Random Walks

Consider an array of points in space, a lattice.

Consider now a "particle" that moves on this lattice (for simplicity we will assume that it is a cubic lattice) but that its motion is completely random: at a given time the walker is at site \( \vec{r} \) and it randomly chooses to hop to one of its nearest-neighbouring sites \( \{ \vec{r} \pm \vec{e}_\mu a \} \) (\( \mu = 1, 2, 3 \)).

Since the probability that it will hop somewhere is \( 1/2^d \) and there are \( 2^d \) neighbors on a cubic lattice (\( d = 3 \)), the probability of any one of these hops is \( \frac{1}{2^d} = \frac{1}{8} \).

Suppose the random walker begins its journey at \( \vec{r} = 0 \), what is the probability that it will arrive at a given site \( \vec{r} \) in \( N \)
hops? If a hop takes a time $\tau \Rightarrow$ after $N$ hops it takes $t = N\tau$

Total displacement: $\vec{R} = \sum_{i=1}^{N} \vec{d}_i$

where $\vec{d}_i$ are the displacements at every intermediate time. These displacements are taken randomly from the set $\{\pm \hat{\epsilon}_\mu a\}$ with equal probability $\frac{1}{2\hat{d}}$.

$\Rightarrow \quad \langle \vec{R} \rangle = \sum_{i=1}^{N} \langle \vec{d}_i \rangle = 0$

But

$\langle \vec{R}^2 \rangle = \langle (\sum_{i=1}^{N} \vec{d}_i)^2 \rangle =$

$= \sum_{i,j=1}^{N} \langle \vec{d}_i \cdot \vec{d}_j \rangle$

$= \sum_{i,j=1}^{N} \delta_{ij} a^2 = Na^2$

since $\langle \vec{d}_i^2 \rangle = \langle \vec{d}_i \cdot \vec{d}_j \rangle = \langle \vec{d}_i \rangle \cdot \langle \vec{d}_j \rangle = 0$

$\Rightarrow$ (uncorrelated and zero average)

and

$\langle \vec{d}_i^2 \rangle = \frac{\sum \vec{d}_i^2 P[\vec{d}_i]}{\sum P[\vec{d}_i]} = \frac{3\hat{d} a^2}{3\hat{d} \cdot \frac{1}{2\hat{d}}} = a^2$
\[ \langle \vec{R}^2 \rangle = N a^2 \]

or \[ \sqrt{\langle \vec{R}^2 \rangle} = a \sqrt{N} \]

which is very slow.

Let \( s(\vec{r}, t / a) \) be the probability to find the walker at \( \vec{r} \) after a step which was completed at time \( t \).

\[ s_L(\vec{r} / a, t / c) = \langle \delta(\vec{R}(t), \vec{r}) \rangle \]

"lattice"

Consider now a region \( \Omega \) centered at \( \vec{r}_0 \) but containing many sites (still small compared to the size of the system).

\[ \int d\vec{r} s(\vec{r}, t) = \sum_{\vec{r} \in \Omega} s_L(\vec{r} / a, t / c) \]

Since a site (or unit cell) has volume \( a^D \)

(D: dimension) \[ s(\vec{r}, t) = \frac{1}{a^D} s_L(\vec{r} / a, t / c) \]

This equation will help us take a continuum limit.
If the walk begins at \( \mathbf{R} = 0 \) at \( t = 0 \)

\[ S_L \left( \frac{\mathbf{r}}{a}, 0 \right) = \delta_{\mathbf{r}, 0} \quad \& \quad \text{(Kronecker delta)} \]

\[ S_L \left( \frac{\mathbf{r}}{a}, \frac{t}{c} \right) = \left\langle \delta \mathbf{R}(t), \mathbf{\hat{r}} \right\rangle = \]

\[ = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \left\langle \frac{e^{i \mathbf{q} \cdot (\mathbf{R}(t) - \mathbf{R})}}{a} \right\rangle \]

\[ = \frac{1}{2m} \text{ Brillouin Zone} \]

\[ = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \left\langle \prod_{i=1}^{N} e^{i \mathbf{q} \cdot \mathbf{d}_i / a} \right\rangle \]

\[ \mathbf{R}(t) = \sum_{i=1}^{N} \mathbf{d}_i \]

\[ t = N \tau \]

Since the \( \mathbf{d}_i \)'s are uncorrelated

\[ \left\langle \prod_{i=1}^{N} e^{i \mathbf{q} \cdot \mathbf{d}_i / a} \right\rangle = \prod_{i=1}^{N} \left\langle e^{i \mathbf{q} \cdot \mathbf{d}_i / a} \right\rangle \]

\[ = \left\langle e^{i \mathbf{q} \cdot \mathbf{d}_i / a} \right\rangle^N \]

any allowed displacement \( \mathbf{d} \)

\[ \left\langle e^{i \mathbf{q} \cdot \mathbf{d}/a} \right\rangle = \sum_{\mu=1}^{2\mathbf{d}} \frac{1}{2\mathbf{d}} \left( e^{i \varphi_\mu} + e^{-i \varphi_\mu} \right) = \frac{1}{2} \sum_{\mu=1}^{2\mathbf{d}} \cos \varphi_\mu \]
\[ \langle \prod_{i=1}^{N} e^{-\frac{\mathbf{q}_i \cdot \mathbf{a}_i}{\alpha}} \rangle = \left( \frac{d}{\alpha} \sum_{\mu=1}^{\infty} \cos \varphi_\mu \right)^N \]

\[ L \left( \frac{\mathbf{r}}{\alpha}, \frac{\mathbf{t}}{\alpha} \right) = \int e^{i \frac{\mathbf{q}}{\alpha} \cdot \mathbf{r}} a^{d} \left( \frac{d}{\alpha} \sum_{\mu=1}^{\infty} \cos \varphi_\mu \right)^N \]

For \( N \) large enough this integral can be estimated by the method of steepest descents, which effectively amounts to the approximation:

\[ \left( \frac{d}{\alpha} \sum_{\mu=1}^{\infty} \cos \varphi_\mu \right)^N \approx e^{-\frac{\mathbf{q}^2 N}{2d}} \]

and

\[ \frac{1}{d} L \left( \frac{\mathbf{r}}{\alpha}, \frac{\mathbf{t}}{\alpha} \right) = \int e^{i \frac{\mathbf{q}}{\alpha} \cdot \mathbf{r}} \left( \frac{d}{\alpha} \sum_{\mu=1}^{\infty} \cos \varphi_\mu \right)^N e^{-\frac{\mathbf{q}^2 N}{2d}} \]

all space

\[ \frac{b}{c} = N \]

\[ S(\mathbf{r}, t) = \left( \frac{2\pi}{\alpha} \right)^{d/2} e^{-\frac{\mathbf{r}^2}{4Dt}} = e^{-\frac{\mathbf{r}^2}{(4\pi Dt)^d}} \]

\[ D = \frac{a^2}{2\pi d} = \text{Diffusion constant} \]
Hence for long time $t = N \Delta t \gg 1$

$$p(r, t) = \frac{e^{-\frac{r^2}{4Dt}}}{(4\pi Dt)^{d/2}}$$

is the probability we were looking for.

Note: $\langle r^2 \rangle = 2Dt d$

or $\langle \frac{r^2}{a^2} \rangle = \frac{2Dt}{a^2} = \frac{2}{a^2} \times \frac{N\Delta t}{\xi}$

$= \frac{N}{a^2}$

$\implies \langle r^2 \rangle = Na^2$

Equation of Motion for $p_L \left( \frac{r}{a}, \frac{t}{\xi} \right)$:

What is the probability to find the walker at $\vec{r}$ after $N$ steps? = equal to the probability to find the walker in any of its neighbors in $(N-1)$ steps $\times$ probability of the last step.

$\implies p_L \left( \frac{r}{a}, \frac{t}{\xi} \right) = \frac{1}{2a} \sum_{\xi} p_L \left( \frac{r}{a} + \xi \vec{a}, \frac{t}{\xi} - 1 \right)$
\( \therefore S(\vec{r}, t + \tau) - S(\vec{r}, t) = \sum_{\vec{r}' \in \text{neighbors}} \frac{S(\vec{r}' - \vec{r}, t) - S(\vec{r}', t)}{2d} \)

\( \therefore s(\vec{r}, t + \tau) - s(\vec{r}, t) \approx \tau \frac{\partial s(\vec{r}, t)}{\partial t} \)

\( s(\vec{r} - \vec{x}, t) - s(\vec{r}, t) = -\vec{x} \cdot \nabla s + \frac{1}{2} (\vec{x} \cdot \nabla) s \)

\( \sum_{i} \xi_i = 0 \quad \text{and} \quad \sum_{i \neq j} \xi_i \xi_j = \delta_{ij} \alpha^2 \)

\( \therefore \tau \frac{\partial s}{\partial t} = \frac{a^2}{2d} \nabla^2 s \)

\( \therefore \frac{\partial s}{\partial t} = \frac{a^2}{2zd} \nabla^2 s \quad \iff \quad \frac{\partial s}{\partial t} = DD^2 s \)

\( D = \frac{a^2}{2zd} \)

Diffusion Equation.
Brownian Motion and Random Walks

Consider a particle of mass M moving in a gas. We will assume that the gas is in equilibrium at some temperature \( T \). The interactions between the particle and the atoms are random: random force, \( \vec{\tau}(t) \).

Hamilton’s Equations:

\[
\frac{d\vec{R}}{dt} = \frac{\vec{P}}{M} = \frac{\partial H}{\partial \vec{P}}
\]

\[
\frac{d\vec{P}}{dt} = -\frac{\partial H}{\partial \vec{R}} + \vec{\tau}(t) + -\frac{\vec{P}}{\tau} \quad \text{“relaxation time”}
\]

The random force \( \vec{\tau}(t) \) will be assumed to be uncorrelated (i.e., a Markov process)

\[
\langle \tau_\mu(t) \tau_\nu(t') \rangle = \Gamma \delta_{\mu\nu} \delta(t-t')
\]

i.e., very short kibbles
Impulse: \( \mathbf{I} = \int_{t}^{t+\delta t} \mathbf{E}(t') \, dt' \)

\[
\implies \langle I_{\mu} I_{\nu} \rangle = \int_{t}^{t+\delta t} \int_{t}^{t+\delta t} \langle \eta_{\mu}(t') \eta_{\nu}(t'') \rangle
\]

\[
= \Gamma \int_{t}^{t+\delta t} \int_{t}^{t+\delta t} \delta_{\mu\nu} \delta t' \delta t''
\]

\[
\langle I_{\mu} I_{\nu} \rangle = \Gamma (\delta t) \delta_{\mu\nu}
\]

and \( \langle I_{\mu} \rangle = 0 \)

\( \Rightarrow \) random walk in momentum space

\( \Rightarrow \) \( I_{\mu} \) are gaussian variables

\( \Rightarrow \) \( \langle e^{i \mathbf{q} \cdot \mathbf{I}} \rangle = e^{-\frac{1}{2} \Gamma \delta t \mathbf{q}^2} \)

An equation s.t.

\[
\frac{d \mathbf{P}}{dt} = -\frac{\mathbf{P}}{\tau} + \eta(t)
\]

is a Langevin Equation

Consider the case \( H = 0 \) (no conservative force)

\( \Rightarrow \) \( \mathbf{P}(t) = \mathbf{P}(t_0) e^{-\frac{t-t_0}{\tau}} + \int_{t_0}^{t} dt' \eta(t') e^{-\frac{t-t'}{\tau}} \)
\[ \langle \tilde{P}(t) \rangle = \tilde{P}(0) e^{-t/\tau} \]

and \( \langle e^{\tilde{q} \cdot \tilde{P}(t)} \rangle = ? \)

\[ (t \to -\infty \Rightarrow \tilde{P}(t) = \int_{-\infty}^{t} dt' \eta(t') e^{-\frac{t-t'}{\tau}} ) \]

\[ \left\langle e^{\tilde{q} \mu \int_{-\infty}^{t} dt' \eta_{\mu}(t')} e^{-\frac{(t-t')}{\tau}} \right\rangle = \]

\[ = e^{-\frac{1}{2} \tilde{q} \mu \tilde{q} \nu \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \left\langle \eta_{\mu}(t') \eta_{\nu}(t'') \right\rangle} \]

\[ \quad \times e^{-\frac{(t-t')}{\tau}} e^{-\frac{(t-t'')}{\tau}} \]

\[ = e^{-\frac{1}{2} \tilde{q} \mu \tilde{q} \nu \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \delta_{\mu \nu} \delta(t' - t'')} \]

\[ = e^{-\frac{t}{2} \tilde{q}^2 \int_{-\infty}^{t} dt' e^{-\frac{2}{\tau} (t-t')}} \]

\[ = e^{-\frac{t \tilde{q}^2}{4}} \quad \text{which is also Gaussian.} \]
If $\bar{P}$ when to satisfy a Maxwell-Boltzmann distribution at temperature $T \Rightarrow$

$$f(\bar{P}) = \frac{1}{(2\pi M_k T)^{d/2}} e^{-\frac{\bar{P}^2}{2M_k T}}$$

$$\Rightarrow \langle e^{i\hat{q} \cdot \bar{P}} \rangle = e^{-\frac{\hat{q}^2}{2} M_k T}$$

$$\Rightarrow \text{both Gaussians agree at}$$

$$M_k T = \frac{\Gamma \sqrt{2}}{2} \quad \text{or} \quad \Gamma \tau = 2 M_k T$$

$$\uparrow \quad \uparrow \quad \uparrow$$

dissipation  
fluctuation

$$\Rightarrow \text{the rate at which momentum is added to}$$

the particle ($\Gamma$) is determined by $kT$ and by $\tau$.

(Fluctuation-Dissipation Theorem.)
Let \( f(\vec{r}, \vec{p}, t) = \langle \sum_{j=1}^{N} \delta(\vec{P}_j(t) - \vec{p}) \delta(\vec{r}_j(t) - \vec{r}) \rangle \) be the particle distribution function.

\[
\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m} = \frac{\partial H}{\partial \vec{p}}
\]

\[
\frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} + \vec{\Omega}(t) - \frac{\vec{F}}{m}
\]

\[
H = \frac{\vec{p}^2}{2m} + U(\vec{r})
\]

Let \( \Theta(\vec{r}, \vec{p}) \) be any observable.

\[
\langle \Theta(\vec{r}, \vec{p}) \rangle(t) = \int d\vec{r} d\vec{p} \Theta(\vec{r}, \vec{p}) f(\vec{r}, \vec{p}, t)
\]

We know that

\[
\frac{df}{dt} + \{H, f\} = \left( \frac{df}{dt} \right)_{\text{coll}}
\]

What expression do we get for \( \frac{df}{dt} \) coll?
Over some interval $\delta t$

$$(\delta \mathbf{p})_{\text{collision}} = -\mathbf{p} \frac{\delta t}{c} + \mathbf{I}$$

$$\mathbf{I} = \int_t^{t+\delta t} dt' \mathbf{\mathcal{I}}(t')$$

$$\delta \langle O(t) \rangle = \int d\mathbf{r} d\mathbf{p} \left[ O(\mathbf{r}, \mathbf{p}) \delta t \frac{df(r,p,t)}{dt} \right]_{\text{all}}$$

$$\delta \langle O \rangle(t) = \langle \int d\mathbf{r} d\mathbf{p} \left[ O(\mathbf{r}, \mathbf{p}+\delta \mathbf{p}) - O(\mathbf{r}, \mathbf{p}) \right] \rangle$$

$$= \int d\mathbf{r} d\mathbf{p} f(\mathbf{r}, \mathbf{p}, t) \left[ \langle O(\mathbf{r}, -\mathbf{p} \frac{\delta t}{c} + \mathbf{I}, \mathbf{p}) \rangle - \langle O(\mathbf{r}, \mathbf{p}) \rangle \right]$$

$$O(p, r) = \int d\mathbf{q} \ e^{i \mathbf{p} \cdot \mathbf{q}} O(\mathbf{q}, \mathbf{r})$$

$$\langle O(\mathbf{p} - \mathbf{p} \frac{\delta t}{c} + \mathbf{I}, \mathbf{r}) \rangle =$$

$$= \int d\mathbf{q} \ \tilde{O}(\mathbf{q}, \mathbf{r}) \langle e^{i \mathbf{p} \cdot \mathbf{q} - i \mathbf{p} \cdot \mathbf{p} \frac{\delta t}{c} + c \frac{\mathbf{I} - \mathbf{q}}{c} \rangle$$

$$= e^{i \mathbf{p} \cdot \mathbf{q}} e^{-\mathbf{p} \cdot \mathbf{p} \frac{\delta t}{c}} e^{-\frac{i}{2} \mathbf{I} \delta t \cdot \mathbf{q}^2}$$
\[
\langle 0 (\vec{p} - \vec{\hat{p}} \frac{d\tau}{c} + \vec{z}, \vec{r}) \rangle - O (\vec{p}, \vec{r}) = \\
= \int d\vec{q} \ (e^{i \vec{p} \cdot \vec{q} / \hbar} - e^{-i \vec{\hat{p}} \cdot \vec{q} / \hbar} - e^{i \vec{p} \cdot \vec{q} / \hbar} - e^{-i \vec{\hat{p}} \cdot \vec{q} / \hbar}) \tilde{O} (\vec{q}, \vec{r}) \\
= \int d\vec{q} \ e^{i \vec{\hat{p}} \cdot \vec{q}} (e^{-i \vec{p} \cdot \vec{q} / \hbar} - \frac{\hbar \dot{p} \vec{q}^2}{2} - 1) \tilde{O} (\vec{q}, \vec{r}) \\
= \int d\vec{q} \ e^{i \vec{\hat{p}} \cdot \vec{q}} \left[ - e^{-i \vec{p} \cdot \vec{q} / \hbar} - \frac{\hbar \dot{p} \vec{q}^2}{2} - \cdots \right] \tilde{O} (\vec{q}, \vec{r}) \\
\vec{p} \cdot \vec{\nabla} \tilde{O} (\vec{p}, \vec{r}) = \vec{p} \cdot \vec{\nabla} \int d\vec{q} \ e^{i \vec{\hat{p}} \cdot \vec{q}} \tilde{O} (\vec{q}, \vec{r}) \\
= \int d\vec{q} \ i \vec{p} \cdot \vec{q} \ e^{i \vec{\hat{p}} \cdot \vec{q}} \tilde{O} (\vec{q}, \vec{r}) \\
\nabla^2 \tilde{O} (\vec{p}, \vec{r}) = \int d\vec{q} \ (-\vec{q}^2) e^{i \vec{\hat{p}} \cdot \vec{q}} \tilde{O} (\vec{q}, \vec{r}) \\
\Rightarrow -\frac{\delta t}{c} \ \vec{\nabla} \tilde{O} (\vec{p}, \vec{r}) + \frac{\hbar \dot{p}}{2} \nabla^2 \tilde{O} (\vec{p}, \vec{r})
\[ \delta \langle 0 \rangle = \int d\mathbf{r} d\mathbf{p} \left[ -\frac{\delta t}{2} \mathbf{p} \cdot \nabla_{\mathbf{p}} f^0 + \frac{\delta t}{2} \mathbf{p} \cdot \nabla_{\mathbf{r}}^2 f^0 \right] \tilde{f}(\mathbf{r}, \mathbf{p}, t) \]

\[ = \int d\mathbf{r} d\mathbf{p} \quad \tilde{f}(\mathbf{r}, \mathbf{p}) \delta t \quad \mathbf{p} \cdot \nabla_{\mathbf{p}} \left\{ \left[ \frac{\mathbf{p}}{c^2} + \frac{\mathbf{p}}{2} \nabla_{\mathbf{p}} \right] \tilde{f}(\mathbf{r}, \mathbf{p}, t) \right\} \]

\[ \Rightarrow \left( \frac{\partial f}{\partial t} \right)_{\text{collisions}} = \mathbf{v}_{\mathbf{p}} \cdot \left\{ \left[ \frac{\mathbf{p}}{c^2} + \frac{\mathbf{p}}{2} \nabla_{\mathbf{p}} \right] f(\mathbf{r}, \mathbf{p}, t) \right\} \]

\[ \Rightarrow \text{Fokker-Planck Equation:} \]

\[ \frac{\partial f}{\partial t} + \{ H, f \} = \mathbf{v}_{\mathbf{p}} \cdot \left\{ \left[ \frac{\mathbf{p}}{c^2} + \frac{\mathbf{p}}{2} \nabla_{\mathbf{p}} \right] f(\mathbf{r}, \mathbf{p}, t) \right\} \]

Equilibrium Solution: \[ \frac{\partial f}{\partial t} = 0 \quad \text{and} \quad f = \bar{f}(H) \]

\[ \Rightarrow \{ H, f \} = 0 \quad \text{but} \]

r.h.s \#0 unless

\[ \mathbf{v}_{\mathbf{p}} \cdot \left\{ \left[ \frac{\mathbf{p}}{c^2} + \frac{\mathbf{p}}{2} \nabla_{\mathbf{p}} \right] \bar{f} [H] \right\} = 0 \]

since \[ \frac{\mathbf{p}}{m} = \frac{\partial H}{\partial \mathbf{p}} \Rightarrow \]
\[ \nabla_p \cdot \left\{ \left[ \frac{M}{c} \left( \nabla_p H \right) + \frac{P}{c} \right] f \right\} = 0 \]

\[ \text{if } \frac{M}{c} \left( \nabla_p H \right) f + \frac{P}{c} \nabla_p f = 0 \]

\[ \Rightarrow \frac{M}{c} \nabla_p H = -\frac{P}{c} \nabla_p \log f \]

\[ \nabla_p \left[ \frac{M}{c} H + \frac{P}{c} \log f \right] = 0 \]

\[ \Rightarrow \frac{M}{c} H + \frac{P}{c} \log f = \text{const.} \]

\[ \log f = \text{const.} + \frac{2M}{Pc} H \]

\[ f = \text{const.} \times e^{-\frac{2M}{Pc} H} \]

\[ \Rightarrow \text{Gibbs } \frac{P \tau}{2M} = kT \]

**Fokker-Planck:**

\[ \frac{df}{dt} + \{H, f\} = \nabla_p \cdot \left\{ \left[ \frac{MkT}{\tau} \nabla_p + \frac{P}{\tau} \right] f \right\} \]
General Langevin Processes

Consider a system described by some coordinate \( q_i(t) \) subject to the action of dissipative forces and random external forces. We will assume that the system is overdamped and write

\[
\frac{d\dot{q}_i(t)}{dt} = -\frac{\xi_i}{\mu} \dot{q}_i(t) + f_i(t) + \eta_i(t)
\]

where \( f_i \) are external forces and \( \eta_i(t) \) are random forces. The random forces obey

(\( \dot{\eta}_i(t) \): friction constant)

\[
\langle \eta_i(t) \rangle = 0
\]

\[
\langle \eta_i(t) \eta_j(t') \rangle = \Gamma \delta(t-t') \delta_{ij}
\]

i.e. are Markov processes ("no memory")

Hence the probability of a particular random process \( \eta_i(t) \) is

\[
dP(\eta) = \Theta_n e^{-\frac{1}{2\Gamma} \int_0\!\! dt \dot{\eta}_i(t)^2} \tag{measure}
\]
What is the Fokker-Planck Equation associated with this problem?

\[ P[\bar{\theta}, t] = \left\langle \prod_{i=1}^{d} \delta(\bar{\theta}_i(t) - \bar{\theta}_i) \right\rangle \]

The same line of reasoning are used before leads to the equation

\[ \frac{\partial P}{\partial t} = \frac{2}{\bar{\theta}} \left[ \sum_{\bar{\theta}} \frac{\partial P}{\partial \bar{\theta}} \right] + \frac{f_i(\bar{\theta})}{\bar{\theta}} P \]

\[ \text{FP} \]

There is an interesting analogy with Quantum Mechanics. Let \( H \) be a QM Hamiltonian, and \( |\bar{\theta}\rangle \) be a final state and \( |\bar{\theta}'\rangle \) be an initial state \( \Rightarrow \)

\[ \left\langle \bar{\theta}' | e^{-tH} | \bar{\theta} \right\rangle = F(\bar{\theta}, \bar{\theta}' ; t) \]

In quantum mechanics, the quantity that is computed is

\[ \left\langle \bar{\theta}' | e^{it\hat{H}} | \bar{\theta} \right\rangle = F(\bar{\theta}, \bar{\theta}' ; t) \]

and it is known as the evolution operator \( \hat{U} \). \( F \) and \( \tilde{F} \) are related through the analytic continuation \( (\hbar \to 1) \; t \to it \) (i.e. "imaginary time")
It is easy to show that, if we define \( \hat{p} = -i \frac{\partial}{\partial \tilde{q}} \), the "Hamiltonian" is

\[
H = \frac{1}{2} \tilde{p}^2 + \frac{i}{\hbar} \tilde{p} \cdot \hat{f}[\tilde{q}]
\]

which is clearly not hermitian.

How do observables evolve in time?

Let \( A[\tilde{q}(t)] \) be an observable, then

\[
\langle A[\tilde{q}(t)] \rangle = \int d\tilde{q} \quad A[\tilde{q}] \quad \langle \tilde{q}' \mid e^{-iHt} \mid \tilde{q} \rangle
\]

where \( \tilde{q}' = \tilde{q}(0) \)

Let us define the "Hermanny representation" of \( A[\tilde{q}(t)] \)

\[
\hat{A}[\tilde{q}(t)] = \int d\tilde{q}' \quad A[\tilde{q}'] \quad \langle \tilde{q}' \mid e^{-iHt} \mid \tilde{q} \rangle
\]

\[
\Rightarrow \quad \frac{\partial A}{\partial t} = \left( \frac{i}{2} \frac{\partial}{\partial \tilde{q}} - \frac{1}{\hbar} \hat{f}[\tilde{q}] \right) \frac{\partial A}{\partial \tilde{q}} \quad \text{is the EQN of motion.}
\]

**Equilibrium:** \( P_0[\tilde{q}] = \lim_{t \to \infty} P[\tilde{q}(t)] \)

Since \( \hat{\sigma}_P = \tilde{q} \cdot \left[ \frac{\hbar}{2} \hat{\sigma}_P \frac{\partial}{\partial \tilde{q}} \hat{f}[\tilde{q}] \right] \)

\[
\Rightarrow \quad \int \hat{\sigma}_P = \int \frac{\hbar}{2} \hat{\sigma}_P \frac{\partial}{\partial \tilde{q}} \hat{f}[\tilde{q}]
\]

where \( \hat{f}[\tilde{q}] \)

\[
\Rightarrow \quad \int \hat{\sigma}_P = 1 \quad \text{is conserved.}
\]
$P_0$ must be a right eigenvalue of $H$
(recall that $H$ is not Hermitian $\Rightarrow$ right and left eigenvectors are $\neq$). In general, the right eigenvectors have a positive real part which guarantees the existence of a "ground state",
 i.e., an equilibrium distribution.

Furthermore, if the form $f_i(\bar{\varphi})$ are conservative, i.e., $f_i = - \frac{\partial}{\partial t} E(\bar{\varphi})$ for some "potential" $E(\bar{\varphi})$ \Rightarrow
\[
\frac{\partial \bar{\varphi}}{\partial t} = -\frac{i}{2\lambda} \bar{\varphi} \frac{\partial E(\bar{\varphi})}{\partial \bar{\varphi}} + \bar{\varphi}(t)
\]

Let us write $P(\bar{\varphi},t)$ in the form:
\[
P(\bar{\varphi},t) = e^{-\frac{i}{2\lambda t} E(\bar{\varphi})} \langle \bar{\varphi} \mid U(\bar{t},0) \mid \bar{\varphi}' \rangle e^{\frac{E(\bar{\varphi}')}{{2\lambda t}}}\\
\]
where $\bar{\varphi}' = \bar{\varphi}(0)$ [i.e., $P(\bar{\varphi},0) = \delta(\bar{\varphi} - \bar{\varphi}')$]

\[
\frac{\partial}{\partial \bar{t}} \langle \bar{\varphi} \mid U(\bar{t},0) \mid \bar{\varphi}' \rangle = -\frac{\lambda}{2} \left( -\frac{\partial^2}{\partial \bar{\varphi}^2} + \frac{1}{2\lambda t} \frac{\partial E(\bar{\varphi})}{\partial \bar{\varphi}} \right) \langle \bar{\varphi} \mid U(\bar{t},0) \mid \bar{\varphi}' \rangle
\]

where the new "Hamiltonian" $\mathring{H}$ is
\[ \tilde{H} = \frac{\hbar}{2} \tilde{p}^2 + \frac{\hbar}{8} \left( \partial^2 \tilde{E}(\tilde{\phi}) \right)^2 - \frac{\hbar}{4} \partial^2 \tilde{E}(\tilde{\phi}) \]

with \( \tilde{E}(\tilde{\phi}) = \frac{1}{\partial^2} E(\tilde{\phi}) \) and \( \tilde{p} = i \frac{\partial}{\partial \tilde{\phi}} \)

\( \tilde{H} \) is clearly Hermitean.

Note: \([\tilde{p}] = \frac{\hbar}{i} \) and \([\tilde{E}] = E / (\hbar^2 / 2) \Rightarrow [\tilde{r}] = E \)

The (normalizable) ground state wave function of \( \tilde{H} \) is \( \tilde{\psi}_0(\tilde{\phi}) = \# e^{-\frac{1}{2} \tilde{E}(\tilde{\phi})} \)

Its eigenvalue is zero

\( \tilde{H} \tilde{\psi}_0 = 0 \)

and in general

\[ \tilde{p}(\tilde{\phi}, t) = e^{-\frac{1}{2} \tilde{E}(\tilde{\phi})} \langle \tilde{\phi} | e^{-i \tilde{H}t} | \tilde{\phi} \rangle e^{\frac{1}{2} \tilde{E}(\tilde{\phi})} \]

Let \( |n\rangle \) the \( n \)-th eigenstate of \( \tilde{H} \) with eigenvalue \( \lambda_n \)

\( \tilde{H} |n\rangle = \lambda_n |n\rangle \)

Clearly \( \langle 0 | \tilde{\phi} \rangle = \tilde{\psi}_0(\tilde{\phi}) \) and \( \lambda_0 = 0 \)

Let \( \tilde{\psi}_n(\tilde{\phi}) = \langle \tilde{\phi} | n \rangle \)

\( \{ \tilde{\psi}_n \} \) are complete (and distinct)

\( \hat{1} = \Sigma_n |n\rangle \langle n| \) is the identity operator.

\( \Rightarrow \)
\[
\langle \tilde{\phi} | e^{-t \tilde{H}} | \tilde{\phi}' \rangle = \\
= \sum_{n=0}^{\infty} \langle \tilde{\phi} | n \rangle e^{-t \lambda_n} \langle n | \tilde{\phi}' \rangle \\
= \sum_{n=0}^{\infty} \psi_n(\tilde{\phi}) \psi_n^*(\tilde{\phi}') e^{-\lambda_n t}
\]

Since \( \tilde{H} \) is not only hermitian but also real and positive, \( \lambda_n \) are real (and positive), and \( \psi_n \) are also real.

\[
\Rightarrow P(\tilde{\phi}, t) = e^{-\frac{1}{2} \tilde{E}(\tilde{\phi})} \sum_{n=0}^{\infty} \psi_n(\tilde{\phi}) \psi_n^*(\tilde{\phi}') e^{-\lambda_n t} e^{\frac{1}{2} \tilde{E}(\tilde{\phi}')}
\]

\[
\Rightarrow P(\tilde{\phi}, t) = e^{-\frac{1}{2} \tilde{E}(\tilde{\phi})} \left[ \psi_0(\tilde{\phi}) \psi_0^*(\tilde{\phi}') + \psi_1(\tilde{\phi}) \psi_1^*(\tilde{\phi}') \right] e^{-\lambda_1 t} \ldots 
\]

But \( \psi_0(\tilde{\phi}) = e^{-\frac{1}{2} \tilde{E}(\tilde{\phi})} \) (up to a constant)

\[
\Rightarrow P(\tilde{\phi}, t) = e^{-\tilde{E}(\tilde{\phi})} + \psi_1(\tilde{\phi}) e^{-\frac{1}{2} \tilde{E}(\tilde{\phi})} \psi_1^*(\tilde{\phi}') e^{\frac{1}{2} \tilde{E}(\tilde{\phi}')} e^{-\lambda_1 t} 
\]

\[
\Rightarrow \lim_{t \to \infty} P(\tilde{\phi}, t) \leq P_0(\tilde{\phi}) = e^{-E(\tilde{\phi})} + \ldots 
\]

(This makes sense if \( \psi_0 \) is normalizable.)
Hence, asymptotically, \( P(\tilde{q},t) \) emerges to a Gibbs distribution with \( kT = \gamma T \). Notice that in general there is a spectrum of relaxation times \( \tau_n = \frac{1}{\lambda_n} \), where \( \lambda_n \) are the eigenvalues of \( \tilde{H} \). Thus, at long times, the correction to \( P_0(\tilde{q}) \) is exponentially small if there is a gap in the spectrum of \( \tilde{H} \), i.e. if \( \lambda > 0 \). If the spectrum is continuous, there is no normalizable solution and the time dependence changes. For instance, for a random walker, \( \tilde{f} = 0 \) and the asymptotic solution is \( P_0 = 0 \). In this case the time dependence was much more singular \( \sim t^{-\frac{1}{2}} e^{-\frac{1}{\gamma t}} \).

Note: For \( \tilde{f} = 0 \) (i.e. free particle) the Fokker-Planck equation becomes: \( \frac{\partial P}{\partial t} = \frac{\nabla^2}{2} P = \text{diffusion equation} \) with \( D = \frac{\nabla}{2} \).

\[
\Rightarrow P(\tilde{q},t) = (2\pi \gamma t)^{-d/2} \exp\left(-\frac{\tilde{q}^2}{2\gamma t}\right) \xrightarrow{t \to \infty} \delta(\tilde{q})
\]