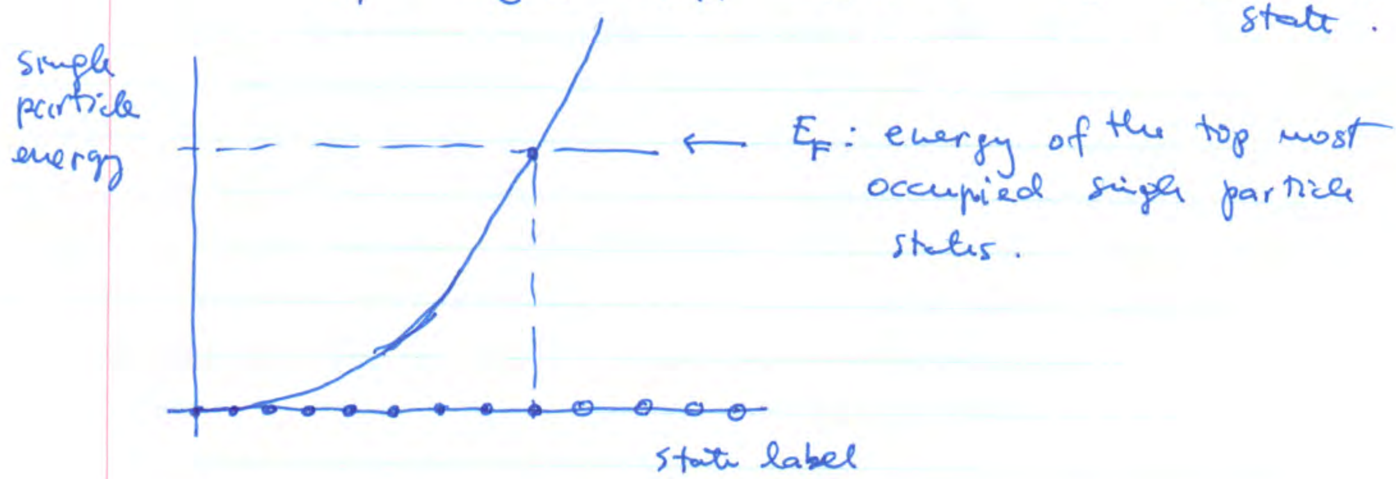
$$|\psi\rangle = |\overbrace{1\cdots 1}^{N-1}010\cdots\rangle \equiv \hat{a}^{+}_{N+1}\,\hat{a}_{N}|G\rangle$$

$$\Rightarrow \quad E_\psi = E_1 + E_2 + \cdots + E_{N-1} + E_{N+1}$$

$$\Rightarrow \quad E_\psi = E_G + \underbrace{(E_{N+1} - E_N)}_{\text{excitation energy}}$$

Since $\quad E_{N+1} > E_N$

$E_\psi - E_G = E_{N+1} - E_N > 0 \quad$ is an excited state.



We will use $|G\rangle$ as the "vacuum" $\Rightarrow$

Since $\hat{a}^{+}(\alpha)\,|G\rangle = 0 \qquad$ if $\quad \alpha \le N$ (or $E_\alpha \le E_F$)

$\Rightarrow$ define $\quad \hat{b}^{+}_\alpha\,|G\rangle = \hat{a}^{+}_\alpha\,|G\rangle \qquad \alpha \le N$

$$\{\hat{a}_\alpha, \hat{a}_{\alpha'}\} = \{\hat{b}_\beta, \hat{b}_{\beta'}\} = 0 \qquad \text{and} \qquad \{\hat{b}_\beta, \hat{b}^{+}_{\beta'}\} = \delta_{\beta\beta'}$$

$\alpha, \alpha' > N \quad , \quad \beta, \beta' \le N \qquad$ (particle-hole transf.)

$$\Rightarrow \ |\alpha_1 \ldots \alpha_m, \beta_1 \ldots \beta_n; G\rangle \equiv \hat{a}^+_{\alpha_1} \ldots \hat{a}^+_{\alpha_m} \hat{b}^+_{\beta_1} \ldots \hat{b}^+_{\beta_n} |G\rangle$$

provided $\quad \hat{a}_\alpha |G\rangle = \hat{b}_\beta |G\rangle = 0$

for $\quad \alpha > N \quad$ and $\beta \leq N$

$$\Rightarrow \ \hat{H} = \sum_\alpha E_\alpha \hat{a}^+_\alpha \hat{a}_\alpha =$$

$$= \sum_{\alpha=1}^{N} E_\alpha \hat{a}^+_\alpha \hat{a}_\alpha + \sum_{\alpha=N+1}^{\infty} E_\alpha \hat{a}^+_\alpha \hat{a}_\alpha$$

$$\equiv \sum_{\beta=1}^{N} E_\beta \hat{b}_\beta \hat{b}^+_\beta + \sum_{\alpha=N+1}^{\infty} E_\alpha \hat{a}^+_\alpha \hat{a}_\alpha$$

$$\equiv \left(\sum_{\beta=1}^{N} E_\beta\right) + \underbrace{\sum_{\alpha>N} E_\alpha \hat{a}^+_\alpha \hat{a}_\alpha}_{\text{particles}} - \sum_{\beta\leq N} E_\beta \underbrace{\hat{b}^+_\beta \hat{b}_\beta}_{\text{holes}}$$

$$\hat{N} = \sum_\alpha \hat{a}^+_\alpha \hat{a}_\alpha = N - \sum_{\beta\leq N} \hat{b}^+_\beta \hat{b}_\beta + \sum_{\alpha>N} \hat{a}^+_\alpha \hat{a}_\alpha$$

$$\hat{H} = E_{gnd} + :\hat{H}:$$

"normal ordered"

etc.

Ground State of a System of free Bosons

Since there is no exclusion pple. for free bosons the ground state is simply

$$|G\rangle = (\hat{a}_0^+)^N |0\rangle$$

i.e. all particles in the lowest energy single particle state, i.e:

$|G\rangle \equiv |0,\ldots,0\rangle$ which if fully symmetrized.

$\Rightarrow$ we have a macroscopic occupancy of a single particle state $\equiv$ Bose Condensation.

We will see that this state is a lot more interesting that it superficially seems to be.

Statistical Mechanics of Identical free Q.Mechanical particles:

Consider a gas of free quantum mechanical particles, either bosons or fermions. Let ε_α be the (single particle) energy levels of the single particle states $|\alpha\rangle$

The Hamiltonian is

$$\hat{H} = \sum_n \varepsilon_\alpha \hat{a}^+_\alpha \hat{a}_\alpha \equiv \sum_\alpha \varepsilon_\alpha \hat{n}_\alpha$$

where $\hat{a}_\alpha$'s obey either fermionic or bosonic commutation relations.

The (canonical) Partition Function is

$$Z_N = \mathrm{tr}\, e^{-\beta \hat{H}} = \sum_{\{n_\alpha\}}{}' e^{-\beta \sum_\alpha n_\alpha \varepsilon_\alpha}$$

\# of particles

where $(')$ is the restriction that

$$\sum_\alpha n_\alpha = N$$

We will use the Grand Canonical Ensemble:

$$\mathcal{Z} = \sum_{N=0}^{\infty} z^N Z_N \qquad , \; z = e^{\beta\mu}$$

$$= \sum_{N=0}^{\infty} z^N \sum_{\{n_\alpha\}}' e^{-\beta \sum_\alpha \varepsilon_\alpha n_\alpha}$$

$$= \sum_{\{n_\alpha\}}' e^{-\beta \sum_\alpha (\varepsilon_\alpha - \mu) n_\alpha}$$

(no restriction

$$= \prod_\alpha \left(\sum_{n_\alpha} e^{-\beta(\varepsilon_\alpha - \mu) n_\alpha} \right)$$

(a) Bosons: $n_\alpha = 0, 1, \ldots \infty$

$$\sum_{n_\alpha = 0}^{\infty} e^{-\beta(\varepsilon_\alpha - \mu) n_\alpha} = \frac{1}{1 - e^{-\beta(\varepsilon_\alpha - \mu)}}$$

$$\Rightarrow \mathcal{Z}(z, T) = \prod_\alpha \left[\frac{1}{1 - e^{-\beta(\varepsilon_\alpha - \mu)}} \right]$$

$$\Rightarrow \frac{PV}{kT} = + \log \mathcal{Z} = - \sum_\alpha \log\left(1 - e^{-\beta(\varepsilon_\alpha - \mu)}\right)$$

$$\Rightarrow \Omega = -kT \log \mathcal{Z} = +kT \sum_\alpha \log\left(1 - e^{-\beta(\varepsilon_\alpha - \mu)}\right)$$ (Bosons)

(5) Fermions

$$\Omega = -\overset{kT}{\downarrow}\log \mathcal{Z} = -\frac{pV}{\cancel{kT}}\cancel{kT} = -pV$$

but $$\mathcal{Z}(z,T) = \prod_\alpha \sum_{n_\alpha=0,1} e^{-\beta(\varepsilon_\alpha-\mu)n_\alpha}$$

$$= \prod_\alpha \left(1 + e^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

$$\Rightarrow \log \mathcal{Z} = \sum_\alpha \log\left(1 + e^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

$$\Omega = -kT\log\mathcal{Z} = -kT\sum_\alpha \log\left(1 + e^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

$\Rightarrow$

$$\Omega = \pm kT\sum_\alpha \log\left(1 \mp e^{-\beta(\varepsilon_\alpha-\mu)}\right)$$ upper sign: bosons, lower sign: fermions

Constraint: fixed density

$$z\frac{\partial}{\partial z}\log\mathcal{Z} = \langle N\rangle$$

$$z\frac{\partial}{\partial z}\frac{1}{V}\log\mathcal{Z} = \frac{\langle N\rangle}{V} = n = \frac{1}{v}$$

since $z = e^{\beta\mu} \Rightarrow \log z = \beta\mu$

$$\Rightarrow \frac{1}{v} = n = -\frac{\partial}{\partial \log z}\left(\frac{\beta\Omega}{V}\right) = -\frac{1}{\beta}\frac{\partial}{\partial\mu}\left(\frac{\beta\Omega}{V}\right)$$

$$\frac{1}{v} = -\frac{kT}{V}\frac{\partial}{\partial\mu}\left[\pm\sum_\alpha \log\left(1 \mp e^{-\beta(\varepsilon_\alpha-\mu)}\right)\right]$$

$$\frac{1}{v} = \mp\frac{1}{V}\sum_\alpha \frac{\mp e^{-\beta(\varepsilon_\alpha-\mu)}}{1 \mp e^{-\beta(\varepsilon_\alpha-\mu)}}$$

$$\Rightarrow \frac{1}{v} = \frac{1}{V}\sum_\alpha \frac{e^{-\beta(\varepsilon_\alpha-\mu)}}{1 \mp e^{-\beta(\varepsilon_\alpha-\mu)}}$$

$$\frac{1}{v} \Rightarrow \langle N\rangle = \sum_\alpha \frac{1}{e^{\beta(\varepsilon_\alpha-\mu)} \mp 1} \equiv \sum_\alpha \langle n_\alpha\rangle$$

$$\Rightarrow \langle n_\alpha\rangle = \frac{1}{e^{\beta(\varepsilon_\alpha-\mu)}-1}$$ Bosons $\Leftrightarrow$ Bose-Einstein Distribution

$$\langle n_\alpha\rangle = \frac{1}{e^{\beta(\varepsilon_\alpha-\mu)}+1}$$ Fermions $\Leftrightarrow$ Fermi Dirac Distribution.

L35

The Ideal Bose–Einstein Gas

$$\Omega = kT \sum_\alpha \log\left(1 - e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

with $$\frac{1}{v} = \frac{1}{V} \sum_\alpha \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}$$

$|\alpha\rangle = |\vec{p}\rangle$ i.e. momentum states

Let us put the system in a box of linear size L

$$\Rightarrow \langle \vec{x} | \vec{p} \rangle = \frac{1}{(L)^{3/2}} e^{i \vec{p}\cdot\vec{x}/\hbar} = \psi_{\vec{p}}(\vec{x})$$

$$\int d^3x \, |\psi_{\vec{p}}(\vec{x})|^2 = \frac{1}{L^3} L^3 = 1 \quad \checkmark$$

$$L^3 = V$$

and the momenta are quantized

$$\vec{p} = \frac{2\pi\hbar}{L}(n_1, n_2, n_3)$$

$$\Delta p_x = \Delta p_y = \Delta p_z = \frac{2\pi\hbar}{L}$$

$$\Rightarrow \sum_\alpha f_\alpha \equiv \sum_{n_1, n_2, n_3} f(n_1, n_2, n_3) \xrightarrow[L\to\infty]{} \left(\frac{L}{2\pi\hbar}\right)^3 \int d^3p \, f(\vec{p})$$

$$\Rightarrow \frac{1}{V}\sum_\alpha f_\alpha \underset{V\to\infty}{\equiv} \int \frac{d^3p}{(2\pi\hbar)^3} f(\vec{p})$$

$$\Rightarrow \Omega = kTV \int \frac{d^3p}{(2\pi\hbar)^3} \log\left(1 - e^{-\beta(\varepsilon(\vec{p})-\mu)}\right)$$

and $$\frac{1}{v} = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)}-1}$$

These last identities are delicate : in taking $V\to\infty$ we assumed that $\dfrac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)}-1}$ is a bounded (integrable) function of $\vec{p}$ $\Rightarrow$ this is violated if there is Bose-Condensation. The same applies to Ω. Also, by definition, the average occupation numbers must be positive

$$\langle n_{\vec{p}}\rangle = \frac{1}{e^{\beta(\varepsilon(\vec{p})-\mu)}-1} \geq 0 \Rightarrow \mu \leq \min_{\vec{p}}\{\varepsilon_{\vec{p}}\}$$

Notice that if μ violates this condition there is an imaginary part in Ω since the argument of the log ~~becom~~ becomes negative (i.e. a branch cut)

Let us define $\min_{\vec{p}} \mathcal{E}(\vec{p}) = 0$ (we will also assume that it is taken by $\vec{p}=0$) which follows from Galilean Invariance

$$\Rightarrow \quad n_0 = \langle \hat{n}_0 \rangle = \frac{1}{e^{-\beta\mu} - 1} \geq 0 \quad \Rightarrow \quad \mu \leq 0$$

$$\Rightarrow \quad n_0 = \frac{z}{1-z} \qquad (z = e^{\beta\mu})$$

bounded

$$\Rightarrow \sum_{\vec{p}} \frac{1}{e^{\beta(\mathcal{E}(\vec{p})-\mu)} - 1} = \frac{z}{1-z} + \sum_{\vec{p}\neq 0} \frac{1}{z^{-1} e^{\beta \mathcal{E}(p)} - 1}$$

$$\frac{1}{v} = \frac{1}{V}\frac{z}{1-z} + \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta\mathcal{E}(\vec{p})} z^{-1} - 1}$$

this term may diverge $\Leftrightarrow$ Bose Condensation.

$$\int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta\mathcal{E}(\vec{p})} z^{-1} - 1} = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{e^{\beta\mu}}{e^{\beta\frac{p^2}{2m}} - e^{\beta\mu}}$$

$$\mathcal{E}(\vec{p}) = \frac{\vec{p}^2}{2m}$$

$$= \int \frac{d^3p}{(2\pi\hbar)^3} \frac{e^{\beta\mu}\, e^{-\beta\vec{p}^2/2m}}{1 - z\, e^{-\beta\vec{p}^2/2m}}$$

Let $z = e^{\beta\mu}$

$$x^2 = \frac{\beta\vec{p}^2}{2m}$$

$$\Rightarrow \frac{1}{v} = \frac{z}{1-z} + \int_0^\infty dx \frac{z\, e^{-x^2}}{1-z e^{-x^2}} \left[\frac{x^2}{2\pi^2\hbar^3}\left(\frac{2m}{\beta}\right)^{3/2}\right]$$

Since $\mu < 0 \Rightarrow |z| < 1$ and $|z| e^{-x^2} < 1 \quad \forall x$

$\Rightarrow$ we can expand (notice that as $z \to 1$ ($\mu \to 0$) there is a potential singularity)

$$\int_0^\infty dx \frac{z\, e^{-x^2}}{1- z e^{-x^2}} = \sum_{n=0}^\infty \int_0^\infty dx\, z^{n+1} e^{-(n+1)x^2}$$

$$= \sum_{n=0}^\infty \frac{z^{n+1}}{\sqrt{\cancel{2}(n+1)}} \frac{\sqrt{\cancel{2}\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^\infty \frac{z^{n+1}}{\sqrt{n+1}} = \frac{\sqrt{\pi}}{2} \sum_{n=1}^\infty \frac{z^n}{n^{1/2}}$$

and $\displaystyle \int_0^\infty dx \frac{z e^{-x^2}}{1 - z e^{-x^2}} x^2 = \frac{\sqrt{\pi}}{4} \sum_{n=0}^\infty \frac{z^{n+1}}{(n+1)^{3/2}} = \frac{\sqrt{\pi}}{4} \sum_{n=1}^\infty \frac{z^n}{n^{3/2}}$

$g_\nu(z) = \displaystyle\sum_{n=1}^\infty \frac{z^n}{n^\nu}$ related to the Riemann zeta-function

$g_\nu(1) = \zeta(\nu)$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda_T^3} g_{3/2}(z) \qquad (1)$$

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2} = \text{de Broglie Thermal wavelength}$$

The function $\zeta_{3/2}(z)$ is finite at $z=1$ ($\mu=0$)

$\zeta_{3/2}(1) = 2.612\ldots$ but its derivative $\to \infty$

We can rewrite (1) as:

$$\lambda_T^3 \frac{\langle n_0 \rangle}{V} = \frac{\lambda_T^3}{v} - g_{3/2}(z)$$

For $\frac{\langle n_0 \rangle}{V} > 0 \Rightarrow$ the temperature must

be such that $\frac{\lambda_T^3}{v} - \underbrace{g_{3/2}(1)}_{\zeta(3/2)} > 0$

If this happens $\Rightarrow$ for $T < T_c$ (to be determined)

$\mu = 0$ and $z = 1$

Critical temperature T_c / $\langle n_0 \rangle = 0$ and $\mu = 0$

$$\Rightarrow \quad \frac{\lambda_{T_c}^3}{v} = \zeta\left(\tfrac{3}{2}\right)$$

$$kT_c = \frac{2\pi\hbar^2}{m} \Big/ \left(v\, \zeta\left(\tfrac{3}{2}\right)\right)^{2/3} \qquad \text{(at fixed } v\text{)}$$

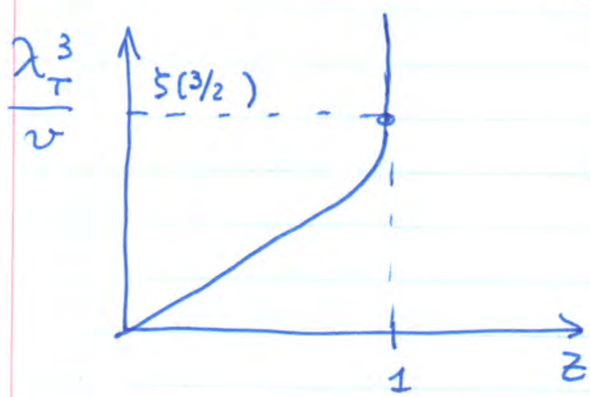
Alternatively

$$v_c = \frac{\lambda_T^3}{\zeta(3/2)}$$

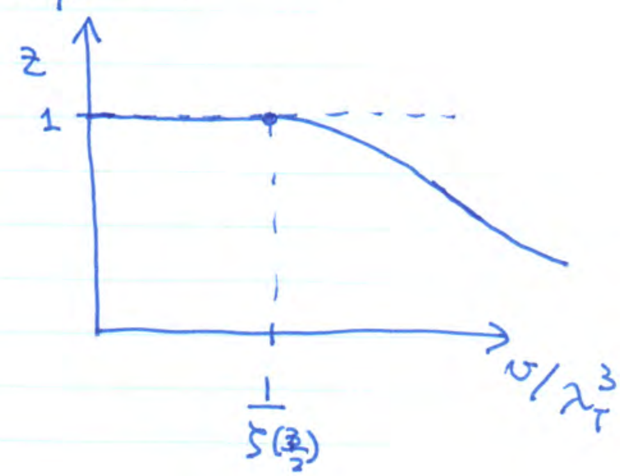
However, for $T > T_c$ (v fixed) $z < 1 \Rightarrow$

$$\Rightarrow \frac{1}{V} \frac{z}{1-z} \xrightarrow[V\to\infty]{} 0$$

$\Rightarrow$ for $T > T_c$, $\quad \frac{1}{v} = \frac{1}{\lambda_T^3} g_{3/2}(z)$;



$\Rightarrow$



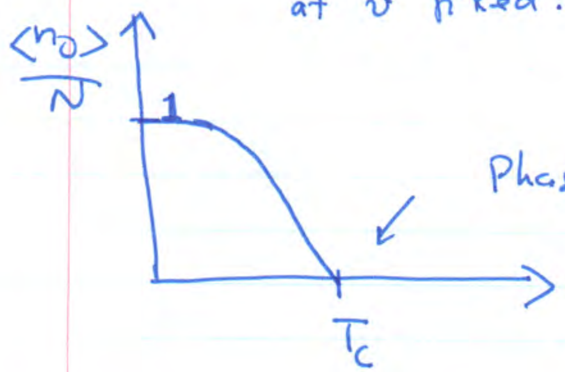
For $T \leq T_c \Rightarrow z = 1$

$$\frac{\langle n_0 \rangle}{V} = \frac{1}{V}\frac{z}{1-z} = \frac{1}{v} - \frac{1}{\lambda_T^3} g(3/2)$$

$$\frac{\langle n_0 \rangle}{V} = \frac{1}{v}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] \qquad T \leq T_c$$

$$\Rightarrow \frac{\langle n_0 \rangle}{N} = \begin{cases} 1 - \left(\frac{T}{T_c}\right)^{3/2} & T \leq T_c \\ 0 & T > T_c \end{cases}$$

at v fixed.



Phase Transition!

$\Rightarrow$ for $T < T_c$ $\exists$ a finite fraction of the total # of particles, N, occupying the single particle state $\vec{p}=0$ ("Bose Condensation")

Note, $1 - \left(\frac{T}{T_c}\right)^{3/2} = 1 - \frac{v}{v_c}$ (T fixed)

Equation of State

$$\frac{PV}{kT} = -\Omega \qquad \text{or}$$

$$\frac{P}{kT} = -\frac{1}{2\pi^2\hbar^3}\int_0^\infty dp\; p^2 \log\left(1 - z\, e^{-\beta p^2/2m}\right)$$

$$- \frac{1}{V}\log(1-z)$$

where z is the solution of

$$\frac{1}{v} = \frac{1}{\lambda_T^3}\, g_{3/2}(z) \qquad T > T_c$$

and $z = 1$ for $T \le T_c$

Since $1-z \sim O(\frac{1}{V}) \Rightarrow$ as $V \to \infty$ $\frac{1}{V}\log(1-z) \to 0$

$$\Rightarrow \frac{P}{kT} = -\frac{1}{\lambda_T^3} \frac{4}{\sqrt{\pi}} \int_0^\infty dx \; x^2 \log(1 - z e^{-x^2})$$

for all $T > T_c$

Furthermore

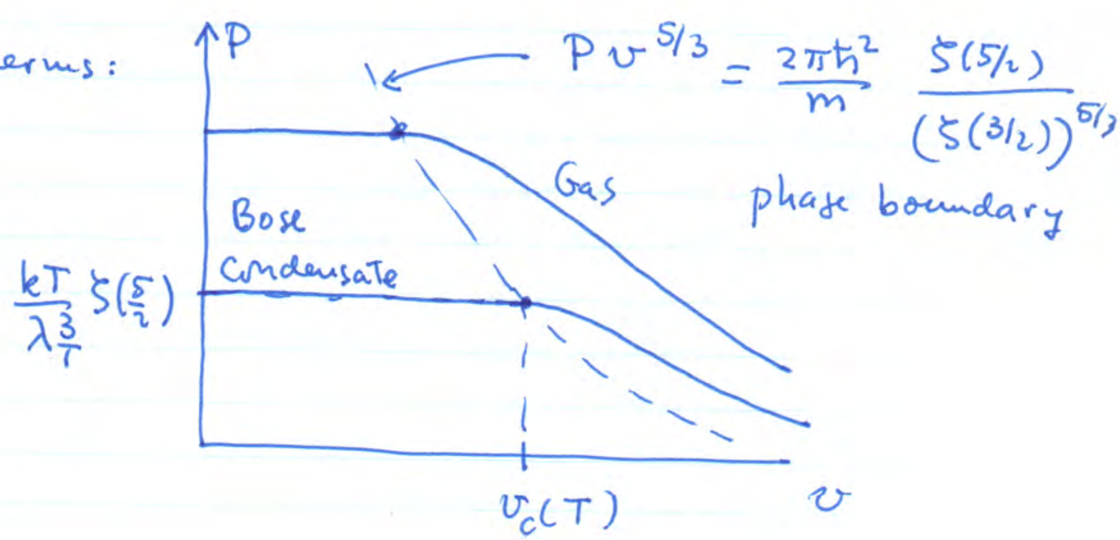
$$-\frac{4}{\sqrt{\pi}} \int_0^\infty dx \; x^2 \log(1 - z e^{-x^2}) = \sum_{n=1}^\infty \frac{z^n}{n^{5/2}} \equiv g_{5/2}(z)$$

$$\Rightarrow \frac{P}{kT} = \frac{1}{\lambda_T^3} g_{5/2}(z) \qquad \text{for } T > T_c \quad (\text{or } v > v_c)$$

and $$\frac{P}{kT} = \frac{1}{\lambda_T^3} \zeta(5/2) \qquad \text{for } T \le T_c$$

since $z = 1$ and $g_{5/2}(1) = \zeta(5/2) = 1.342\ldots$

$\Rightarrow$ Isotherms:



L36

Is this a second order or a first order transition?
We saw that the condensate fraction $\frac{\langle n_0 \rangle}{V}$ vanishes continuously at T_c (as $T \to T_c$)

Moreover it can be seen that the specific heat C is continuous at T_c and that its derivative has a discontinuity $\Rightarrow$ there is a "cusp" in the specific heat at T_c. However the details of this behavin can not be understood correctly ~~for~~ in the context of an ideal Bose gas as these effects are modified by interactions (no matter how weak they are).

<u>Long Range Order and Spontaneous Symmetry Breaking</u>

Let us examine the one-particle density matrix

$$\rho_1(\vec{x},\vec{y}) = \langle \hat{a}^+(\vec{x})\,\hat{a}(\vec{y})\rangle = \frac{1}{\mathcal{Z}} \operatorname{tr}\left[\hat{a}^+(\vec{x})\hat{a}(\vec{y})\, e^{-\beta(\hat{H}-\mu\hat{N})}\right]$$

In a Hilbert space with N particles and at $T=0$ this expression reduces to a ground state expectation value

$$\rho_1(\vec{x},\vec{y}) = \langle G|\, \hat{a}^+(\vec{x})\, \hat{a}(\vec{y})\, |G\rangle =$$

$$= N \int d^3r_2 \dots d^3r_N \; \Psi_G^*(\vec{x},\vec{r}_2,\dots \vec{r}_N)\, \Psi_G(\vec{y},\vec{r}_2,\dots \vec{r}_N)$$

which is indeed a one-particle density matrix.

In Fourier space we can write

$$\rho_1(\vec{x},\vec{y}) = \frac{1}{V} \sum_{\vec{k},\vec{q}} e^{i(\vec{k}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \langle \hat{a}^+(\vec{k})\, \hat{a}(\vec{q}) \rangle$$

For a translationally invariant system the total linear momentum $\hat{\vec{P}}$ is conserved, i.e.

$$[\hat{\vec{P}}, \hat{H}] = 0$$

$$\Rightarrow \langle [\hat{\vec{P}}, \hat{a}^+(\vec{k})\, \hat{a}(\vec{q})] \rangle =$$

$$= \frac{1}{Z} \mathrm{tr} \left\{ [\vec{P}, \hat{a}^+(\vec{k})\, \hat{a}(\vec{q})]\, e^{-\beta(\hat{H}-\mu\hat{N})} \right\} = 0$$

if $[\hat{\vec{P}}, \hat{H}] = 0$ (check it!)

But $\left[\hat{\vec{P}},\ \hat{a}^{\dagger}(\vec{k})\,\hat{a}(\vec{q})\right] = \hbar(\vec{k}-\vec{q})\ \hat{a}^{\dagger}(\vec{k})\,\hat{a}(\vec{q})$

$$\Rightarrow \langle \hat{a}^{\dagger}(\vec{k})\ \hat{a}(\vec{q})\rangle = (2\pi)^3\ \delta^3(\vec{k}-\vec{q})\ \underset{\substack{\uparrow \\ \text{Bose factor!}}}{\langle \hat{n}(\vec{k})\rangle}$$

$\hat{n}(\vec{k}) = \hat{a}^{\dagger}(\vec{k})\,\hat{a}(\vec{k})$

$$\Rightarrow \rho_1(\vec{x},\vec{y}) = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\ \langle \hat{n}(\vec{k})\rangle$$

$$= \frac{n_0}{V} + \int \frac{d^3k}{(2\pi)^3}\ \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\frac{1}{z}e^{\beta\varepsilon(\vec{k})}-1}$$

Hence

(a) If $T > T_c$ $\frac{n_0}{V} \approx 0$

$$\Rightarrow \rho_1(\vec{x},\vec{y}) \equiv \rho_1(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3}\ \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\frac{1}{z}e^{\beta\varepsilon(\vec{k})}-1}$$

where $z = z(T, v)$

and

(b) If $T \leq T_c \Rightarrow z = 1$ and

$$\rho_1(\vec{x}-\vec{y}) = \frac{n_0}{V} + \int \frac{d^3k}{(2\pi)^3}\ \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{e^{\beta\varepsilon(\vec{k})}\ldots}$$

Let us examine these integrals

(a) $g_>(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{e^{\beta\mathcal{E}(\vec{k})} - z} \, z$

$$\beta\mathcal{E}(\vec{k}) = \frac{\vec{k}^2}{2mkT} \qquad (z<1)$$

Clearly $e^{\beta\mathcal{E}(\vec{k})} - z$ is a monotonically increasing function of $\vec{k}^2$ ~~which~~ which has no zeros at any value of $\vec{k}$. Thus the large contributions to $g_>(\vec{x}-\vec{y})$ come from $\vec{k}\to 0$. Moreover this integral is real and positive. Since

$$e^{\beta\mathcal{E}(\vec{k})} - z \geq 1 - z + \beta\mathcal{E}(\vec{k}) \qquad (\text{"convexity"})$$

$$\Rightarrow \quad g_>(\vec{x}-\vec{y}) \leq z \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{1 - z + \frac{\beta k^2\hbar^2}{2m}}$$

$$g_>(\vec{x}-\vec{y}) \leq \frac{2mkT}{\hbar^2} z \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\frac{2mkT}{\hbar^2}(1-z) + \vec{k}^2}$$

whereas (b) for $T \le T_c$ $\quad z = 1$

and $\rho(\vec{x}-\vec{y}) = g_<(\vec{x}-\vec{y}) \le \frac{2mkT}{\hbar^2}\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2}$

Let $\quad a^2 = \frac{2mkT}{\hbar^2}(1-z)$

dimensionally $\quad [a] = [\vec{k}] = \frac{[\text{momentum}]}{[\hbar]} = \frac{1}{[\text{length}]}$

de Broglie Thermal wavelength $\quad \lambda_T^2 = \frac{2\pi\hbar^2}{mkT}$

$\Rightarrow \quad a^2 = \frac{4\pi(1-z)}{\lambda_T^2} \equiv \frac{1}{\xi^2}$

where $\quad \xi = \frac{\lambda_T}{\sqrt{4\pi(1-z)}}$

Notice that λ_T is a smooth function of T but since $z \to 1$ as $T \to T_c^+$ $\Rightarrow$ $\xi \to \infty$ as $T \to T_c^+$

L37 Integral:

$$\int \frac{d^dk}{(2\pi)^d}\frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+a^2} = \frac{1}{2}\int_0^\infty d\alpha \int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{r} - \frac{1}{2}\alpha(\vec{k}^2+a^2)}$$

$$\int \frac{d^dk}{(2\pi)^d}\, e^{-\frac{\alpha\vec{k}^2}{2} + i\vec{k}\cdot\vec{r}} = \frac{e^{-\frac{\vec{k}^2}{2\alpha}}}{(2\pi\alpha)^{d/2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + a^2} = \frac{1}{2}\int_0^\infty \frac{d\alpha}{(2\pi\alpha)^{d/2}} e^{-\frac{\alpha a^2}{2} - \frac{\vec{r}^2}{2\alpha}}$$

scale $\alpha = \frac{\lambda}{s}$ such that

$$\frac{\lambda a^2}{2s} + \frac{\vec{r}^2}{2\lambda} s = \frac{\lambda a^2}{2}\left(s + \frac{1}{s}\right)$$

$$\Rightarrow \frac{\lambda a^2}{\cancel{2}} = \frac{\vec{r}^2}{\cancel{2}\lambda} \Rightarrow \lambda = \frac{|\vec{r}|}{a} = |\vec{r}|\xi$$

$$\frac{\lambda a^2}{2} = \frac{|\vec{r}|\cancel{\xi}}{2}\frac{1}{\xi^{\cancel{2}}} = \frac{|\vec{r}|}{2\xi}$$

$$\frac{d\alpha}{\alpha^{d/2}} = -\lambda^{1-\frac{d}{2}} \frac{ds}{s^2} s^{d/2} = -\lambda^{1-\frac{d}{2}} ds\, s^{\frac{d}{2}-2}$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + a^2} = \frac{\lambda^{1-\frac{d}{2}}}{(2\pi)^{d/2}} \frac{1}{2}\int_0^\infty ds\, s^{\frac{d}{2}-2} e^{-\frac{|\vec{r}|}{2\xi}\left(s+\frac{1}{s}\right)}$$

$$\frac{1}{2}\int_0^\infty ds\, s^{\nu-1} e^{-\frac{x}{2}\left(s+\frac{1}{s}\right)} = K_\nu(x)$$ Bessel function.

$$\int \frac{d^dk}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + a^2} = \frac{1}{(2\pi)^{\frac{d}{2}} \left(|\vec{r}|\xi\right)^{\frac{d}{2}-1}} K_{\frac{d}{2}-1}\left(\frac{|\vec{r}|}{\xi}\right)$$

For $x \gg 1$ (i.e. $\frac{|\vec{r}|}{\xi} \gg 1 \Leftrightarrow |\vec{r}| \gg \xi$)

$$K_\nu(x) \sim \sqrt{\frac{\pi \xi}{2|\vec{r}|}}\, e^{-|\vec{r}|/\xi} + \dots \qquad (\forall \nu)$$

$\Rightarrow$ at long distances

$$\rho_1(\vec{x}-\vec{y}) \simeq \frac{4\pi z}{\lambda_T^2} \frac{1}{(2\pi)^{3/2}} \frac{K_{1/2}(|\vec{r}|/\xi)}{(|\vec{r}|\xi)^{1/2}}$$

$$\rho_1(\vec{x}-\vec{y}) = \frac{\cancel{4\pi}^{\,2}\, z(T,v)}{\lambda_T^2\, \cancel{2\pi}\sqrt{2\pi}} \frac{1}{\sqrt{|\vec{r}|\xi}} K_{1/2}(|\vec{r}|/\xi)$$

$$= \sqrt{\frac{2}{\pi}} \frac{z(T,v)}{\lambda_T^3} \left(\frac{\lambda_T^2}{|\vec{r}|\xi}\right)^{1/2} K_{1/2}(|\vec{r}|/\xi)$$

$$\approx \sqrt{\frac{2}{\pi}} \frac{z(T,v)}{\lambda_T^3} \left(\frac{\lambda_T^2}{|\vec{r}|\cancel{\xi}}\right)^{1/2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\cancel{\xi}}{|\vec{r}|}}\, e^{-|\vec{r}|/\xi}$$

$$= \frac{z(T,v)}{\cancel{\pi}\,\lambda_T^2} \frac{1}{|\vec{r}|} e^{-|\vec{r}|/\xi}$$

How does it behaves for $|\vec{r}| \gg \xi$?

$$K_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^{-x} \left[1 + O\left(\frac{1}{x}\right) \right] \quad (\forall \nu)$$

for $x \gg 1$

$$\Rightarrow \quad \rho_1(\vec{x}-\vec{y}) = \sqrt{\frac{2}{\pi}} \frac{z(T,v)}{\lambda_T^3} \left(\frac{\lambda_T^2}{|\vec{r}|\xi} \right)^{1/2} K_{1/2}\left(\frac{|\vec{r}|}{\xi}\right)$$

$$\Rightarrow \rho_1(\vec{x}-\vec{y}) \approx \frac{z(T,v)}{\lambda_T^2} \frac{e^{-|\vec{r}|/\xi}}{|\vec{r}|} \qquad \text{(i.e. a "Yukawa potential")}$$

$|\vec{r}| = |\vec{x}-\vec{y}|$

(b) For $T \leq T_c$, $z = 1$ and

$$\rho_1(\vec{x}-\vec{y}) = \frac{n_0}{V} + g_<(\vec{x}-\vec{y})$$

where

$$g_<(\vec{x}-\vec{y}) = \frac{4\pi}{\lambda_T^2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2} = \frac{1}{\lambda_T^2 |\vec{r}|}$$

Hence ~~by~~, at long distances $|\vec{r}| \gg \lambda_T$ or ξ the one-particle density matrix behaves like

$$\rho_1(\vec{x}-\vec{y}) \underset{|\vec{x}-\vec{y}| \gg \lambda_T}{\longrightarrow} \frac{n_0(T,v)}{V} + \frac{1}{\lambda_T^2 |\vec{r}|} + \cdots$$

for $T \le T_c$

whereas for $T > T_c$ and $|\vec{r}| \gg \xi > \lambda_T$

$$\rho_1(\vec{x}-\vec{y}) \longrightarrow \frac{z(T,v)}{\pi \lambda_T^2} \frac{e^{-|\vec{r}|/\xi}}{|\vec{r}|} + \cdots$$

for $|\vec{r}| \gg \xi$

Hence at high temperatures, $T > T_c$, we find an exponential decay of correlations while at low temperatures, $T < T_c$, the correlation function approaches a constant as $|\vec{r}| \to \infty$ with power law precision (for this problem). Hence, for $T < T_c$, as $|\vec{x}-\vec{y}| \to \infty$

$$\lim_{|\vec{x}-\vec{y}|} \rho_1(\vec{x}-\vec{y}) = \frac{\langle n_0 \rangle}{V}$$ (the # of particles at $\vec{p}=0$ per unit volume)

= "condensate fraction"

Since $\rho_1(\vec{x},\vec{y}) = \langle \hat{a}^\dagger(x)\, a(\vec{y}) \rangle$

we find that in the condensed state the operator $\hat{a}(\vec{x})$ behaves as if it had a c-number piece

i.e. as if $\langle \hat{a}(\vec{x}) \rangle = \sqrt{\dfrac{\langle n_0 \rangle}{V}} \neq 0$

This is a surprising result. The Hamiltonian has a conserved quantity, the number of particles $[\hat{N}, \hat{H}] = 0$ but $[\hat{a}(\vec{x}), \hat{N}] \neq 0$

$\Rightarrow$ the ground state cannot be an "eigenstate of $\hat{a}$"

Furthermore $\hat{a}(\vec{x})$ is not hermitean and since it changes the # of particles by 1 is is not diagonal in the number basis. What is going on?

What happens is that these results are correct

only in the thermodynamic limit $V \to \infty$ (v fixed).
In this limit it is correct to say that $\hat{a}(x)$ has a c-number piece since $n_0 \to \infty$ (with $\frac{n_0}{V}$ finite). The field $\hat{a}(\vec{x})$ is an example of an order parameter field since it picks up a non-vanishing expectation value for $T \leq T_c$. In fact this ground state exhibits spontaneous symmetry breaking.

Since $[\hat{N}, \hat{H}] = 0 \Rightarrow U = e^{i\theta \hat{N}}$ is a unitary operator with the property

$$\hat{U}(\theta)\, \hat{H}\, \hat{U}(\theta)^{-1} = \tilde{\hat{H}} = \hat{H} \qquad \Rightarrow \hat{H} \text{ is invariant}$$

but $\hat{U}(\theta)\, \hat{a}(\vec{x})\, \hat{U}(\theta)^{-1} = e^{i\theta \hat{N}}\, \hat{a}(\vec{x})\, e^{-i\theta \hat{N}}$

where $\hat{N} = \int d^3y\; \hat{a}^+(y)\, \hat{a}(y)$

$$\Rightarrow [\hat{N}, \hat{a}(\vec{x})] = -\, \hat{a}(\vec{x})$$

$$[\hat{N}, \hat{a}^+(\vec{x})] = +\, \hat{a}^+(\vec{x})$$

$$e^{i\theta \hat{N}}\, \hat{a}(x)\, e^{-i\theta \hat{N}} \equiv \hat{a}_\theta(\vec{x})$$

$$\frac{d\hat{a}_\theta(x)}{d\theta} = i\,[\hat{N}, \hat{a}_\theta(\vec{x})] = -\, i\, \hat{a}_\theta(\vec{x})$$

$$\Rightarrow \quad \hat{a}_\theta(\vec{x}) = e^{-i\theta}\, \hat{a}(\vec{x})$$

$$\Rightarrow \quad e^{i\theta \hat{N}}\, \hat{a}(\vec{x})\, e^{-i\theta \hat{N}} = e^{-i\theta}\, \hat{a}(\vec{x})$$

$\Rightarrow$ $\hat{N}$ generates (infinitesimal) shifts of the phase of $\hat{a}(\vec{x})$. This is a symmetry transformation. Moreover, to say that

$\langle \hat{a}(\vec{x}) \rangle \neq 0 \Rightarrow |G\rangle$ may not be an eigenstate of $\hat{N}$! . This is what one means by spontaneous symmetry breaking: the ground state is not invariant under the symmetries of $\hat{H}$. However this is only possible in the thermodynamic limit. It also implies that the ground state is degenerate since if $|G\rangle$ is a ground state $\Rightarrow e^{i\theta\hat{N}}|G\rangle$ is just as good as a ground state.

However to make these arguments fully consistent we need to consider a non-ideal Bose gas, i.e. interactions.

(L37) The weakly Interacting Bose Gas

Consider the following highly simplified model of an interacting Bose gas

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

$$\hat{H}_0 = \sum_{\vec{p}} \varepsilon_0(\vec{p})\, \hat{a}^+(\vec{p})\, \hat{a}(\vec{p})$$

and

$$\hat{H}_{int} = \frac{U}{2V} \sum_{\vec{p}_1+\vec{p}_2=\vec{p}_3+\vec{p}_4} \hat{a}^+(\vec{p}_4)\, \hat{a}^+(\vec{p}_3)\, \hat{a}(\vec{p}_2)\, \hat{a}(\vec{p}_1)$$

where U is a coupling constant and it is related to the s-wave scattering amplitude (length) a between a pair of bosons by

$$\boxed{U = \frac{4\pi a}{m}}$$

This formula should be correct at very long distances. It also holds for hard spheres.

At $T=0$ we expect that, at least if U is small enough, there will be a "macroscopic occupancy of the $\vec{p}=0$ single particle state" (i.e. a B.E.C.) $\Rightarrow$ In this limit it is as if

$\hat{a}(\vec{p}=0) \equiv \hat{a}_0$ had a c-number piece

$$\hat{a}_0 \sim \sqrt{\langle n_0 \rangle} \gg 1$$

Let us separate in $\hat{H}$ the terms that contain $\hat{a}_0$ from those that do not. Using the fact that

$$N = n_0 + \frac{1}{2} \sum_{\vec{p} \neq 0} \left(\hat{a}^+(\vec{p})\, \hat{a}(\vec{p}) + \hat{a}^+(-\vec{p})\, \hat{a}(-\vec{p}) \right)$$

we can write

$$H = \frac{UN^2}{2V} + \frac{1}{2} \sum_{\vec{p} \neq 0} \left(\frac{p^2}{2m} + \frac{UN}{V} \right) \left(\hat{a}^+(\vec{p})\, \hat{a}(\vec{p}) + \hat{a}^+(-\vec{p})\, a(-\vec{p}) \right)$$

$$+ \text{ quartic terms in } \hat{a}(\vec{p})$$

The terms quartic in $\hat{a}(\vec{p})$ are neither $O(N)$ or $O(1)$ but are small $\sim \frac{1}{N} \Rightarrow$ if $\frac{N}{V}$ is small these terms should give a small contribution (this is true only to some extent; they are very important for $T \sim T_c$).

This effective (Bogoliubov) Hamiltonian is hermitian (better be!) but it does not conserve the # of particles since they can go in and of the condensate.

Let us diagonalize $\hat{H}$ by a Bogoliubov-Valatin (BV) transformation

$$\hat{a}(\vec{p}) = \frac{1}{\sqrt{1-A^2(\vec{p})}} \left(\hat{\alpha}(\vec{p}) + A(\vec{p})\, \hat{\alpha}^+(-\vec{p}) \right)$$

$$\hat{a}^+(\vec{p}) = \frac{1}{\sqrt{1-A^2(\vec{p})}} \left(\hat{\alpha}^+(\vec{p}) + A(\vec{p})\, \hat{\alpha}(-\vec{p}) \right)$$

where the amplitude $A(\vec{p})$ is chosen to be real. We will choose $A(\vec{p})$ so that all off-diagonal terms, i.e. terms $\sim \hat{\alpha}^+(\vec{p})\hat{\alpha}^+(-\vec{p})$, have vanishing coefficients.

The B-V transformation is canonical, i.e.

$$\begin{aligned} [\hat{a}(\vec{p}), \hat{a}(\vec{p}\,')] &= 0 \\ [\hat{a}(\vec{p}), \hat{a}^+(\vec{p}\,')] &= \delta_{\vec{p},\vec{p}\,'} \end{aligned} \iff \begin{aligned} [\hat{\alpha}(\vec{p}), \hat{\alpha}(\vec{p}\,')] &= 0 \\ [\hat{\alpha}(\vec{p}), \hat{\alpha}^+(\vec{p}\,')] &= \delta_{\vec{p},\vec{p}\,'} \end{aligned}$$

We find:

$$H = \left\{ \frac{UN^2}{2V} + \sum_{\vec{p}\neq 0} \frac{1}{1-A^2(\vec{p})} \left[\left(\frac{p^2}{2m} + \frac{UN}{V} \right) A^2(\vec{p}) + \frac{UN}{V} A(\vec{p}) \right] \right\}$$

$$+ \frac{1}{2} \sum_{\vec{p}\neq 0} \frac{1}{1-A^2(\vec{p})} \left[\left(\frac{p^2}{2m} + \frac{UN}{V} \right) (1 + A^2(\vec{p})) + 2\frac{UN}{V} A(\vec{p}) \right] \underbrace{}_{(\alpha^+(\vec{p})\alpha(\vec{p}) + \vec{p} \leftrightarrow -\vec{p})} +$$

$$+ \frac{1}{2} \sum_{\vec{p} \neq 0} \frac{1}{1 - A^2(\vec{p})} \left[\left(\frac{p^2}{2m} + \frac{UN}{V} \right) 2A(\vec{p}) + \frac{UN}{V} \left(1 + A^2(\vec{p}) \right) \right]$$

$$\left(\hat{\alpha}^{+}(\vec{p})\, \hat{\alpha}^{+}(-\vec{p}) + \hat{\alpha}(\vec{p})\, \hat{\alpha}(-\vec{p}) \right)$$

Choose:

$$\left(\frac{p^2}{2m} + \frac{UN}{V} \right) 2A(\vec{p}) + \frac{UN}{V} \left(1 + A^2(\vec{p}) \right) = 0$$

$$\Rightarrow \quad A(\vec{p}) = \frac{V}{UN} \left[- \frac{p^2}{2m} - \frac{UN}{V} + \sqrt{\left(\frac{p^2}{2m} + \frac{UN}{V} \right)^2 - \left(\frac{UN}{V} \right)^2} \right]$$

needed for a positive definite Hamiltonian.

$\Rightarrow$

$$H = E_0 + \frac{1}{2} \sum_{\vec{p} \neq 0} \varepsilon(\vec{p}) \left(\hat{\alpha}^{+}(\vec{p})\, \hat{\alpha}(\vec{p}) + \hat{\alpha}^{+}(-\vec{p})\, \hat{\alpha}(-\vec{p}) \right)$$

where

$$E_0 = \frac{UN^2}{2V} - \frac{1}{2} \sum_{\vec{p} \neq 0} \left[\left(\frac{p^2}{2m} + \frac{UN}{V} \right) - \sqrt{\left(\frac{p^2}{2m} + \frac{UN}{V} \right)^2 - \left(\frac{UN}{V} \right)^2} \right]$$

is the ground state energy and ~~and~~

$$\varepsilon(\vec{p}) = \sqrt{\left(\frac{p^2}{2m} + \frac{UN}{V} \right)^2 - \left(\frac{UN}{V} \right)^2}$$

is the dispersion, which has the following limiting behaviors

(a) For $|p| \to 0$ $\quad \varepsilon(\vec{p}) \approx \sqrt{\dfrac{U}{m v}}\, |\vec{p}| = \left(\dfrac{4\pi a}{v}\right)^{1/2} \dfrac{|\vec{p}|}{m}$

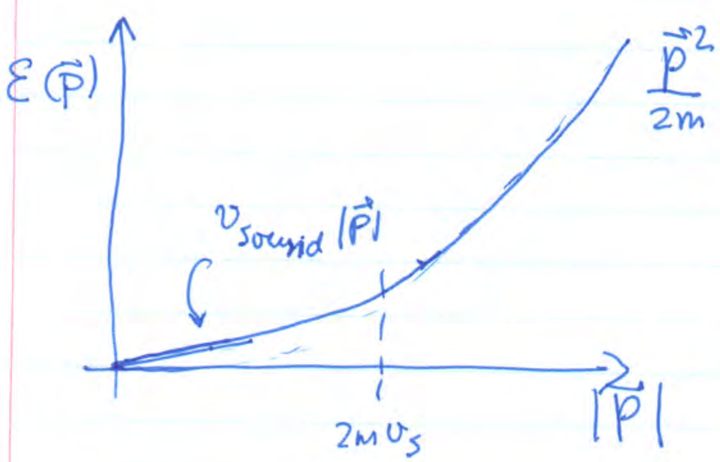
$$v \equiv \frac{V}{N}$$

i.e. it is a sound wave(!) with $v_{sound} = \frac{1}{m}\left(\frac{4\pi a}{v}\right)^{1/2}$

(b) For $|p|$ large $\left(|p| \gg \left(\dfrac{8\pi a}{v}\right)^{1/2} \right)$

$$\mathcal{E}(\vec{p}) \approx \frac{p^2}{2m}$$

i.e. it behaves as a free particle.



Finally there are two subtleties:

(1) The expression for E_0, the ground state energy, contains momentum integrals which diverge for large $|p|$. This ultra-violet divergence is due to the fact that the Bogoliubov model

is correct only at long distances, i.e. momenta small compared with the characteristic (length) scale of the interaction between the particles. In fact the relation between the coupling constant U and the scattering length is valid only at the level of the Born approximation (i.e. first order perturbation theory) To second order in perturbation theory in which two particles from the condensate are scattered into a pair with momenta $\vec{p}$ and $-\vec{p}$ the relation is corrected to

$$\frac{4\pi a}{m} = U - \frac{U^2}{V} \sum_{p\neq 0} \frac{1}{\vec{p}^2/m} + \cdots$$

which has an formal singularity at $\vec{p}$-large, we now write $U = U(a)$ in the formula for E_0, we get that

$$E_0 = \frac{2\pi a}{m} \frac{N^2}{V} + \frac{8\pi^2 a^2}{m^2}\left(\frac{N}{V}\right)^2 \sum_{p\neq 0} \frac{1}{\vec{p}^2/m} +$$

$$- \frac{1}{2}\sum_{\vec{p}\neq 0} \left(\frac{p^2}{\ldots} + \frac{4\pi a N}{\ldots}\right)\left(1-\sqrt{1-\left[\frac{4\pi a N/mV}{\ldots}\right]^2}\right)$$

which is UV finite. (see AGD)

$$\frac{E}{V} = \frac{2\pi a}{m}\left(\frac{N}{V}\right)^2\left[1 + \frac{128}{15\sqrt{\pi}}\, a^{3/2}\left(\frac{N}{V}\right)^{1/2}\right]$$

i.e. this is an expansion in powers of $\left[a\left(\frac{N}{V}\right)^{1/3}\right]^{3/2}$

Note: $v_s = \sqrt{\frac{V^2}{mN}\frac{\partial^2 E}{\partial V^2}} = \frac{\sqrt{4\pi a N/V}}{m} \equiv \left|\frac{\partial \varepsilon}{\partial \vec{p}}\right|_{p\to 0}$

<u>Superfluid fraction:</u>

The momentum distribution now is $(T \leq T_c)$

$$\langle n(\vec{p})\rangle = \langle \hat{\alpha}^{+}(\vec{p})\,\hat{\alpha}(\vec{p})\rangle = \frac{1}{e^{\beta\varepsilon(\vec{p})}-1}$$

However $\langle \hat{N}(\vec{p})\rangle = \langle \hat{a}^{+}(\vec{p})\,\hat{a}(\vec{p})\rangle =$

$$= \frac{\langle \hat{n}(\vec{p})\rangle + A^2(\vec{p})\left[\langle \hat{n}(\vec{p})\rangle + 1\right]}{1 - A^2(\vec{p})} \qquad (\vec{p}\neq 0)$$

$$N_0 = N - \sum_{\vec{p}\neq 0}\langle \hat{N}(\vec{p})\rangle$$

$\Rightarrow$ At $T=0$, $\langle \hat{n}(\vec{p})\rangle = 0$

$$\Rightarrow \langle \hat{N}(\vec{p})\rangle = \frac{A^2(\vec{p})}{1 - A^2(\vec{p})} = \frac{\frac{8\pi^2 a^2}{m^2}\left(\frac{N}{V}\right)^2}{\varepsilon(\vec{p})\left[\varepsilon(\vec{p}) + \frac{p^2}{2m} + \frac{4\pi a N}{mV}\right]}$$

Note: $\langle \hat{N}(\vec{p}) \rangle \neq 0$ for $\vec{p} \neq 0$ but

$$\lim_{a \to 0} \langle \hat{N}(\vec{p}) \rangle = 0$$

$$\Rightarrow \frac{N_0}{N} = -\frac{V}{N} \int \frac{d^3p}{(2\pi)^3} \langle \hat{N}(\vec{p}) \rangle$$

$$\Rightarrow \frac{N_0}{N} = 1 - \frac{8}{3\sqrt{\pi}} a^{3/2} \left(\frac{N}{V}\right)^{1/2} \qquad \text{(AGD)}$$

Hence, even at $T = 0$ $\quad \frac{N_0}{N} \neq 1$ for $U \neq 0$