

~~classical~~

Phase Transitions in Classical Statistical Mechanics

We now turn our attention to the problem of phase transitions in systems at finite temperature. Since it turns out that quantum effects only enter in the determination of the value of the critical temperature, something that is hard to compute with accuracy anyway, we will consider only classical equilibrium systems.

Although there is a wide variety of systems which undergo second order or continuous transitions we will consider only some representative cases. There are two reasons for doing this: (a) simplicity and (b) the fact that ~~the~~ many systems ~~that~~ ^{which} are very different microscopically have transitions that are quite similar. We will use transitions in magnetic systems (i.e. spin systems) as our prototype.

(1) The Ising Model

The Ising Model is a magnetic system in which a spin degree of freedom $S_z = \uparrow$ or \downarrow (± 1) resides at every site of a lattice. The energy of a configuration of spins $\{\sigma(\vec{r})\}$ is

$$\mathcal{H} = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \sigma(\vec{r}) \sigma(\vec{r}')$$

where $\sigma(\vec{r}) = \pm 1$ and J is the exchange constant, $[J] = \text{energy}$. For simplicity we consider only nearest-neighbor interactions.

This is a good model for, among other things, magnetic systems with uniaxial ^(“easy axis”) anisotropy.

In $J > 0$ this is a ferromagnet whereas for $J < 0$ it is an antiferromagnet.

(2) The XY Model

This is a model for magnetic systems with an “easy plane”. It turns out that is also a model for a superfluid and/or a superconductor.

transition. The "spins" are now two-component unit vectors \vec{S} , $\|\vec{S}\|=1$, $\vec{S} = (S_x, S_y)$

The energy (or Hamiltonian) has the same form

$$\mathcal{H} = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}')$$

③ Isotropic Heisenberg model

Now the spins have three components $\vec{S} = (S_x, S_y, S_z)$ and have $\|\vec{S}\|=1$. This is the " $O(3)$ "

ferromagnet with

$$\mathcal{H} = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}')$$

→ The order parameter is a three-component vector.

Sometimes an $O(N)$ model ~~is~~ is considered

where $\vec{S} = (S_1, \dots, S_N)$ with $\|\vec{S}\| = \sqrt{\sum_{i=1}^N S_i^2} = 1$

The XY model is then the $O(2)$ model and

the Ising Model is the $O(1)$ model. (We will

discuss this notation in a moment).

④ Liquid Crystals: A model of ~~the~~^a liquid

crystal begins with the fact that the molecules look like rigid rods, i.e. like headless vectors.

Hence there is no observable change if a rod is rotated by 180° , i.e. $\vec{S} \equiv -\vec{S}$

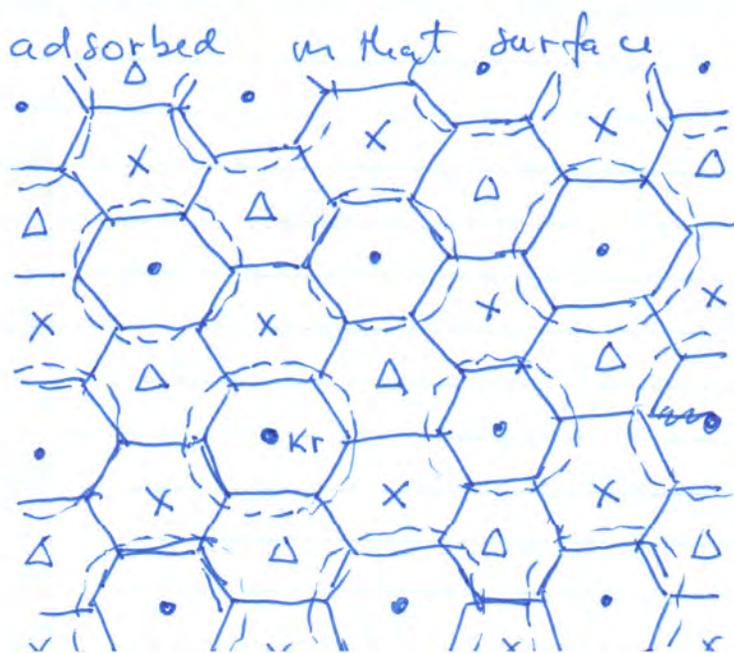
\Rightarrow the rods are directors, not vectors. \Rightarrow the energy cannot change under $\vec{S} \rightarrow -\vec{S}$ for each rod separately. A crude model of a

(nematic) liquid crystal has the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle \vec{r}_i, \vec{r}_{i'} \rangle} (\vec{S}(\vec{r}_i) \cdot \vec{S}(\vec{r}_{i'}))^2$$

⑤ Potts Models and \mathbb{Z}_N models

Consider a clean surface of graphite, i.e. a regular hexagonal lattice, and imagine that simple atoms (e.g. Argon or Krypton) are



There are 3 inequivalent (triangular) sublattices

\Rightarrow 3 degenerate ground states.

We still have 2 energies since all atoms are the same.

This is an example of a q -state Potts model in which there is a degree of freedom with q states but there are only two energies = same or different, $\sigma = 1, \dots, q$

$$\mathcal{H} = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \delta_{\sigma(\vec{r}), \sigma(\vec{r}')}$$

Another example is a discretization of the XY model where $\vec{S} = (\cos \theta, \sin \theta)$ with

$$\theta = \frac{2\pi}{N} n, \quad n = 1, \dots, N. \quad \text{This is}$$

the \mathbb{Z}_N or "clock" model with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \cos [\theta(\vec{r}) - \theta(\vec{r}')]$$

For $q = 2$ (and $N = 2$), this is the Ising Model once again.

In all cases of interest we must a partition function of the form

$$Z = \sum_{\{\text{configurations}\}} e^{-\beta \mathcal{H}[\text{configuration}]}$$

We will want to compute expectation values of observables such as the local magnetization

$$M(\vec{r}) = \langle \sigma(\vec{r}) \rangle \quad \text{or} \quad \text{the local}$$

energy density

$$E(\vec{r}) = \frac{1}{2d} \sum_{\vec{r}'} \langle \sigma(\vec{r}) \sigma(\vec{r}') \rangle$$

\uparrow $2d$ \rightarrow neighbors of \vec{r}
 \uparrow $2 \times$ dimension (or coordination number)

From now on we will discuss the simplest case, the Ising Model.

In all cases the system has a symmetry, e.g. in the Ising Model we can flip all spins simultaneously

$$[\sigma(\vec{r})] \rightarrow [-\sigma(\vec{r})]$$

and $\mathcal{H}[\sigma] = \mathcal{H}[-\sigma] \Rightarrow$ this is a symmetry

There are two elements $\sigma \rightarrow \sigma$ and $\sigma \rightarrow -\sigma$

As a symmetry operation this is the permutation group of two elements or \mathbb{Z}_2 . For the XY model the symmetry is $R\vec{S} = \vec{S}'$ where R is a rotation with fixed axis $\Rightarrow \vec{S} = (S_x, S_y)$
 $= (\cos\alpha, \sin\alpha)$

Alternatively, we can use a complex unit vector $e^{i\theta(\vec{r})}$. The symmetry is $\theta \rightarrow \theta + \alpha$
 $(-\alpha \leq \alpha \leq 2\pi)$ or $e^{i\alpha} e^{i\theta(\vec{r})} = e^{i\theta'(\vec{r})}$

This group of transformations is known as $O(2)$ or $U(1)$.

For the Heisenberg model the symmetry is the simultaneous rotation of all spins that the same rotation matrix, parametrized by the Euler angles θ, ϕ ; this is the $O(3)$ symmetry group. For the Potts model the symmetry is the permutation group of q elements S_q whereas for the \mathbb{Z}_N model, the symmetry is the ~~symmetric~~ cyclic group \mathbb{Z}_N .

In particular the group \mathbb{Z}_2 has two elements, the identity operation I , $I[\sigma] = [\sigma]$, and the inversion operation V ; $V[\sigma] = [-\sigma]$. This is a group under composition " \circ "

$$\mathbb{Z}_2 = \{ I, V; \circ \}$$

$$\begin{aligned} I \circ I &= I \\ I \circ V &= V \\ V \circ I &= V \\ V \circ V &= I \end{aligned}$$

In general the order parameter transforms non-trivially under the ~~symmetry~~ symmetry group of the system. Let us call the group G .

<u>Model</u>	<u>Group G</u>	<u>Order Parameter</u>
Ising	\mathbb{Z}_2	$\langle \sigma \rangle$
XY	$U(1)$	$\langle e^{i\theta} \rangle$ <small>explain "p" with with</small>
Heisenberg	$O(N)$	$\langle \vec{S} \rangle$ <small>(or other representations)</small>
Potts	S_N (permutations)	$\langle \sigma \rangle$
clock (\mathbb{Z}_N)	\mathbb{Z}_N	$\langle e^{i\theta} \rangle$
Solids on-solids (SOS)	\mathbb{Z}	$\langle e^{in(x)\alpha} \rangle$ with <small>$0 < \alpha \leq 2\pi$</small>

Symmetry Breaking Fields

we can also add terms to $H[\sigma]$ that break the symmetry explicitly, e.g. a magnetic field

$$H[\sigma] = -J \sum_{\langle r, r' \rangle} \sigma(r) \sigma(r') + h \sum_{\vec{r}} \sigma(r)$$

The last term is not invariant under $\sigma \rightarrow -\sigma$

In the absence of symmetry breaking fields configurations that differ by the action of the symmetry have the same weight in the partition function \Rightarrow If we take a finite system \Rightarrow the expectation value of any observable that's odd (i.e. not invariant) under the symmetry must be equal to zero. How can we possibly get a magnetized state if $h=0$?

To illustrate this issue let us consider the case of the $d=2$ (two-dimensional)

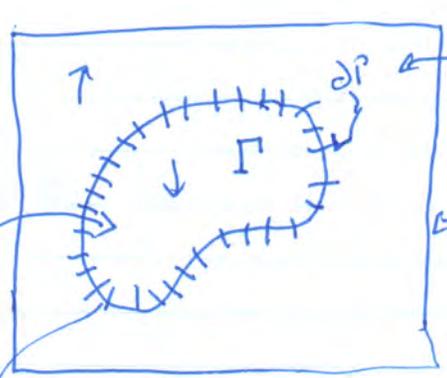
(L39)

Ising Model at $T \neq 0$ and $h \neq 0$

$$Z = \sum_{[\sigma]} e^{\beta J \sum_{\langle \vec{r}, \vec{r}' \rangle} \sigma(\vec{r}) \sigma(\vec{r}') + \beta h \sum_{\vec{r}} \sigma(\vec{r})}$$

↑
all configurations

Consider a configuration of the form



here all spins are up
boundary

Γ: region
∂Γ: boundary of Γ

all spins ↓

What is the energy?

broken bonds

→ area (perimeter)
→ volume

$$E = E_0 + 2J S[\partial\Gamma] + 2h V[\Gamma]$$

↑
ground state energy

↑
energy cost per "broken bond"

↑
energy cost of one overturned spin.

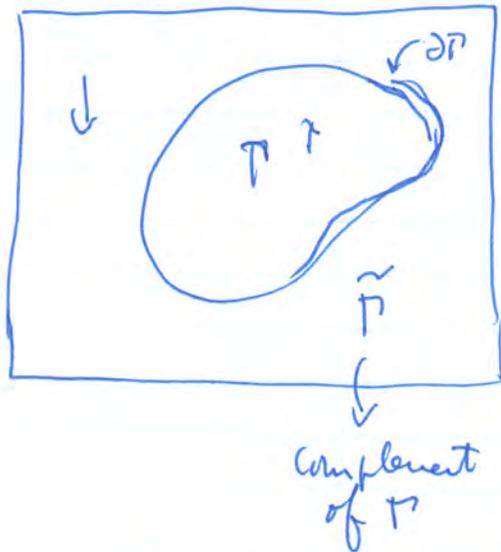
(in any dimension $d > 1$)

$$\Rightarrow E_0 = \text{circles} - JN2d - hN \quad (N = L^d)$$

$N = \#$ of sites

$d = \text{dimension}$. (hypercube lattice with coordination number $2d$)

For the opposite configuration (same region but $\sigma \rightarrow -\sigma$ everywhere)



The energy now is

$$E' = E_0 + 2J S(\partial\Gamma) - 2h V(\tilde{\Gamma})$$

$$V(\tilde{\Gamma}) = L^d - V(\Gamma)$$

$$\Rightarrow E' = E_0 + 2J S(\partial\Gamma) - 2h [L^d - V(\Gamma)]$$

$$\Rightarrow E' = E - 2h L^d$$

$$\Rightarrow E' - E = -2h L^d \rightarrow -\infty \quad L \rightarrow \infty$$

\Rightarrow In the thermodynamic limit the configuration with $\sigma \rightarrow -\sigma$ is suppressed for all $h \neq 0$

(no matter how small). Thus for any $h \neq 0$ only $\frac{1}{2}$ of the configurations survive.

Thus if we calculate $\langle \sigma \rangle$ with L and h finite and send $L \rightarrow \infty$ (thermodynamic limit)

at fixed h and then send $h \rightarrow 0$, the result may be a non-zero value for $\langle \sigma \rangle$.

$$M = \text{spontaneous magnetization} = \lim_{h \rightarrow 0} \left[\lim_{L \rightarrow \infty} \langle \sigma \rangle \right]$$

If this procedure results in a non-zero M we have a phase in which the symmetry (\mathbb{Z}_2 for the Ising Model) has been broken spontaneously: Spontaneous Symmetry Breaking (recall the discussion in the Bose Gas)

However for a finite system this cannot happen since if we set $h=0$ at fixed L the average will vanish.

Correlation Functions

Consider the object

$$G(\vec{r}, \vec{r}') = \langle \sigma(\vec{r}) \sigma(\vec{r}') \rangle \equiv G(\vec{r} - \vec{r}') \quad (\text{if } H \text{ is translationally invariant})$$

Clearly, this quantity measures the probability to find $\sigma(\vec{r}')$ to be, say, up if $\sigma(\vec{r})$ is also up.

How does $G(\vec{r}, \vec{r}')$ behave? If there is no symmetry breaking \Rightarrow this probability must decrease as $|\vec{r} - \vec{r}'| \rightarrow \infty$. Conversely, if the phase is magnetized, i.e. if $\langle \sigma \rangle \neq 0$

$$\Rightarrow \langle \sigma(r) \sigma(r') \rangle \rightarrow \langle \sigma \rangle^2 + f(|r-r'|)$$

$\xrightarrow{|r-r'| \rightarrow \infty} 0$

whereas $\langle \sigma(r) \sigma(r') \rangle \rightarrow 0$ for a disordered (a symmetric) phase as $|r-r'| \rightarrow \infty$

How do we compute $\langle \sigma \rangle$?

Since $Z = \sum_{[\sigma]} e^{\beta J \sum_{\langle r, r' \rangle} \sigma(r) \sigma(r') + \beta h \sum_r \sigma(r)}$

$$\Rightarrow \frac{1}{Z} \frac{\partial Z}{\partial \beta h} = \left\langle \sum_r \sigma(r) \right\rangle = L^d \langle \sigma(\vec{r}) \rangle$$

$$M = L^d \mathcal{M}$$

$$\Rightarrow \mathcal{M} = \frac{1}{L^d} \frac{\partial \log Z}{\partial \beta h}$$

local magnetization or magnetization per spin

$$Z = e^{-\beta F}$$

$$f = \frac{1}{L^d} F \quad (\text{Free energy density})$$

$$\Rightarrow \mathcal{M} = -\frac{\beta}{L^d} \frac{\partial F}{\partial \beta h} = -\frac{1}{L^d} \frac{\partial F}{\partial h} = -\frac{\partial f}{\partial h}$$

⇒ Local magnetization

$$m(\vec{r}) = \langle \sigma(\vec{r}) \rangle = - \frac{\partial f}{\partial h} \quad \left(\begin{array}{l} \text{constant for} \\ \text{a uniform} \\ \text{system} \end{array} \right)$$

Magnetic susceptibility:

$$\chi = \frac{\partial M}{\partial h} = L^d \frac{\partial m}{\partial h}$$

$$\chi = \frac{\partial}{\partial h} \frac{\partial \log Z}{\partial \beta h}$$

$$\begin{aligned} \chi &= \beta \frac{\partial^2 \log Z}{\partial (\beta h)^2} = \frac{1}{kT} \frac{\partial}{\partial \beta h} \left[\frac{1}{Z} \frac{\partial Z}{\partial \beta h} \right] \\ &= \beta \left[\frac{1}{Z} \frac{\partial^2 Z}{\partial (\beta h)^2} - \left[\frac{1}{Z} \frac{\partial Z}{\partial \beta h} \right]^2 \right] \end{aligned}$$

$$\frac{1}{Z} \frac{\partial Z}{\partial \beta h} = \left\langle \sum_r \sigma(\vec{r}) \right\rangle$$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial (\beta h)^2} = \left\langle \left(\sum_r \sigma(\vec{r}) \right)^2 \right\rangle$$

$$\Rightarrow \frac{1}{Z} \frac{\partial^2 Z}{\partial (\beta h)^2} - \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta h} \right)^2 = \left\langle \left(\sum_r \sigma(\vec{r}) \right)^2 \right\rangle - \left\langle \sum_r \sigma(\vec{r}) \right\rangle^2$$

In Fourier space we have

$$\begin{aligned}\tilde{G}(\vec{k}) &= \int d^d R G(\vec{R}) e^{i\vec{k}\cdot\vec{R}} \\ &= \int d^d R [m^2 + G_c(\vec{R})] e^{i\vec{k}\cdot\vec{R}}\end{aligned}$$

$$\Rightarrow \tilde{G}(\vec{k}) = m^2 (2\pi)^d \delta^d(\vec{k}) + \tilde{G}_c(\vec{k})$$

\Rightarrow The order parameter contributes with a δ -function to $\tilde{G}(\vec{k})$

$$\text{Since } \chi = \beta L^d \sum_{\vec{R}} G_c(\vec{R})$$

$$\Rightarrow \boxed{\chi = \beta L^d \tilde{G}_c(0)}$$

\uparrow zero momentum

The High Temperature Expansion (Ising Model)

Let us consider an Ising model on a hypercubic lattice in d dimensions. For the moment we will set $h=0$.

$$\begin{aligned}\Rightarrow Z &= \sum_{[\sigma]} e^{\beta J \sum_{\langle r, r' \rangle} \sigma(r) \sigma(r')} \\ &= \sum_{[\sigma]} \prod_{\langle r, r' \rangle} e^{\beta J \sigma(r) \sigma(r')}\end{aligned}$$

Since $e^{a\sigma} = \cosh a + \sigma \sinh a \equiv \cosh a (1 + \sigma \tanh a)$

we get

$$Z = \sum_{[\sigma]} (\cosh \beta J)^{2Nd} \prod_{\langle r, r' \rangle} (1 + (\tanh \beta J) \sigma(r) \sigma(r'))$$

$2Nd = \#$ of ~~bonds~~ bonds on this lattice.

$$Z = (\cosh \beta J)^{2Nd} \tilde{Z}(\beta J, N)$$

$$\tilde{Z} = \sum_{[\sigma]} \prod_{\langle r, r' \rangle} (1 + (\tanh \beta J) \sigma(r) \sigma(r'))$$

Since we will sum over all configurations
 \Rightarrow terms in the product in which a given spin, say $\sigma(r)$, appears once will vanish, ~~since~~ and it is $\neq 0$ if the spin appears twice (in ~~total~~ which case we get a factor of 2). Since we get a contribution per bond which is either 1 or $\tanh(\beta J) \sigma(r) \sigma(r')$ (for the bond $\langle r, r' \rangle$) we find that the non-zero terms are closed paths on this

lattice since the endpoints of each bond must appear twice and a given bond can only appear once. If we denote by Γ a set of closed ^{non-backtracking} paths (or loops) on the lattice and by $L(\Gamma)$ the total length (in lattice units) of these loops, we see that

$$\tilde{Z} = 2^{2Nd} \sum_{\Gamma} \left(\tanh(\beta J) \right)^{L(\Gamma)} N_L(\Gamma)$$

$$\Rightarrow Z = (2 \cosh \beta J)^{2Nd} \sum_{\Gamma} x^{L(\Gamma)} N_L(\Gamma)$$

where $x = \tanh \beta J$ and $N_L(\Gamma)$ is the number of loops of length L in Γ .

This is a rapidly convergent series. (see D. Rouelle)

$$\tilde{Z} = \sum_{L=0}^{\infty} N_L x^L = 1 + \binom{d}{2} \frac{d}{2} N x^4 + O(x^6)$$

$$N_L = \binom{d}{2} N \text{ for a } \square$$

contribution from the shortest loop \square

Correlation Function

$$G(\vec{R}, \vec{R}') = \frac{\sum_{\Gamma} \text{[Diagram: path from } \vec{R} \text{ to } \vec{R}' \text{ with a loop]} }{\sum_{\Gamma} \text{[Diagram: loop]}}$$

The reason is that a spin downstairs is a source of an open ~~loop~~ ^{path} which must then end at the other spin.

$$\Rightarrow G(\vec{R}, \vec{R}') = (\tanh(\beta J))^{d(\vec{R}, \vec{R}')} N_{d(\vec{R}, \vec{R}')} + \dots$$

where $d(\vec{R}, \vec{R}') = \sum_{i=1}^d |\vec{R}_i - \vec{R}'_i|$ ("Manhattan distance")

Along the same row $d(\vec{R}, \vec{R}') = |\vec{R} - \vec{R}'|$

$$\begin{aligned} \Rightarrow G(\vec{R}, \vec{R}') &= x^{|\vec{R} - \vec{R}'|} \cdot 1 + \dots \\ &\equiv e^{-|\vec{R} - \vec{R}'| \ln(\frac{1}{x})} + \dots \end{aligned}$$

$$\xi = \text{correlation length} = \frac{1}{\ln \frac{1}{x}} = \frac{1}{|\ln \tanh(\beta J)|}$$

\Rightarrow If the high temperature series converges

$\Rightarrow G(\vec{R}, \vec{R}')$ decays exponentially fast.

The Low Temperature Expansion

At $T=0$ there are just two (for an Ising model) configurations for the ground state: either all spins up or all down. The excitations are overturned spins. If we begin with the ground state $\uparrow\uparrow\dots\uparrow$, then at low T there will be a few \downarrow spins. Thus the system (or the lattice) can be divided into regions of \downarrow spins surrounded by a background of \uparrow spins. In the absence of an external symmetry breaking field, the only energy cost is paid at the boundaries of the regions of \downarrow spins. Let $\{\Gamma\}$ be a set of closed (hyper) surfaces in d dimensions on a hypercubic lattice; and let us denote by $\partial\Gamma$ the # of links piercing these surfaces. At the endpoints of each one of these links the spins are antiparallel to each other.

The energy of a configuration with a set of regions Γ with boundary hyper-area $\partial\Gamma$

$$E[\Gamma] = E_0 + 2J \underbrace{S[\partial\Gamma]}_{\substack{\# \text{ of lines on} \\ \text{the boundaries.}}}$$

These boundaries are domain walls, defects of the spin order.

$$\Rightarrow Z = \sum_{\substack{[\Gamma] \\ \uparrow \\ \text{sum over} \\ \text{regions}}} e^{-\beta E_0} e^{-2\beta J S[\partial\Gamma]}$$

$$= \sum_A e^{-\beta E_0 - 2\beta J A} N_A$$

(hyperareas)



N_A = # of regions whose surface "area" adds up to A .

Note: In $d=2$ the regions are areas and their boundaries are closed loops! (Same as the high Temp. expansion).

Let us consider first the case $d=1$, which is quite special.

Let us begin with the ground state $\uparrow \dots \uparrow$ with N spins and periodic boundary conditions (PBC's). The lowest energy excitation has one overturned spin

$\uparrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow \dots \uparrow$

with an excitation energy $= 2J$. But the configuration:

$\uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \dots \uparrow$ also has an excitation energy of $2J$. They have the same # of domain walls (2) with domain walls of any length.

Let us consider a system with N sites and periodic boundary conditions. ~~The~~ An arbitrary configuration has the form and it is uniquely $\uparrow \uparrow \downarrow \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \uparrow \dots$ determined by the position of the domain walls.

The energy of a configuration with k domain walls is $E = E_0 + 2Jk$, where k must be even since we have PBC's. The Boltzmann weight is $e^{-2\beta Jk}$ and the number of configurations with k walls is

$$N(N-1)(N-2)\dots(N-k) = \binom{N}{k}, \text{ with } k \text{ even.}$$

The configurations with $[0] \rightarrow -[0]$ have the same weight and ^{would seem to} contribute with a factor of 2. However if we do not pose restrictions on the coords. of the domain walls, it is easy to see that we have counted all configurations once.

$$\Rightarrow Z_N = 2 \sum_{k \text{ even}} \binom{N}{k} (e^{-2\beta J})^k e^{-\beta E_0}$$

$$= \sum_{k=0}^N \binom{N}{k} \{ e^{-2\beta Jk - \beta E_0} + \sum_{k=0}^N \binom{N}{k} e^{-2\beta Jk - \beta E_0} (-1)^k \}$$

(i.e. we are cancelling out k odd)

$$= e^{-\beta E_0} [(1 + e^{-2\beta J})^N + (1 - e^{-2\beta J})^N]$$

$Z_N = (2 \cosh \beta J)^N (1 + (\tanh \beta J)^N)$

which is exact.

Although this is the exact result let us look at the entropy-energy balance. First of all

$$\forall T, \tanh \beta J < 1 \Rightarrow (\tanh \beta J)^N \xrightarrow[N \rightarrow \infty]{} 0$$

$$Z_N \approx (2 \cosh \beta J)^N \text{ as } N \rightarrow \infty$$

what is the energy^{cost} of a config with k walls?

$$E \approx 2Jk + E_0 \text{ (} k \text{ walls)}$$

Entropy: $S \approx \log \binom{N}{k} = k \log N - k \log k$
 $= k \log N$

\Rightarrow The entropy diverges like $\log N$ while the energy cost is finite \Rightarrow domain walls proliferate and destroy long range order. We also see that, for large $N \gg 1$,

$$\log Z_N \approx N \log(2 \cosh \beta J) = N \beta J + N \log(1 + e^{-2\beta J})$$
$$\approx N \beta J + N e^{-2\beta J} + \dots$$

which has an essential singularity as $T \rightarrow 0$

Special Symmetry for $d=2$: Self-duality.

The low T expansion for $d=2$ is

$$Z = 2 e^{-2\beta E_0} \sum_{\Gamma \text{ (loops)}} e^{-2\beta J L(\Gamma)} \quad \# \text{ of loops on the dual lattice}$$

while the high T expansion is $Z = (2 \cosh \beta J)^{2N} \sum_{\Gamma} (\tanh \beta J)^{L(\Gamma)} \frac{264}{(\# \text{ of loops})}$

Let $\beta J = K \Rightarrow$ let us define $\tilde{K} / \tanh K = e^{-2\tilde{K}}$

$$\Rightarrow \frac{Z[K]}{(2 \cosh K)^{2N}} = \frac{Z[\tilde{K}]}{2 e^{2N\tilde{K}}} \quad (\text{Kramers-Wannier})$$

\Rightarrow it is possible to relate the free energy at temperature $T = T_{\pm}(K)$ with the free energy at $\tilde{T} = T(\tilde{K})$ though $\tanh K = e^{-2\tilde{K}}$

If \exists a unique transition $\Rightarrow \exists K^*$ dual system

s.t. $\tanh K^* = e^{-2K^*}$ (i.e. invariant)

$$\Rightarrow K_c = \tilde{K}_c = \frac{1}{2} \ln(\sqrt{2} + 1)$$

$$\Rightarrow \beta_c J = \frac{1}{2} \ln(\sqrt{2} + 1)$$

$$kT_c = \frac{2J}{\ln(\sqrt{2} + 1)}$$

Onsager Temperature

The Gaussian Approximation and Mean Field Theory

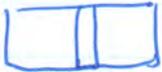
Let us look more closely the high T expansion.

We saw that it has the form

$$Z = (2 \cosh \beta J)^{2Nd} \sum_{L=0}^{\infty} (\tanh \beta J)^L N_L$$

N_L
 \downarrow
 # of loops
 of total length L

The problem with this expansion is that ~~the~~ not ~~all~~ all configurations of loops are allowed: if a link is occupied by a loop it cannot be shared by another loop \Rightarrow the loops may cross but may not overlap. Likewise a loop cannot backtrack.

\Rightarrow  and  are allowed but  is excluded.

Let us imagine for the moment that we ignore this restriction and that we sum freely over all loops configurations without

restriction. (We will be clearly overcounting configurations). Let N_L be the number of simple closed, non-retracing contours of length L .

$$\frac{Z}{(2 \cosh \beta J)^{2Nd}} \approx 1 + \sum_L N_L (\tanh \beta J)^L$$

$$+ \frac{1}{2!} \sum_{L, L'} N_L N_{L'} (\tanh \beta J)^{L+L'}$$

pairs of simple contours of lengths L, L'

$$+ \dots + \frac{1}{n!} \sum_{n \text{ simple contours}} N_{L_1} \dots N_{L_n} (\tanh \beta J)^{L_1 + \dots + L_n} + \dots$$

$$= e^{\sum_{L=0}^{\infty} N_L (\tanh \beta J)^L}$$

$N_L = \#$ of ~~simple~~ simple contours of length L

= to the # of paths on the lattice that

begin at an arbitrary point \vec{x} and end

at \vec{x} after L steps / $2L =$

$\Rightarrow N_L = \frac{1}{2L} W_L(\vec{x}, \vec{x})$ closely related to a random

walk! In fact, let $W_L(\vec{x}, \vec{y})$ be the # of ways of going from \vec{x} to \vec{y} in L steps.

$$\Rightarrow W_L(\vec{x}, \vec{y}) = \sum_{i=1}^{2d} W_{L-1}(\vec{x}, \vec{y} + \vec{b}_i) - W_{L-1}(\vec{x}, \vec{y})$$

(non-retracing!)

where $\{\vec{b}_i\}$ are the lattice vectors connecting \vec{y}

to its $2d$ neighbors. We have found this

problem before when we looked at the problem of random walks!

Let us solve this equation by Fourier transforms

$$W_L(\vec{x}, \vec{y}) = \sum_{\vec{p}} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} W_L(\vec{p})$$

$$\Rightarrow W_L(\vec{p}) = W_{L-1}(\vec{p}) \left(\sum_{i=1}^{2d} e^{i\vec{p} \cdot \vec{b}_{i-1}} \right)$$

$$W_L(\vec{p}) = W_{L-1}(\vec{p}) \left(2 \sum_{i=1}^d \cos p_i \right)$$

If we iterate we find

$$W_L(\vec{p}) = W_0(\vec{p}) \left(2 \sum_{i=1}^d \cos p_i \right)^L$$

$$W_L(\vec{x}, \vec{y}) = \sum_{\vec{p}} W_0(\vec{p}) \left(2 \sum_{i=1}^d \cos p_i \right)^L e^{i\vec{p} \cdot (\vec{x} - \vec{y})}$$

with the "initial condition" $W_0(\vec{x}, \vec{y}) = \delta_{\vec{x}, \vec{y}}$

$$\delta_{\vec{x}, \vec{y}} = \sum_{\vec{p}} w_0(\vec{p}) e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \Rightarrow w_0(\vec{p}) = 1$$

$$\Rightarrow w_L(\vec{p}) = \left(2 \sum_{i=1}^d \cos p_i \right)^L$$

$$w_L(\vec{x}, \vec{x}) = \sum_{\vec{p}} w_L(\vec{p}) = \sum_{\vec{p}} \left(2 \sum_{i=1}^d \cos p_i \right)^L$$

$$\text{and} \quad \sum_{L=1}^{\infty} \frac{1}{2L} w_L(\vec{x}, \vec{x}) (\tanh \beta J)^L =$$

$$= \frac{1}{2} \sum_{L=1}^{\infty} \frac{1}{L} \sum_{\vec{p}} \left(2 \sum_{i=1}^d \cos p_i \right)^L (\tanh \beta J)^L$$

$$\ln(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n}$$

\Rightarrow

$$= -\frac{1}{2} \sum_{\vec{p}} \ln \left[1 - (\tanh \beta J) \left(2 \sum_{\mu=1}^d \cos p_{\mu} \right) \right]$$

$$Z_N^{\text{Gaussian}} = e^{-\beta F_G} = e^{-\frac{N}{2} \left(\frac{1}{N} \sum_{\vec{p}} \ln \left[1 - (\tanh \beta J) \left(2 \sum_{\mu=1}^d \cos p_{\mu} \right) \right] \right)}$$

$$\Rightarrow \frac{F_G}{N} = \frac{1}{2} kT \int_{\text{BZ}} \frac{d^d \vec{p}}{(2\pi)^d} \ln \left[1 - (\tanh \beta J) \left(2 \sum_{\mu=1}^d \cos p_{\mu} \right) \right]$$

BZ = First Brillouin Zone, i.e. $|k_{\mu}| \leq \pi$
 $\mu = 1, \dots, d$

$$\cos p = 1 - 2 \sin^2 \frac{p}{2}$$

$$1 - (\tanh \beta J) \left(2 \sum_{\mu=1}^d \left(1 - 2 \sin^2 \left(\frac{p_{\mu}}{2} \right) \right) - 1 \right)$$

$$= \left(1 - \cancel{2} \tanh \beta J \right) + 4 \tanh(\beta J) \sum_{\mu=1}^d \sin^2 \left(\frac{p_{\mu}}{2} \right)$$

$$\text{Def: } m^2 = \frac{1 - \cancel{2} \tanh \beta J}{\tanh \beta J}$$

$$\frac{F_G}{N} = \frac{1}{2} kT \ln \tanh(\beta J) + \frac{1}{2} kT \int_{\text{BZ}} \frac{d^d \vec{p}}{(2\pi)^d} \ln \left[m^2 + 4 \sum_{\mu=1}^d \sin^2 \left(\frac{p_{\mu}}{2} \right) \right]$$

$$\approx \frac{1}{2} kT \ln \tanh(\beta J) + \frac{1}{2} kT \int_{\text{BZ}} \frac{d^d \vec{p}}{(2\pi)^d} \ln (m^2 + \vec{p}^2)$$

For $\beta J \ll 1$ ($kT \gg J$) the loops are very rare and are unlikely to overlap \Rightarrow this approx. should be correct here.

$$m^2 \approx \frac{(2d-1) \beta J}{\beta J} = \frac{kT}{J} - (2d-1)$$

$$\exists T_c / m^2 = 0, \text{ i.e. } 1 = \tanh \beta_c J$$

Also these expressions become singular (i.e. there is an imaginary part) for $T < T_c$

$$\Rightarrow kT_c = \frac{2J}{\ln\left(\frac{d}{d-1}\right)} \quad \left(\text{for } d \gg 1, kT_c \approx 2Jd + 0\left(\frac{1}{d}\right) \right)$$

This result is exact on a tree lattice (Bethe-Peierls)
(Note: $T_c = 0$ for $d=1$ which is correct)

Correlation Function in the Gaussian Approximation:

$$G(\vec{R}, \vec{R}') = \langle \sigma(\vec{R}) \sigma(\vec{R}') \rangle$$

$$= \frac{1}{Z[\sigma]} \sum \sigma(\vec{R}) \sigma(\vec{R}') e^{-\beta H[\sigma]}$$

We can write a high T expansion for $G(\vec{R}, \vec{R}')$. Here too we need only to keep connected diagrams: there is a linked cluster theorem for these loop configurations as well (i.e. cancellation of "non-conn.")

The linked graphs here are graphs (or paths) beginning at \vec{R} and ending at \vec{R}' . These paths may have all possible lengths but are linked.

The leading ~~term~~ contribution does not have any (~~linked~~) closed paths linked to the paths going from \vec{R} to \vec{R}' .

\Rightarrow Within the Gaussian approximation all these linked graphs (which are there to eliminate the counting of overlapping paths) are simply

$$G_{\text{Gaussian}}(\vec{R}, \vec{R}') = \sum_{L=0}^{\infty} x^L N_L(\vec{R}, \vec{R}')$$

$$x = \tanh \beta J$$

$$N_L(\vec{R}, \vec{R}') = W_L(\vec{R}, \vec{R}')$$

$$= \int_{\mathcal{B}^d} \frac{d^d p}{(2\pi)^d} W_L(\vec{p}) e^{i \vec{p} \cdot (\vec{R} - \vec{R}')} = \int_{\mathcal{B}^d} \frac{d^d p}{(2\pi)^d} \left(\prod_{\mu=1}^d 2 \cos p_{\mu} \right)^L e^{i \vec{p} \cdot (\vec{R} - \vec{R}')}$$

$$\Rightarrow G(\vec{R}, \vec{R}') = \sum_{L=0}^{\infty} x^L \int_{\mathcal{B}^d} \frac{d^d p}{(2\pi)^d} \left(\prod_{\mu=1}^d 2 \cos p_{\mu} \right)^L e^{i \vec{p} \cdot (\vec{R} - \vec{R}')}$$

We now have a geometric sum to do:

$$G(\mathbf{R}, \mathbf{R}') = \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\vec{p} \cdot (\vec{R} - \vec{R}')} }{1 - \tanh(\beta J) \left(2 \sum_{\mu=1}^d \cos p_{\mu} - 1 \right)}$$

is the correlation function in the Gaussian approximation.

For $|\vec{R} - \vec{R}'| \gg a_0$ (the lattice spacing)

we can approximate $\cos p \approx 1 - \frac{p^2}{2} + \dots$

$$G(\mathbf{R}, \mathbf{R}') \approx \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\vec{p} \cdot (\vec{R} - \vec{R}')} }{1 - \tanh(\beta J) \left(2 \sum_{\mu=1}^d \left(1 - \frac{p_{\mu}^2}{2} \right) - 1 \right)}$$

$$= \frac{1}{\tanh(\beta J)} \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\vec{p} \cdot (\vec{R} - \vec{R}')} }{p^2 + m^2}$$

(Ornstein-Zernike)

(a) for $m|\vec{R} - \vec{R}'| \gg 1$ we found before that

$$G(\mathbf{R}, \mathbf{R}') \approx \left(\frac{1}{\tanh(\beta J)} \right) \frac{m^{d-2}}{2(2\pi)^{\frac{d}{2}-1}} \frac{e^{-m|\vec{R} - \vec{R}'|}}{[m|\vec{R} - \vec{R}'|]^{\frac{d}{2}-1}}$$

(b) for $m|\vec{R} - \vec{R}'| \ll 1$

$$G(\mathbf{R}, \mathbf{R}') \approx \left(\frac{1}{\tanh(\beta J)} \right) \frac{\Gamma\left(\frac{d}{2}-1\right)}{2(2\pi)^{\frac{d}{2}-1} m^{d-2}}$$

Let us define the correlation length ξ

$$\xi = \left| \frac{1}{m} \right| = \sqrt{\frac{\tanh \beta J}{1 - (2d-1) \tanh \beta J}}$$

If $d > 1 \Rightarrow$ as $T \rightarrow T_c^+ = \frac{2J}{\ln(d/d-1)}$

We find

$$\xi \sim \frac{A}{(T - T_c)^{\nu}} \quad \text{with } \nu = 1/2$$

where $A^2 = \frac{kJ}{d(d-1) \ln(d/d-1)}$

Mean Field Theory

We will now look at the problem using a variational principle. Let H be the Hamiltonian and H_0 be some other Hamiltonian, typically one that our intuition tells us is relevant to the problem.

$$\Rightarrow Z = \text{tr} e^{-\beta H} = \text{tr} e^{-\beta(H_0 + V)}$$

where $V = H - H_0$

In a classical system $[H_0, V] = 0 \Rightarrow$

$$e^{-\beta(H_0 + V)} = e^{-\beta H_0} e^{-\beta V}$$

For a quantum system this is not generally true but there exist a variant of the argument we will use here (see Feynman).

$$\Rightarrow Z = \text{tr} e^{-\beta H} = \text{tr} e^{-\beta H_0} e^{-\beta V} = \langle e^{-\beta V} \rangle_0 Z_0$$

$$\text{where } Z_0 = \text{tr} e^{-\beta H_0} = e^{-\beta F_0}$$

Using the convexity of the exponential function we get

$$\langle e^{-\beta V} \rangle_0 \geq e^{-\beta \langle V \rangle_0}$$

$$\Rightarrow e^{-\beta F} \geq e^{-\beta F_0} e^{-\beta \langle V \rangle_0}$$

$$\text{or } F \leq F_0 + \langle V \rangle_0$$

$$F \leq F_0 + \langle H - H_0 \rangle_0$$

This result turns out to be correct even in quantum systems.

$$\text{Let } H = - \frac{1}{2} \sum_{\vec{r}, \vec{r}'} J(\vec{r}, \vec{r}') \sigma(\vec{r}) \sigma(\vec{r}')$$

$$\text{and } H_0 = - \sum_{\vec{r}} m(\vec{r}) \sigma(\vec{r})$$

where the functions $m(\vec{r})$ are still undetermined

What is F_0 ?

$$\begin{aligned} Z_0 &= \text{tr } e^{-\beta H_0} = \sum_{\{\sigma\}} e^{\beta \sum_{\vec{r}} m(\vec{r}) \sigma(\vec{r})} = \\ &= \prod_{\vec{r}} [e^{\beta m(\vec{r})} + e^{-\beta m(\vec{r})}] \end{aligned}$$

$$F_0 = -kT \sum_{\vec{r}} \log [2 \cosh \beta m(\vec{r})]$$

$$\langle H - H_0 \rangle_0 = \langle H \rangle_0 - \langle H_0 \rangle_0$$

$$\langle H \rangle_0 = -\frac{1}{2} \sum_{\vec{r}, \vec{r}'} J(\vec{r}, \vec{r}') \langle \sigma(\vec{r}) \sigma(\vec{r}') \rangle_0$$

$$\langle H_0 \rangle_0 = - \sum_{\vec{r}} m(\vec{r}) \langle \sigma(\vec{r}) \rangle_0$$

$$kT \frac{1}{Z_0} \frac{\partial Z_0}{\partial m(\vec{r})} = \langle \sigma(\vec{r}) \rangle_0$$

$$\langle \sigma(\vec{r}) \rangle_0 = \frac{\partial \ln Z_0}{\partial m(\vec{r})} = \tanh \beta m(\vec{r})$$

$$\langle \sigma(r) \sigma(r') \rangle = \tanh(\beta m(r)) \tanh(\beta m(r')) \quad (r \neq r')$$

$$\begin{aligned} \langle H - H_0 \rangle &= -\frac{1}{2} \sum_{r, r'} J(r, r') \tanh(\beta m(r)) \tanh(\beta m(r')) \\ &\quad + \sum_r m(r) \tanh(\beta m(r)) \end{aligned}$$

$$\begin{aligned} F &\lesssim -kT \sum_r \log(2 \cosh(\beta m(r))) + \\ &\quad + \sum_r m(r) \tanh(\beta m(r)) \\ &\quad - \frac{1}{2} \sum_{r, r'} J(r, r') \tanh(\beta m(r)) \tanh(\beta m(r')) \end{aligned}$$

Let us now look for the lowest upper bound:

$$\frac{\partial F}{\partial m(r)} = 0 \quad (\text{extremal condition})$$

$$\begin{aligned} 0 &= -kT \cancel{\beta} \tanh(\beta m(r)) + \tanh(\beta m(r)) \\ &\quad + \frac{\beta m(r)}{\cosh^2(\beta m(r))} - \sum_{r'} J(r, r') \frac{\tanh(\beta m(r')) \beta}{\cosh^2(\beta m(r))} \end{aligned}$$

$$\Rightarrow \boxed{m(r) = \sum_{r'} J(r, r') \tanh(\beta m(r'))}$$

Uniform state $\Rightarrow m(r) = m$

$$\Rightarrow m = \left(\sum_{r'} J(r, r') \right) \tanh(\beta m)$$

$$J(r, r') = J(r - r')$$

$$\Rightarrow \sum_{r'} J(r - r') \equiv \sum_R J(R)$$

Clearly since

$$\langle \sigma(\mathbf{r}) \rangle = \tanh(\beta m(\mathbf{r}))$$

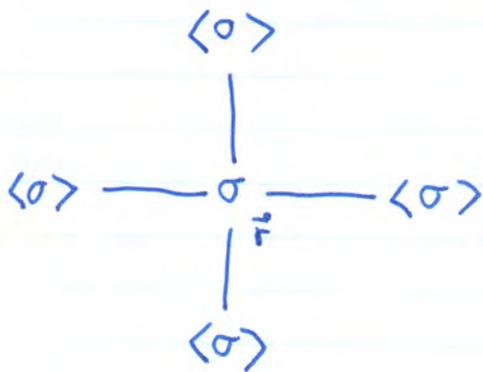
$$\Rightarrow \langle \sigma \rangle = \tanh(\beta m) \quad (\text{uniform state})$$

However, since $m = \left(\sum_{\mathbf{R}} J(\mathbf{R}) \right) \tanh(\beta m)$, we get

$$\langle \sigma \rangle = \frac{m}{\sum_{\mathbf{R}} J(\mathbf{R})}$$

$$\Rightarrow \langle \sigma \rangle = \tanh \left[\beta \left(\sum_{\mathbf{R}} J(\mathbf{R}) \right) \langle \sigma \rangle \right]$$

Curie
Weiss

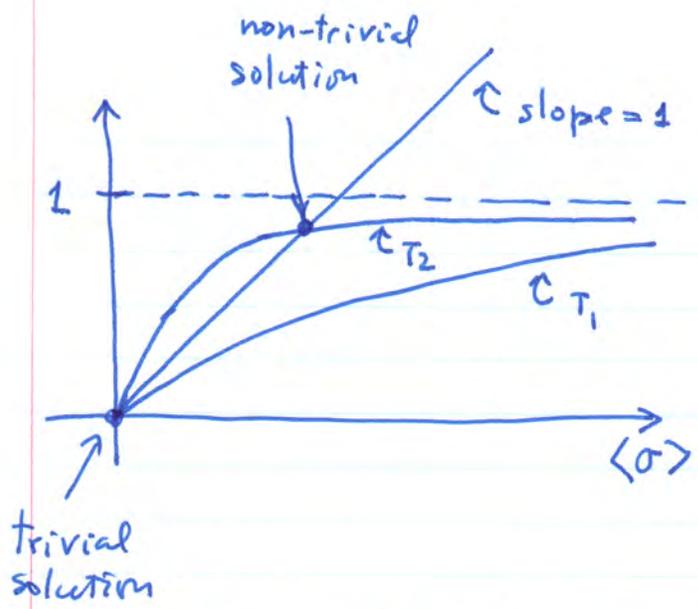


It is as if a spin at site \vec{r} sees the surrounding spins as if they took their expectation value

$\langle \sigma \rangle = 0$ is always a solution: disordered phase

$\langle \sigma \rangle \neq 0$ is possible only if the slope of the

r.h.s. at $\langle \sigma \rangle = 0$ is > 1



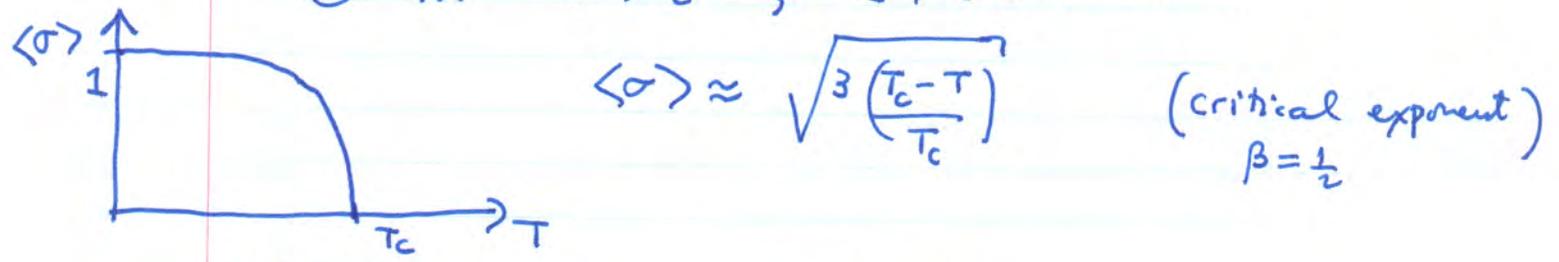
$T_1 > T_2$
 The non-trivial solution first appears when the slope = 1 ; T_c

slope = $\beta \left(\sum_R J(R) \right)$ (at $\langle \sigma \rangle = 0$)
 $k T_c = \sum_R J(R)$
 $\Rightarrow \beta_c \left(\sum_R J(R) \right) = 1 \Rightarrow$ ~~$k T_c = \sum_R J(R)$~~

nearest neighbors: $\sum_R J(R) = 2d J$ (hypercubic lattice)
 $\Rightarrow k T_c = 2d J$

- Solutions:
- (a) For $T > T_c \Rightarrow \langle \sigma \rangle = 0$
 - (b) For $T \leq T_c \Rightarrow \langle \sigma \rangle \neq 0$

(1) For $T \rightarrow T_c^-$, $\langle \sigma \rangle \ll 1$ and



(2) For $T \rightarrow 0$, $\langle \sigma \rangle \rightarrow 1$

$\langle \sigma \rangle \approx 1 - 2e^{-2 \frac{T_c}{T}} + \dots$ as $T \rightarrow 0$

The Limit of Large dimension $d \rightarrow \infty$

Let us consider now an Ising model on a lattice with a very large coordination number, i.e. an "infinite range" model

$$\begin{aligned}
 H &= -\frac{1}{2} \sum_{\langle i, j \rangle} J_{ij} \sigma(i) \sigma(j) \\
 &\equiv -\frac{J}{2} \sum_{\langle i, j \rangle} \sigma(i) \sigma(j) \equiv -\frac{J}{2} \left(\sum_i \sigma(i) \right)^2
 \end{aligned}$$

↑
all!

$$\begin{aligned}
 Z &= \sum_{[\sigma]} e^{-H[\sigma]/kT} \\
 &= \sum_{[\sigma]} e^{+\beta \frac{J}{2} \left(\sum_i \sigma(i) \right)^2} \\
 &\equiv \sum_{[\sigma]} \int_{-\infty}^{+\infty} d\phi e^{-\frac{\beta}{2J} \phi^2} \sum_{[\sigma]} e^{-\beta \phi \sum_i \sigma(i)} \\
 &\equiv \int_{-\infty}^{+\infty} d\phi e^{-\beta \phi^2 / 2J} \left[2 \cosh(\beta \phi) \right]^N \frac{\sqrt{\beta}}{\sqrt{2\pi J}}
 \end{aligned}$$

To get a meaningful $N \rightarrow \infty$ we rescale J

$$J = \tilde{J}/N$$

$$Z = \int_{-\infty}^{+\infty} d\phi \sqrt{\frac{N\beta}{2\pi\tilde{J}}} e^{-\frac{N\beta\phi^2}{2\tilde{J}}} (2\cosh(\beta\phi))^N$$

$$\equiv \int_{-\infty}^{+\infty} d\phi \sqrt{\frac{N\beta}{2\pi\tilde{J}}} e^{-N\mathcal{U}(\phi)}$$

$$\mathcal{U}(\phi) = \beta \frac{\phi^2}{2\tilde{J}} - \ln(2\cosh(\beta\phi)) - \frac{1}{2N} \ln\left(\frac{\beta N}{2\pi\tilde{J}}\right)$$

$N \rightarrow \infty \Rightarrow$ the integral is dominated by the extrema of $\mathcal{U}(\phi)$

$$\mathcal{U}'(\phi)|_{\bar{\phi}} = 0 \Rightarrow 0 = \beta \frac{\bar{\phi}}{\tilde{J}} - \beta \tanh(\beta\bar{\phi}) = 0$$

$$\Rightarrow \bar{\phi} = \tilde{J} \tanh(\beta\bar{\phi})$$

$$\text{or } \bar{\phi} = N\tilde{J} \tanh(\beta\bar{\phi})$$

(mean-field theory!)

Landau Theory of Phase Transitions

Landau (and Ginzburg) proposed a phenomenological theory of phase transitions.

This theory is based on the concept of an ~~#~~ order parameter which transforms according to some global symmetry.

e.g. in the case of the Ising model

the order parameter is the local magnetization which changes sign under a reversal of all the spins. Let M be the (uniform)

magnetization. Near ~~the~~ a continuous phase transition, where the magnetization will be small, we can expand the free energy in powers of M , i.e.

$$F[M] = F_0(T) + a(T)M^2 + b(T)M^4 + \dots$$

Here $F_0(T)$ is the free energy of the paramagnetic phase (i.e. with $M=0$). The

coefficients $a(T)$ and ~~also~~ $b(T)$ are smooth functions of the Temperature T .

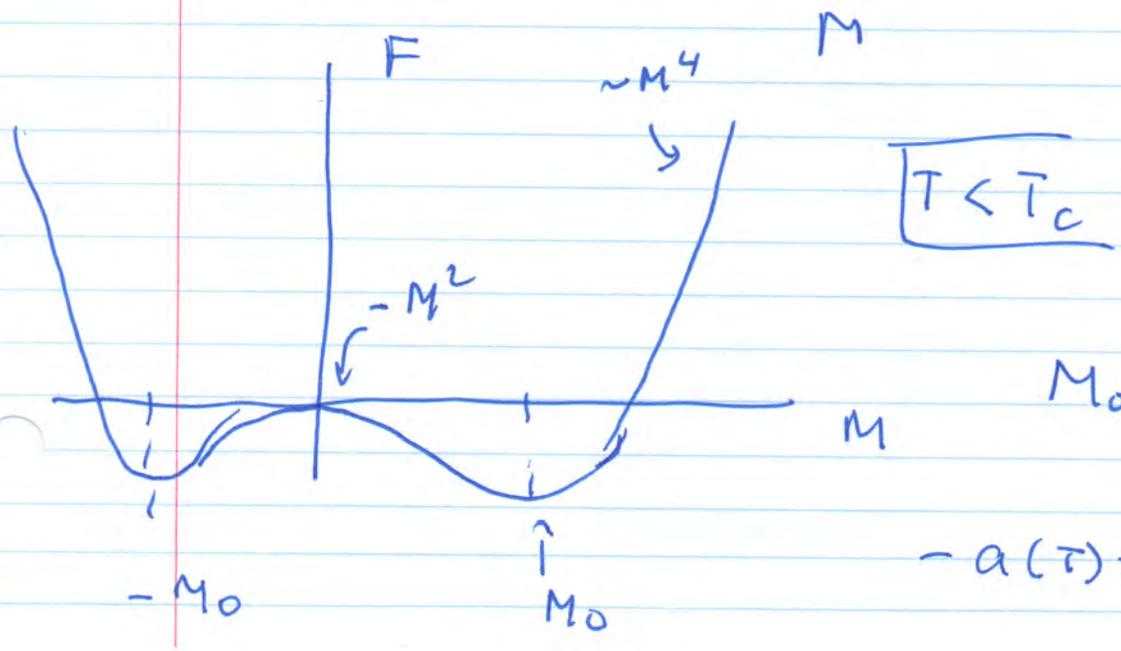
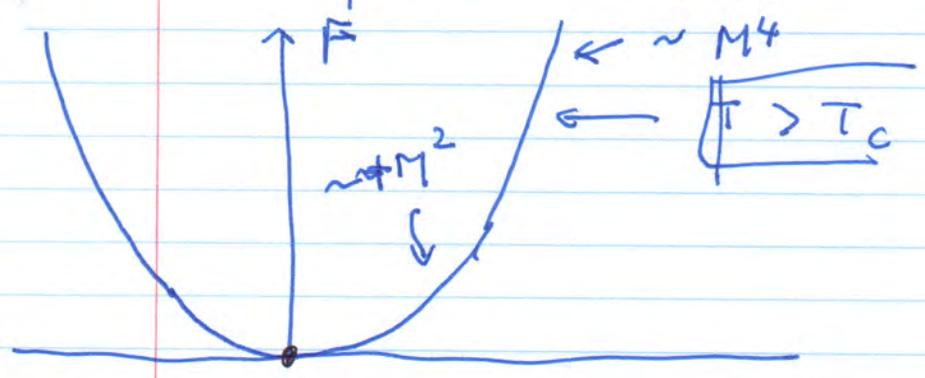
If $a(T)$ changes sign near T_c ,

$$a(T) = a_0 (T - T_c)$$

↑
constant

and $b(T) \equiv b_0 > 0$ (positive for thermodynamic stability). We can

now plot $F(T, M)$



$$M_0 = I \sqrt{\frac{-a(T)}{2b(T)}}$$

$$-a(T) = a_0 (T_c - T) \quad (T < T_c)$$

The free energy of the paramagnetic phase (with $M=0$) is $F_0(T)$. The free energy of the ferromagnetic phase is $F_0 - \frac{a^2}{4b} < F_0$

\Rightarrow For $T < T_c$ the system is a ferromagnet and the \mathbb{Z}_2 symmetry of reversing all spins is broken spontaneously.

$$M_0 = \sqrt{\frac{a_0(T_c - T)}{2b}} = \text{const} \cdot \sqrt{T_c - T} \\ \equiv \text{const} (T_c - T)^\beta$$

$\beta = \frac{1}{2}$ critical exponent. This exponent is the same for all ferromagnets (like the Landau theory!) while T_c changes from one system to another.

How do we describe a first order transition in this picture? In general a first order transition occurs if the free energy is a continuous function of the temperature (as it should ~~be~~) but with a discontinuous slope at T_c .

In the Landau ~~theory~~ theory this transition is pictured as follows. Let us consider

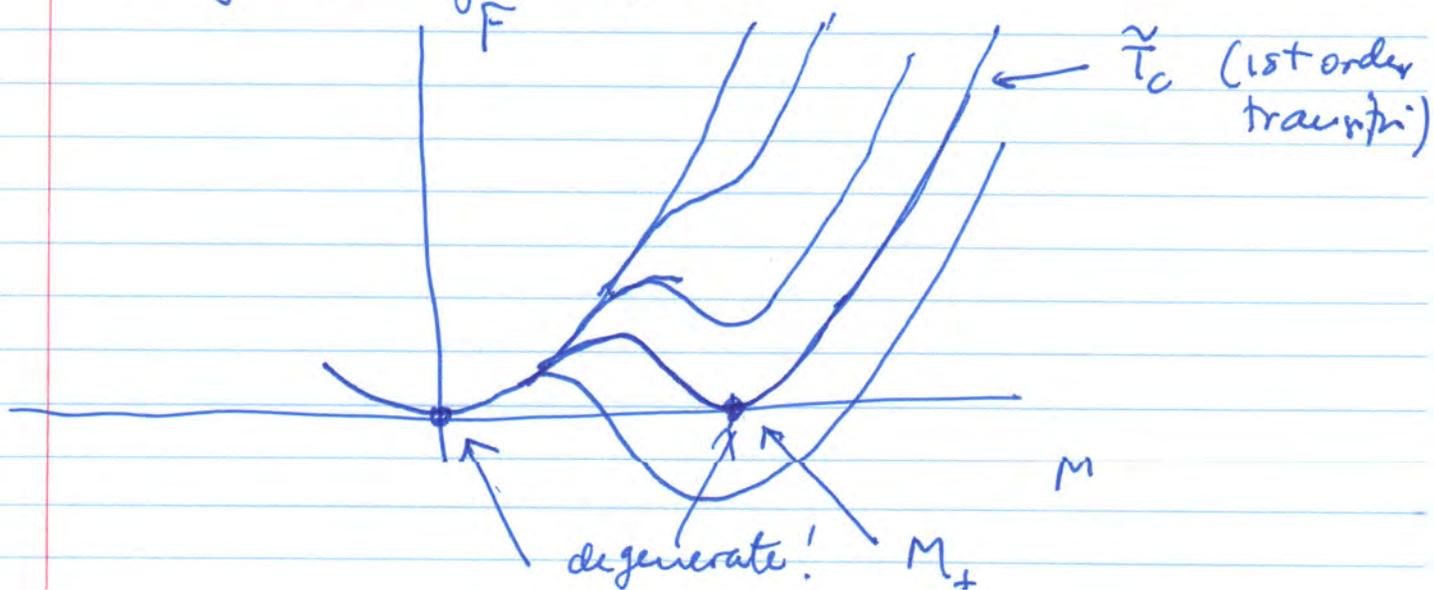
a free energy of the form

$$F = F_0 + aM^2 + bM^4 + cM^6$$

Thermodynamic stability ~~requires~~ ^{requires} that $c > 0$

Suppose that $a > 0$ but now $b(T)$

may change sign:



What is the value of \tilde{T}_c for the first order transition?

$$F_0 = F_0 + a M_+^2 + b M_+^4 + c M_+^6 \quad (1)$$

and

$$0 = 2a M_+ + 4b M_+^3 + 6c M_+^5 \quad (2)$$

~~These~~ These two equations determine the value of \tilde{T}_c (since $a = a_0(T - T_c)$)

and of M_+ as a function of b ,

$$(1) \Rightarrow 0 = a + b M_+^2 + c M_+^4 \quad (1')$$

$$(2) \Rightarrow 0 = a + 2b M_+^2 + 3c M_+^4 \quad (2')$$

$$(1') \Rightarrow M_{\pm}^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

\Rightarrow if $b < 0 \Rightarrow b = -|b|$

$$\text{and } M_{\pm}^2 = \frac{|b|}{2c} \pm \frac{1}{2c} \sqrt{b^2 - 4ac}$$

plugging this result into (2')

$$0 = a + 2b \left(\frac{|b|}{2c} + \frac{1}{2c} \sqrt{b^2 - 4ac} \right)$$

$$+ \frac{3b}{4c} \left(|b| + \sqrt{b^2 - 4ac} \right)^2$$

$$\Rightarrow a = \frac{b^2}{4c} + \frac{b^2}{4c} \sqrt{1 - \frac{4ac}{b^2}}$$

Solve for a

$$\left(a - \frac{b^2}{4c} \right)^2 = \frac{b^4}{16c^2} \left(1 - \frac{4ac}{b^2} \right)$$

$$a^2 + \frac{b^4}{16c^2} - \frac{ab^2}{2c} = \frac{b^4}{16c^2} - \frac{b^4}{16c^2} - \frac{4ac}{b^2}$$

$$a^2 - \frac{ab^2}{2c} = - \frac{ab^2}{4c}$$

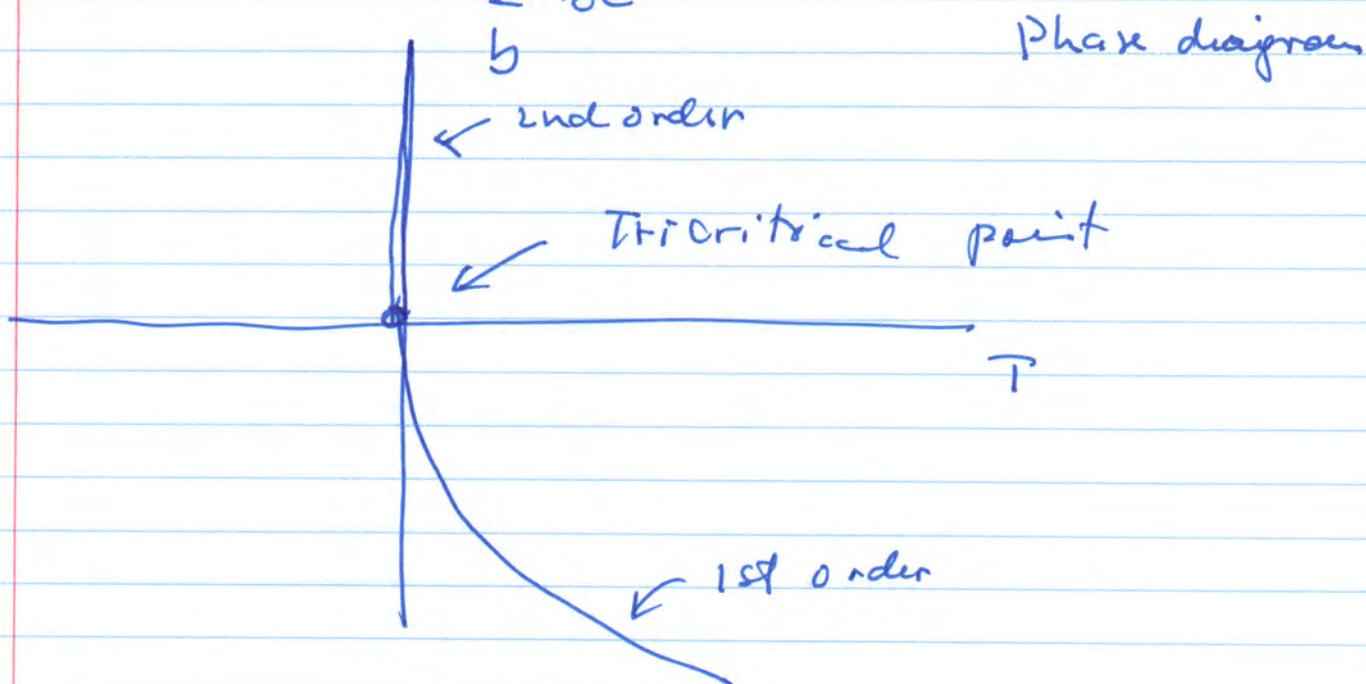
$$\Rightarrow a^2 = \frac{ab^2}{2c} \Rightarrow a=0 \text{ or } a = \frac{b^2}{2c}$$

If $a=0 \Rightarrow$ we are at the 2nd order transition and $\frac{1}{2} M_+ = 0$

$$\text{If } a \neq 0 \Rightarrow a = \frac{b^2}{2c} \quad (b < 0)$$

$$\Rightarrow a_0 (\tilde{T}_c - T_c) = \frac{b^2}{2c}$$

$$\tilde{T}_c = T_c + \frac{b^2}{2a_0 c}$$



The Transfer Matrix

In the case of a classical system with short range interactions we can recast the partition function in a form that is reminiscent of a quantum mechanical problem in one dimensionless. Let us consider the case of an Ising model (although this method is very general)

The partition function is

$$Z = \sum_{[\sigma]} e^{-\beta H[\sigma]}$$

where $H[\sigma]$ involves only nearest neighbor ferromagnetic interactions J and an external field h . In the simplest case of a 1D

system

$$H(\sigma) = -J \sum_{n=1}^N \sigma(n) \sigma(n+1) - h \sum_{n=1}^N \sigma(n)$$

For a system with PBC's, $\sigma(N+1) \equiv \sigma(1)$

We will now show that $Z = \text{tr} \hat{T}^N$

where \hat{T} is a hermitian matrix. ($\hat{T}^\dagger = \hat{T}$)

Let us denote by $|\uparrow\rangle = |\uparrow\rangle, |\downarrow\rangle$ the basis states.

$$\Rightarrow \text{tr } \hat{T}^N = \langle \uparrow | \hat{T}^N | \uparrow \rangle + \langle \downarrow | \hat{T}^N | \downarrow \rangle$$

$$\equiv \sum_{\sigma} \langle \sigma | \hat{T}^N | \sigma \rangle$$

$$\langle \sigma | A B | \sigma' \rangle = \sum_{\sigma''} \langle \sigma | A | \sigma'' \rangle \langle \sigma'' | B | \sigma' \rangle$$

$$\Rightarrow \text{tr } \hat{T}^N = \sum_{\sigma_1, \dots, \sigma_N} \langle \sigma_1 | \hat{T} | \sigma_2 \rangle \langle \sigma_2 | \hat{T} | \sigma_3 \rangle \dots \langle \sigma_N | \hat{T} | \sigma_1 \rangle$$

I will now choose \hat{T} /

$$\langle \sigma | \hat{T} | \sigma' \rangle = e^{\beta J \sigma \sigma' + \beta h \frac{\sigma + \sigma'}{2}} \quad \text{Transfer Matrix}$$

$$\Rightarrow \text{tr } \hat{T}^N = \sum_{\{\sigma_n\}} e^{\beta J \sum_{n=1}^N \sigma_n \sigma_{n+1} + \beta h \sum_{n=1}^N \sigma_n} = Z!$$

~~Since~~ since $\hat{T}^\dagger = \hat{T}$ and it is real \Rightarrow

$$\hat{T} = a I + b \sigma_3 + c \sigma_1$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow a + b = e^{\beta J + \beta h}$$

$$a - b = e^{\beta J - \beta h}$$

$$c = e^{-\beta J}$$

$$\Rightarrow \begin{cases} a = e^{\beta J} \cosh(\beta h) \\ b = e^{\beta J} \sinh(\beta h) \\ c = e^{-\beta J} \end{cases}$$

$$\Rightarrow Z = \text{tr} \hat{T}^N = \lambda_+^N + \lambda_-^N$$

λ_{\pm} are the eigenvalues of \hat{T}

$$\lambda_{\pm} = a \pm \sqrt{b^2 + c^2}$$

$$\lambda_{\pm} = e^{\beta J} \cosh(\beta h) \pm \sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}}$$

$$\equiv e^{\beta J} \cosh(\beta h) \pm \sqrt{e^{2\beta J} \cosh^2(\beta h) - 2 \frac{\sinh(\beta h)}{\cosh(\beta J)}}$$

clearly $\lambda_+ > \lambda_- \Rightarrow \left(\frac{\lambda_-}{\lambda_+} \right)^N \xrightarrow{N \rightarrow \infty} 0$

In the thermodynamic limit $N \rightarrow \infty$

$Z \approx \lambda_+^N$ (i.e. only the largest eigenvalue of the Transfer matrix contributes)

$$\Rightarrow_{N \rightarrow \infty} e^{-\beta F} = \lambda_+^N$$

and $F = -kTN \ln \lambda_+(T, h)$

\Rightarrow F is extensive ($\propto N$) and the free energy density is

$$f(T, h) = \lim_{N \rightarrow \infty} \frac{F(T, h)}{N} = -kT \ln \lambda_+(T, h)$$

$$f(T, h) = -kT \ln \left[e^{\beta J} \cosh(\beta h) + \sqrt{e^{2\beta J} \underbrace{\cosh^2(\beta h) + e^{-2\beta J}}_{\sinh^2(\beta h)}} \right]$$

$$h \rightarrow 0$$

$$f(T) = -kT \ln(2 \cosh(\beta J)) \quad (\text{as we found earlier})$$

Clearly $f(T)$ is continuous and differentiable $\forall T > 0$ except at $T = 0$.

We can alternatively write the transfer matrix as a product of the form

$$\hat{T} = \hat{T}_1^{1/2} \hat{T}_2 \hat{T}_1^{1/2} \quad \text{with} \quad \hat{T}_1^\dagger = \hat{T}_1$$

$$\hat{T}_2^\dagger = \hat{T}_2$$

where

$$\hat{T}_1 = e^{\alpha \hat{\sigma}_3}$$

$$\hat{T}_2 = A e^{\mu \hat{\sigma}_1}$$

Since $\det \hat{T} = a^2 - b^2 - c^2$

$$\Rightarrow A = \sqrt{a^2 - b^2 - c^2} = \sqrt{2 \sinh(2\beta J)}$$

After some easy algebra we get

$$e^{-2\mu} = \tanh(\beta J) \Rightarrow \mu = -\frac{1}{2} \ln(\tanh(\beta J))$$

and $\alpha = \frac{1}{2} \ln\left(\frac{a+b}{a-b}\right) = \beta h$

$$\Rightarrow \hat{T}_1 = e^{\beta h \hat{\sigma}_3} \quad \text{and}$$

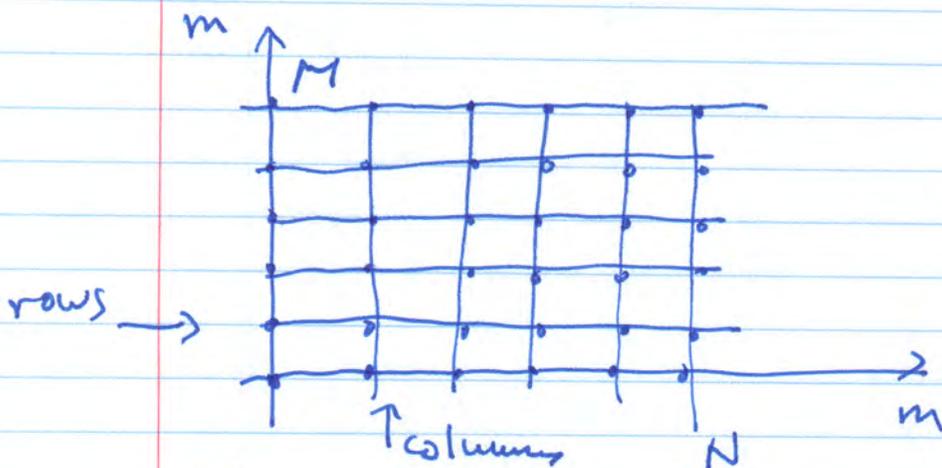
$$\hat{T}_2 = \sqrt{2 \sinh(2\beta J)} e^{-\frac{1}{2} \ln(\tanh(\beta J)) \hat{\sigma}_1}$$

The Transfer Matrix for dimension $d > 1$

We will now use the same idea for $d > 1$. For concreteness we will take $d=2$ and a square lattice with ferromagnetic interactions. We will label the lattice sites with two integers (n, m)

with $n=1, \dots, N$ and $m=1, \dots, M$

The total # of sites is NM . We will assume periodic boundary conditions at least in one direction (say m)



The Hamiltonian now is

$$H(\sigma) = -J \sum_{n,m} \left[\sigma(n,m) \sigma(n+1,m) + \sigma(n,m) \sigma(n,m+1) \right]$$

Thus this looks like an array of 1D chains each being a row of the 2D lattice. The partition function now is

$$Z = \sum_{\{\sigma(n,m)\}} e^{-\beta H[\sigma]}$$

$$\equiv \sum_{\{\sigma(n,m)\}} e^{\beta J \sum_{n,m} (\sigma(n,m) \sigma(n+1,m) + \sigma(n,m) \sigma(n,m+1))}$$

We will choose \underline{m} as the direction of transfer and write

$$Z = \text{tr} \hat{T}^M$$

Now \hat{T} has to act on the configurations of each row. So we can define a ~~the~~ Hilbert space ~~for~~ for each row of dimension 2^N whose states are configurations $|\sigma(1), \dots, \sigma(N)\rangle$ where l ~~table~~ labels site l , etc. Hence the Transfer Matrix will now be $2^N \times 2^N$

Actually if we compare the configurations of spins on the row labelled by n with the configurations on the next row (labelled by $n+1$) they will differ in that some spins are flipped. Thus the changes from row-to-row can be viewed as a "time evolution" of a quantum state of a chain of spins! In this picture the transfer matrix plays a role similar to the evolution operator in quantum mechanics!

We can repeat the procedure used in 1D and write the transfer matrix as a product of matrices. Let \hat{T}_1 and \hat{T}_2 be two $2^N \times 2^N$ matrices. We will choose \hat{T}_1 to be a diagonal matrix in the basis $|\sigma(1), \dots, \sigma(N)\rangle$ and \hat{T}_2

to be off-diagonal and generate spin flips
~~the~~ from one row to the next. So, we
 will encode the intra-row interactions
 in \hat{T}_1 and the inter-row interactions in \hat{T}_2 .

We get (repeating what we did in 1D)

$$\hat{T}_1 = e^{\beta J_1 \sum_n \hat{\sigma}_3(n) \hat{\sigma}_3(n+1)}$$

and

$$\hat{T}_2 = (2 \sinh(2\beta J_2))^{N/2} e^{-\frac{1}{2} \ln(\tanh(\beta J_2)) \sum_{n=1}^N \hat{\sigma}_1(n)}$$

where J_1, J_2 are the couplings in the two directions.

The partition function now is

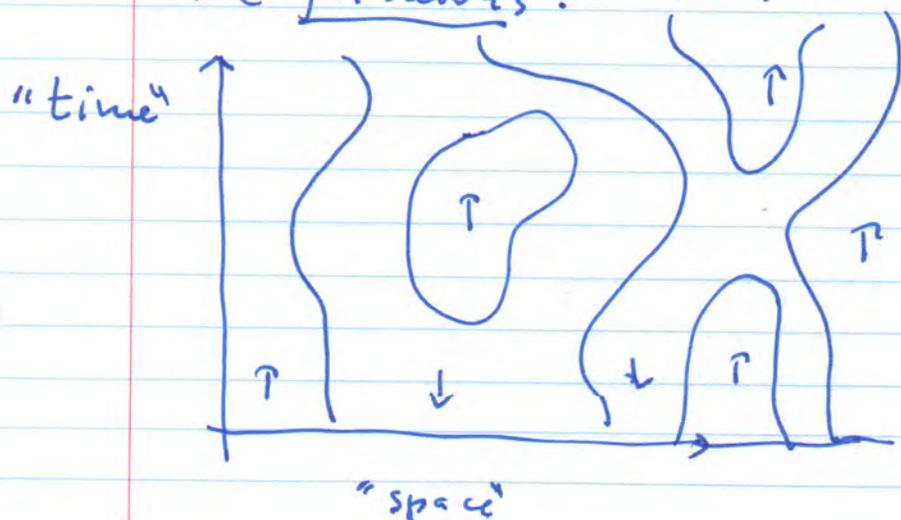
$$Z = \text{tr} \hat{T}^M$$

$$\text{with } \hat{T} = \hat{T}_1^{1/2} \hat{T}_2 \hat{T}_1^{1/2}$$

This procedure can be repeated in all dimensions
 i.e. in $d=3$ ~~the~~ the transfer matrix connects
 planes (instead of rows, etc.)

It turns out that the $d=2$ problem can be solved exactly (Onsager, 1944) by diagonalizing the Transfer Matrix (there are many other ways).

In the picture of the transfer matrix a configuration of the 2D Ising model can be regarded as the "time evolution" (i.e. the changes from row-to-row) of the configuration of the spins of the first ($m=1$) row. Alternatively we can focus on the domain walls as the histories of some "particles" in this discrete space-time. We will see that these particles are fermions. The partition function is then



a sum over all such histories, i.e. a path integral

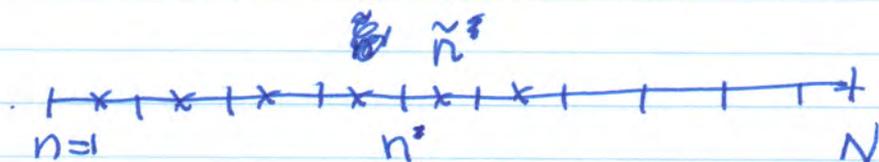
Since \hat{T}_2 flips spins, it creates pairs of domain walls.

Here we have chosen the direction of transfer ("time") to be arbitrarily the vertical direction (label by the row index). We could have also chosen the horizontal direction as the direction of transfer and the partition function should be the same.

Duality

We saw before that the partition function of the 2D Ising model has a symmetry called duality which relates the high temperature expansion to its low temperature expansion. It is very easy (and instructive) to see what this symmetry is in the transfer matrix picture. To this end we will consider a one-dimensional

lattice and its dual lattice (at the midpoints)



The operator that flips all spins ~~also~~ simultaneously is $\hat{R} = \prod_{n=1}^N \hat{\sigma}_1(n)$

Since $\hat{R} |\sigma_1(1), \dots, \sigma(N)\rangle = |-\sigma(1), \dots, -\sigma(N)\rangle$
 \uparrow
 eigenstates of $\hat{\sigma}_3$

However the operator

$$\prod_{n=1}^{n'} \hat{\sigma}_1(n)$$

flips all spins from $n=1$ up to (and including)

$$\prod_{n=1}^{n'} \hat{\sigma}_1(n) |\sigma(1) \dots \sigma(N)\rangle = |-\sigma(1), \dots, -\sigma(n'), \sigma(n'+1), \dots, \sigma(N)\rangle$$

\Rightarrow it creates a kink or domain wall

~~at~~ between $n=n'$ and $n=n'+1$.

I will denote $\prod_{n=1}^{n'} \hat{\sigma}_1(n) \equiv \hat{\tau}_3(n^*)$ where

\tilde{n} is the midpoint between the sites n and $n+1$. This operator satisfies the condition $\hat{\tau}_3^2(\tilde{n}) = 1$ (trivially).

In addition we also have that

$$\hat{\tau}_3(\tilde{n}-1) \hat{\tau}_3(\tilde{n}) = \hat{\sigma}_1(n)$$

i.e. one spin flip creates two domain walls. On the other hand we

can also define ~~the~~ the operator

$$\hat{\tau}_1(\tilde{n}) = \sigma_3(n) \sigma_3(n+1)$$

which also satisfies $\hat{\tau}_1^2(\tilde{n}) = 1$ and anticommutes with $\hat{\tau}_3(\tilde{n})$ since

$\hat{\tau}_3(\tilde{n})$ contains $\hat{\sigma}_1(n)$ which anticommutes with $\hat{\sigma}_3(n)$. $\Rightarrow \{ \hat{\tau}_3(\tilde{n}), \hat{\tau}_1(\tilde{n}') \} = 0$

Hence they satisfy the same algebra as the operators $\hat{\sigma}_1(n)$ and $\hat{\sigma}_3(n)$ (the Pauli algebra). Clearly we now can make the following identifications

$$\sum_{n=1}^N \hat{\sigma}_1(n) \equiv \sum_{\tilde{n}=1}^N \hat{\tau}_3(\tilde{n}) \hat{\tau}_3(\tilde{n}+1)$$

$$\sum_{n=1}^N \hat{\sigma}_3(n) \hat{\sigma}_3(n+1) \equiv \sum_{\tilde{n}=1}^N \hat{\tau}_1(\tilde{n})$$

where we assumed PBC's.

If we now look at the transfer matrix we see that this mapping (or identification) is equivalent to exchanging \hat{T}_1 with \hat{T}_2

provided we define ($K = \beta J$)

$$\tilde{K} \equiv -\frac{1}{2} \ln \tanh K$$

$$\text{and } K = -\frac{1}{2} \ln \tanh \tilde{K}$$

where \tilde{K} is the dual of K . (If we set

$$k_B = 1 \text{ and } J = 1 \Rightarrow K = \frac{1}{T} \text{ and } \tilde{K} = \frac{1}{\tilde{T}})$$

$$\Rightarrow \text{duality requires } \tanh K = e^{-2\tilde{K}}$$

$$\text{Self-duality: } K = \tilde{K} \Leftrightarrow \tanh K = e^{-2K}$$

$$\Rightarrow K_c = \frac{1}{2} \ln(1 + \sqrt{2}) \Rightarrow K_c = \frac{2J}{\ln(1 + \sqrt{2})}$$

(Onsager)

Exact solution of the 2D Ising Model

We will now show that it is possible to diagonalize the transfer matrix. As in the 1D case we will be interested only in its largest eigenvalue since it yields the extensive part of the free energy in the thermodynamic limit. This problem was solved first by Lars Onsager in 1944 using a somewhat different method. Here we will show that the 2D classical problem is equivalent to a 1D quantum system of free fermions. This approach was first developed by T. Schultz, D. Mattis and E. Lieb (Rev. Mod. Phys. 36, 856 (1964)). The Schultz-Mattis-Lieb solution is somewhat involved algebraically. Here we will solve ~~an~~ a physically equivalent problem.

The matrices \hat{T}_1 and \hat{T}_2 do not commute with each other. However if we assume that the coupling βJ_2 is large and βJ_1 is small (while keeping a relation fixed) the problem will simplify.

$$\text{Indeed if } \beta J_2 \gg 1 \Rightarrow \tanh \beta J_2 = \frac{e^{2\beta J_2} - 1}{e^{2\beta J_2} + 1} \\ \approx 1 - 2e^{-2\beta J_2} + \dots$$

$$\text{and } -\frac{1}{2} \ln \tanh(\beta J_2) \approx -\frac{1}{2} \ln(1 - 2e^{-2\beta J_2} + \dots) \\ \approx +e^{-2\beta J_2} + O(e^{-4\beta J_2})$$

$$\Rightarrow \hat{T}_1^{1/2} \hat{T}_2 \hat{T}_1^{1/2} \approx e^{\beta J_1 \sum_n \hat{\sigma}_3(n) \hat{\sigma}_3(n) + e^{-2\beta J_2} \sum_n \hat{\sigma}_1(n)}$$

We now demand that $\beta J_1 = \lambda e^{-2\beta J_2}$ with λ fixed

$$\Rightarrow \beta J_1 \ll 1$$

$$\text{and } \hat{T} = e^{-e^{-2\beta J_2} \hat{H}}$$

$$\hat{H} = - \sum_{n=1}^N \hat{\sigma}_1(n) - \lambda \sum_{n=1}^N \hat{\sigma}_3(n) \hat{\sigma}_3(n)$$

The partition function now becomes

$$Z = \text{tr} \hat{T}^M = \text{tr} e^{- (M e^{-2\beta J_2}) \hat{H}}$$

This problem is then equivalent ~~to~~ to the partition function of the quantum spin chain known as the 1D Ising model in a transverse field at an effective

temperature $\frac{1}{T_{\text{eff}}} = M \underbrace{e^{-2\beta J_2}}_{\text{imaginary time step}}$

The thermodynamic ~~limit~~ limit ($M \rightarrow \infty$)

will then require that $T_{\text{eff}} \rightarrow 0$

Hence the partition function is then reduced to finding the ground state energy of the 1D quantum spin chain.

We noted before that the partition function can be viewed as ~~a set of~~ the histories of domain walls. We will now see that the domain walls are represented

by fermions. We will also see that in this case the number of fermions will not be conserved but that the parity of their number is conserved. This reflects the fact that ~~the~~ domain walls can be created and destroyed in pairs.

A spin system in 1D ~~looks~~ is equivalent to a system of bosons ~~with~~ with hard cores: spin operators commute with each other on $n \neq$ sites but anticommute on the same site (~~if~~ you can flip a spin only once). To relate this problem to a system of fermions we will

introduce the Jordan-Wigner transformation

$$\text{def } \hat{K}(n) = \prod_{m=1}^n (-\hat{\sigma}_m^z) \hat{\sigma}_n^x$$

$$\text{Let } \hat{K}(n) = \prod_{n' \leq n} \left((-1)^{\sum_{n''=1}^{n'} \hat{\sigma}_1(n'')} \right) \\ \equiv (-1)^n \hat{\tau}_3(\tilde{n})$$

be the operator that creates a kink or domain wall at dual site \tilde{n} (between sites n and $n+1$).

Let us now consider the operators

$$\hat{\chi}_1(n) = \hat{K}(n-1) \hat{\sigma}_3(n)$$

$$\hat{\chi}_2(n) = i \hat{K}(n) \hat{\sigma}_3(n)$$

(with $\hat{\chi}_1(1) = \hat{\sigma}_1(1)$, $\hat{\chi}_2(1) = -\hat{\sigma}_2(1)$)

These operators have the properties

$$\hat{\chi}_1^2(n) = 1 \quad \hat{\chi}_2^2(n) = 1 \quad (\forall n)$$

$$\text{and } \{\hat{\chi}_1(n), \hat{\chi}_1(n')\} = 2 \delta_{nn'}$$

$$\{\hat{\chi}_2(n), \hat{\chi}_2(n')\} = 2 \delta_{nn'}$$

$$\{\hat{\chi}_1(n), \hat{\chi}_2(n')\} = 0$$

\Rightarrow these operators are hermitian and

obey anti-commutative relations \Rightarrow Majorana fermions.

We now define

Jordan Wigner

$$\begin{cases} \hat{\Psi}^\dagger(n) = \hat{K}(n-1) \hat{\sigma}^+(n) = \frac{1}{2} (\hat{\chi}_1(n) + i\hat{\chi}_2(n)) \\ \hat{\Psi}(n) = \hat{K}(n-1) \hat{\sigma}^-(n) = \frac{1}{2} (\hat{\chi}_1(n) - i\hat{\chi}_2(n)) \end{cases}$$

(with $\hat{\sigma}^\pm \equiv \frac{1}{2} (\hat{\sigma}_3 \mp i\hat{\sigma}_2)$)

$$\Rightarrow \frac{1}{2} \langle \hat{\Psi}(n), \hat{\Psi}(n') \rangle = \langle \hat{\Psi}^\dagger(n), \hat{\Psi}^\dagger(n') \rangle = 0$$

and $\langle \hat{\Psi}(n), \hat{\Psi}^\dagger(n') \rangle = \delta_{n,n'}$ (fermions!)

It is now easy to show that

$$\hat{\Psi}^\dagger(n) \hat{\Psi}(n) = \frac{1}{2} (1 + \hat{\sigma}_1(n)) \quad (\text{is the fermion \#})$$

$$\begin{aligned} \Rightarrow -\hat{\sigma}_1(n) &= 1 - 2 \hat{\Psi}^\dagger(n) \hat{\Psi}(n) \\ &\equiv e^{i\pi \hat{\Psi}^\dagger(n) \hat{\Psi}(n)} \end{aligned}$$

$$\Rightarrow \hat{\sigma}^+(n) = e^{i\pi \sum_{n' < n} \hat{\Psi}^\dagger(n') \hat{\Psi}(n')} \hat{\Psi}^\dagger(n)$$

$$\hat{\sigma}^-(n) = e^{i\pi \sum_{n' < n} \hat{\Psi}^\dagger(n') \hat{\Psi}(n')} \hat{\Psi}(n)$$

and

$$\hat{\sigma}_3(n) = e^{i\pi \sum_{n' < n} \hat{\psi}^\dagger(n') \hat{\psi}(n')} (\hat{\psi}^\dagger(n) + \hat{\psi}(n))$$

$$\hat{\sigma}_2(n) = e^{i\pi \sum_{n' < n} \hat{\psi}^\dagger(n') \hat{\psi}(n')} \frac{1}{i} (-\hat{\psi}^\dagger(n) + \hat{\psi}(n))$$

Boundary conditions

If $\hat{\sigma}_3(N+1) = \hat{\sigma}_3(1)$ (and same with $\hat{\sigma}_1$)

$$\Rightarrow \hat{\psi}(N+1) = \hat{Q} \hat{\psi}(1)$$

where $\hat{Q} = e^{i\pi \sum_{n=1}^N \hat{\psi}^\dagger(n) \hat{\psi}(n)}$

$$\equiv \prod_{n=1}^N \hat{\sigma}_1(n) \quad (\text{if } N \text{ is even})$$

However \hat{Q} is the spin flip operator

$$[\hat{Q}, \hat{H}] = 0 \quad \text{and hence can be}$$

diagonalized simultaneously with \hat{H} .

The transformed Hamiltonian is:

$$\hat{H} = +N + \sum_{n=1}^N 2 \hat{\psi}^\dagger(n) \hat{\psi}(n)$$

$$- \lambda \sum_{n=1} (\hat{\psi}^\dagger(n) - \hat{\psi}(n)) (\hat{\psi}^\dagger(n+1) + \hat{\psi}(n+1))$$

+ boundary term,

$$\begin{aligned} \text{boundary term} &= -\lambda \hat{\sigma}_3(N) \hat{\sigma}_3(1) \\ &= -\lambda \hat{Q} (\hat{\Psi}^\dagger(N) - \hat{\Psi}^\dagger(1)) \frac{(\hat{\Psi}^\dagger(N))}{(\hat{\Psi}^\dagger(1) + \hat{\Psi}(1))} \end{aligned}$$

$$\hat{Q} = e^{i\pi N_F}$$

$$N_F = \sum_{n=1}^N \hat{\Psi}^\dagger(n) \hat{\Psi}(n) \quad \text{is the \# of fermions.}$$

\Rightarrow a system of spins with PBC's

~~has~~ is equivalent to a system of fermions whose boundary ~~conditions~~ conditions are

periodic if $N_F = \text{even}$ and antiperiodic if N_F is odd. It will turn out that

$$\begin{array}{ccc} E_0^- & > & E_0^+ \\ \uparrow & & \uparrow \\ \text{antiperiodic} & & \text{periodic.} \end{array}$$

Notice that this Hamiltonian does not conserve N_F but it conserves the parity of N_F . It is similar to the pairing Hamiltonian of the BCS theory of superconductors

Diagonalization

To diagonalize the Hamiltonian we will use its translation invariance and go to Fourier (momentum) space.

$$\hat{\Psi}(n) = \frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} e^{i 2\pi \frac{kn}{N}} \tilde{a}(k)$$

$$\tilde{a}(k) = \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} e^{-i 2\pi \frac{kn}{N}} \hat{\Psi}(n)$$

(we shifted the origin)

Such that $\{\tilde{a}(k), \tilde{a}^\dagger(k')\} = N \delta_{k,k'}$
and $\{\tilde{a}(k), \tilde{a}(k')\} = \{\tilde{a}^\dagger(k), \tilde{a}^\dagger(k')\} = 0$

Also $\frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} e^{-i 2\pi \frac{kn}{N}} = \delta_{n,0}$

$$\frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} e^{i 2\pi \frac{kn}{N}} = \delta_{k,0}$$

Thermodynamic limit: $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} \frac{L}{2\pi} \delta_{k,0} \equiv \delta(k)$$

\hookrightarrow Dirac

δ -function

$$2\pi \delta(k) = \lim_{N \rightarrow \infty} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} e^{ikn}$$

$L = Na$
 \uparrow
Spacing

notice that

$$2\pi\delta(0) = N \Rightarrow N = \frac{\delta(0)}{2\pi}$$

likewise $\tilde{a}(k) \equiv \hat{a}(k)$
 $\int_{\text{continuum}}$

$$\text{and } \{ \hat{a}(k), \hat{a}^\dagger(k') \}_{|k| \leq \pi} = 2\pi\delta(k)$$

We also find

$$\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{\Psi}^\dagger(n) \hat{\Psi}(n) \stackrel{N \rightarrow \infty}{=} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \hat{a}^\dagger(k) \hat{a}(k)$$

$$\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{\Psi}^\dagger(n) \hat{\Psi}(n+1) \stackrel{N \rightarrow \infty}{=} \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{\pm ik} \hat{a}^\dagger(k) \hat{a}(k)$$

$$\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{\Psi}^\dagger(n) \hat{\Psi}^\dagger(n+1) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ik} \hat{a}^\dagger(k) \hat{a}^\dagger(-k)$$

$$\sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} \tilde{\Psi}(n) \tilde{\Psi}(n+1) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ik} \hat{a}(k) \hat{a}(-k)$$

\Rightarrow Collecting terms we get a "pairing Hamiltonian"

$$H = -N + 2 \int_0^\pi \frac{dk}{2\pi} (1 + \lambda \cos k) (\hat{a}^\dagger(k) \hat{a}(k) + \hat{a}^\dagger(-k) \hat{a}(-k)) + \int_{-\pi}^{\pi} \frac{dk}{2\pi} 2\lambda \sin k i (\hat{a}^\dagger(k) \hat{a}^\dagger(-k) + \hat{a}(k) \hat{a}(-k))$$

We can now diagonalize this Hamiltonian by means of a Bogoliubov transformation.

$$\hat{a}(k) = u(k) \hat{\eta}(k) + i v(k) \hat{\eta}^\dagger(-k)$$

$$\hat{a}(-k) = u(k) \hat{\eta}(-k) + i v(k) \hat{\eta}^\dagger(k)$$

where $u(k)$ and $v(k)$ are real functions

$$\begin{aligned} \hat{\eta}(k) &= u(k) \hat{a}(k) + i v(k) \hat{a}^\dagger(-k) \\ \hat{\eta}(-k) &= u(k) \hat{a}(-k) - i v(k) \hat{a}^\dagger(k) \end{aligned}$$

This is a canonical transf. since

$$\begin{aligned} \{ \hat{a}(k), \hat{a}^\dagger(k') \} &= 2\pi \delta(k-k') \Rightarrow \{ \hat{\eta}(k), \hat{\eta}^\dagger(k') \} \\ &= 2\pi \delta_{kk'} \end{aligned}$$

of and only of $u^2(k) + v^2(k) = 1$

⇒ We parametrize the transformation by an angle $\theta(k)$ (and $\theta(-k)$)

$$u(k) = \cos \theta(k)$$

$$v(k) = \sin \theta(k)$$

We will choose $\theta(k)$ / the fermion non-conserving terms cancel.

$$H_k = \alpha(k) (\hat{a}^\dagger(k) \hat{a}(k) + \hat{a}^\dagger(-k) \hat{a}(-k)) + i\beta(k) (\hat{a}^\dagger(k) \hat{a}^\dagger(-k) + \hat{a}(k) \hat{a}(-k))$$

$$\alpha(k) = 2(1 + \lambda \cos k)$$

$$\beta(k) = 2\lambda \sin k$$

If we choose $\tan 2\theta(k) = \frac{\beta(k)}{\alpha(k)}$

→ the fermion non-commuting terms cancel

$$\tan \theta(k) = \frac{\lambda \sin k}{1 + 2\lambda \cos k}$$

and

$$H = \int_0^\pi \frac{dk}{2\pi} \omega(k) (\hat{\eta}^\dagger(k) \hat{\eta}(k) + \hat{\eta}^\dagger(-k) \hat{\eta}(-k)) + \epsilon_0 N$$

$$\epsilon_0 = -1 + \int_0^\pi \frac{dk}{2\pi} [4\lambda \sin k \sin^2 \theta(k) - 2 \sin(2\theta(k)) (1 + \lambda \cos k)]$$

$$\omega(k) = 2 [(1 + \lambda \cos k) \cos(2\theta(k)) + \lambda \sin k \sin(2\theta(k))]$$

$$\omega(k) \geq 0 \text{ if } \cos(2\theta(k)) \geq 0 \quad \text{and} \quad \sin(2\theta(k)) \geq 0 \quad \text{313}$$

$$\Rightarrow \text{sgn} \cos(2\theta(k)) = \text{sgn} \alpha(k)$$

$$\text{sgn} \sin(2\theta(k)) = \text{sgn} \beta(k)$$

$$\Rightarrow \omega(k) = |\alpha(k)| |\cos 2\theta(k)| + |\beta(k)| |\sin(2\theta(k))|$$

$$\Rightarrow \omega(k) = 2 \sqrt{1 + \lambda^2 + 2\lambda \cos k}$$

\Rightarrow ground state $|0\rangle$

$$\hat{\eta}(k)|0\rangle = \hat{\eta}(-k)|0\rangle = 0$$

and

$$E_0(\lambda) = -1 + \int_0^\pi \frac{dk}{2\pi} \left[4\lambda \sin k \sin^2(2\theta(k)) - 2 \sin(2\theta(k)) (1 + \lambda \cos k) \right]$$

$$\Rightarrow E_0(\lambda) \equiv - \int_0^\pi \frac{dk}{2\pi} \omega(k) = -\frac{1}{2} \int_{-\pi}^\pi \frac{dk}{2\pi} \omega(k) < 0$$

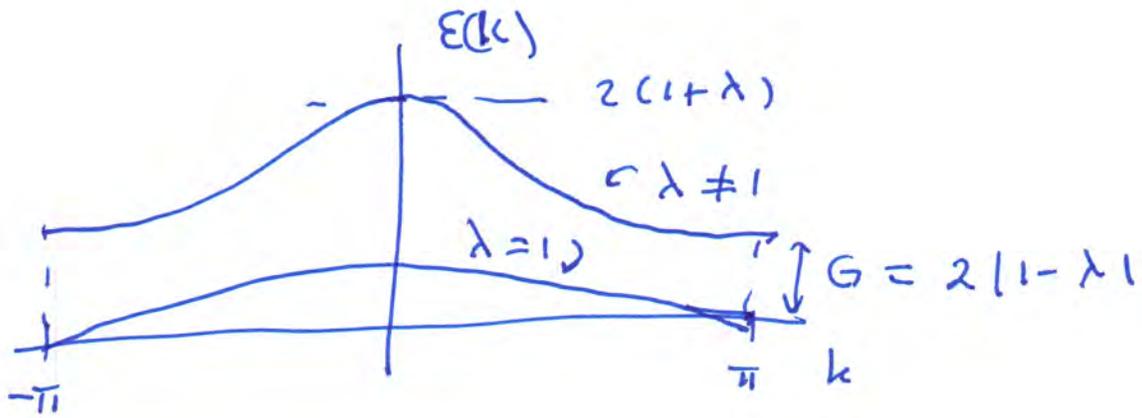
First excited state

$$|\pm k\rangle = \hat{\eta}^\dagger(\pm k)|0\rangle$$

Excitation energy $E(k) = \omega(k) > 0$

Energy gap

$$G(\lambda) = \min_{|k| \leq \pi} E(k)$$



The gap closes as $\lambda \rightarrow 1$.

$$E(k) = \omega(k) = 2 \sqrt{1 + \lambda^2 + 2\lambda \cos k} = 2 \sqrt{(1-\lambda)^2 + 4\lambda \cos^2(\frac{k}{2})}$$

Since

$$Z = \text{tr} \hat{T}^N = \lim_{\beta \rightarrow \infty} \text{tr} e^{-\beta H}$$

and $Z = e^{-N N f}$

$\Rightarrow f \approx E_0(\lambda)$ is the free energy of the 2D classical problem!

and $\lambda - 1 \approx \frac{T - T_c}{T_c}$ in the classical 2D problem.

$$E_0(\lambda) = -2 \int_0^\pi \frac{dk}{2\pi} \sqrt{(1+\lambda)^2 - 4\lambda^2 \sin^2(\frac{k}{2})}$$

$$= -\frac{2}{\pi} |1+\lambda| \int_0^{\frac{\pi}{2}} dk \sqrt{1 - (1-\gamma^2) \sin^2 k}$$

$$|\gamma| = \left| \frac{1-\lambda}{1+\lambda} \right|$$

$$\Rightarrow \epsilon_0(\lambda) = -\frac{2}{\pi} |1+\lambda| E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right)$$

↑
elliptic integral.

$$\text{If } \lambda \rightarrow 1 \Rightarrow \gamma \rightarrow 0$$

$$\text{and } E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right) \underset{\gamma \rightarrow 0}{\approx} 1 + \frac{\gamma^2}{4} \left(\ln \frac{16}{\gamma^2} - 1 \right) + \dots$$

$$\Rightarrow \epsilon_0(\lambda) = \epsilon_0^{\text{sing}}(\lambda) + \epsilon_0^{\text{reg}}(\lambda)$$

where

$$\epsilon_0^{\text{sing}}(\lambda) = -\frac{4}{\pi} \left[1 + \frac{(1-\lambda)^2}{16} \left(\ln \left(\frac{64}{1-\lambda} \right)^2 - 1 \right) + \dots \right]$$

$$\Rightarrow \epsilon_0^{\text{sing}}\left(\frac{T-T_c}{T_c}\right) = -\frac{4}{\pi} \left[1 + \frac{1}{8} \left(\frac{T-T_c}{T_c}\right)^2 \left(\ln \left(\frac{8}{\left|\frac{T-T_c}{T_c}\right|} \right) - \frac{1}{2} \right) \right]$$

The specific heat ~~ε~~ has a singular term!

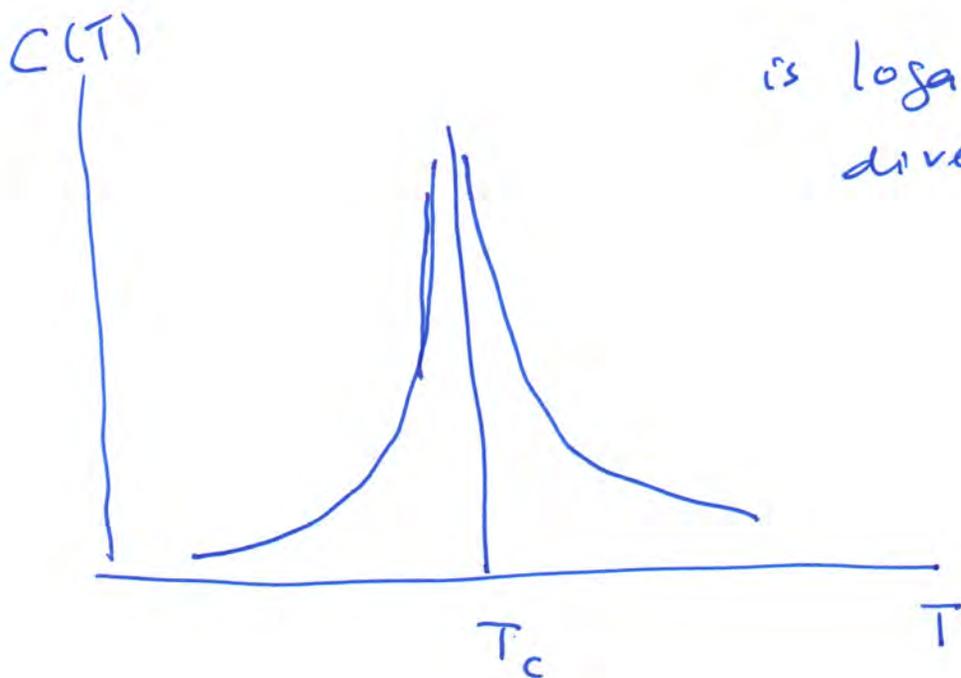
$$C_{\text{sing}} = - \frac{\partial^2 \epsilon_0^{\text{sing}}}{\partial t^2} = + \frac{1}{2\pi} \ln \left(\frac{8}{|t|} \right) - \frac{3}{4\pi} + \dots$$

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$$t = \frac{T - T_c}{T_c}$$

$$\rightarrow C\left(\frac{T - T_c}{T}\right) \approx \frac{1}{2\pi} \ln \left| \frac{8 T_c}{T - T_c} \right| + \dots$$

is logarithmically divergent at T_c !



Gap: $G(\lambda) = |\lambda - \lambda_c|^{v_\sigma}$

$\lambda_c = 1$ and $v_\sigma = 1$

$\Rightarrow \xi\left(\frac{T - T_c}{T_c}\right) = \frac{\tau \text{ const.}}{\left|\frac{T - T_c}{T_c}\right|^v}$
 (correlation length)

$v = 1$!

How about correlation functions?

A more involved calculation yields an expression for the magnetization

$$M(\lambda) = \text{const. } |\lambda - 1|^\beta \quad \beta = \frac{1}{8}$$

$$\Rightarrow M(T) = \text{const. } (T_c - T)^{1/8} \quad (\text{P.N. Yang})$$

and the spin correlation function at T_c

↙ 2D classical ↘ 1D quantum

$$G(R) = \langle \sigma(0) \sigma(R) \rangle \equiv \langle 0 | \sigma_z(0) \sigma_z(R) | 0 \rangle$$

$$G(R) \approx \frac{\text{const.}}{R^\eta} \quad \eta = \frac{1}{4}$$

These results look very different from mean field theory ($\beta = \frac{1}{2}$) and Landau theory ($\eta = 0 \frac{1}{2}$ as $d=2$) Similarly we found $\nu = 1$ whereas the Landau theory predicts $\nu = \frac{1}{2}$.

Scaling and the Renormalization Group

In the past few lectures we discussed the properties of a system at a critical point using two approaches: (a) mean-field theory (and Landau theory) and (b) exact solutions.

Although the results were different they agreed on the following. As a continuous transition at T_c is approached, $T \rightarrow T_c$ (or

$$t = \frac{T - T_c}{T_c}, t \rightarrow 0) \text{ thermodynamic quantities}$$

such as the magnetization (the order parameter) and the magnetic susceptibility, the heat capacity (or the specific heat) obey scaling

laws i.e.

magnetization
 $(h=0) \quad m(t, 0) = \underset{\substack{\uparrow \\ \text{constant}}}{A} (-t)^\beta$

($\beta = \frac{1}{2}$ in mean-field theory)
 $\frac{1}{8}$ in the 2D Ising model)

Susceptibility:

$$\chi = \left. \frac{\partial m}{\partial h} \right|_{h=0} \sim \frac{\text{const.}}{|t|^\gamma}$$

($\gamma = 1$ in Landau theory)

Specific heat ($h=0$)

$$C(t,0) \sim \frac{\text{const}}{|t|^\alpha} + \text{non-singular terms.}$$

in mean-field
Landau theory
($\alpha=0$
 $\alpha=2-\frac{d}{2}$ Landau)

Correlation function @ T_c

$$G(R) = \langle \sigma(R) \sigma(0) \rangle \sim \frac{\#}{R^{d-2+\eta}} \quad (\eta: \text{anomalous dimension})$$

($\eta=0$ Landau theory
 $\eta = \frac{1}{4}$ 2d Ising model)

Correlation Length ($h=0$)

$$\xi(t) \sim \frac{\#}{|t|^\nu}$$

($\nu = \frac{1}{2}$ Landau theory
 $\nu = 1$ 2D Ising)

Magnetization @ T_c

$$m \sim h^{1/\delta}$$

($\delta=3$ Landau theory)

Q1: How do we explain these power-laws?

Q2: How do we explain that the exponents are

universal? (i.e. depend only on dimension and symmetry and not on microscopic details while T_c depends on everything)

The renormalization group is a theory of critical phenomena constructed ~~to~~ to explain ~~the~~ (and predict) the existence of universal scaling laws for systems close enough to a critical point, i.e. a continuous phase transition. This is a subject of semester-long courses. Here we will just give some of the flavor of these ideas. (K. G. Wilson, L. P. Kadanoff)

The reason universal phenomena is observed near a continuous phase transition is that as $T \rightarrow T_c$ the correlation length $\xi \rightarrow \infty$ (in lattice units) (in the thermodynamic limit). Hence in this limit the underlying lattice system can be replaced by an effective continuum (as in hydrodynamics or as in QFT).

Since ξ measures the characteristic length scale of fluctuations, as $\xi \rightarrow \infty$ we get fluctuations

on all length scales. Hence at a critical point the system should become scale invariant (on length scales long compared with the lattice constant a_0).

The scale-invariant behavior (or, more generally, the scaling behavior) should then be expressed in terms of homogeneous functions.

Homogeneous function: $f(x)$ is homogeneous

of degree k (not necessarily an integer) if

$$f(\lambda x) = \lambda^k f(x) \quad (\lambda \in \mathbb{R}^+)$$

The correlation function $G(R)$ ^(@ T_c) is an example of a homogeneous function

$$G(\lambda R) = \lambda^{-(d-2+\eta)} G(R)$$

$k = d-2+\eta$ in this case. We can interpret

this result by saying that the units of the order parameter field are $L^{-(d-2+\eta)/2}$

scaling dimension: $\Delta = \frac{d-2+\eta}{2} \Rightarrow \eta$: anomalous dimension.

In the Landau ~~the~~ theory $\Delta = \frac{d-2}{2}$. This can be seen from our calculation of $G(k)$ and from Landau - Ginzburg theory which postulates that the free energy for a slowly position-dependent magnetization is

$$F = \int d^d x \left[\frac{\kappa}{2} (\nabla m)^2 + a(t) m^2 + b m^4 - h m \right]$$

$a(t) = \text{const} \times \frac{T-T_c}{T} \equiv \text{const} \times t$; κ is the stiffness of the order parameter m .

Dimensionally: $1 = L^d L^{-2} [m]^2$

$\uparrow \quad \uparrow \quad \uparrow$
 units phase space derivatives

$$\Rightarrow [m] = L^{-(d-2)/2} \quad (\text{Landau})$$

$$\Rightarrow \langle m(0) m(r) \rangle \sim \frac{\#}{r^{d-2}} \quad (\text{Landau})$$

$q(k) \sim \xi^{-2}$ since it has units of L^{-2}

$$\Rightarrow \xi(t) \sim \frac{1}{\sqrt{|a|}} = \frac{\text{const}}{|t|^{1/2}} \quad \nu = \frac{1}{2} \quad (\text{Landau})$$

More generally the free energy of the equilibrium state at temperature T (close to T_c) and magnetic field h ("small") should contain a singular piece $f_s(T, h)$

$$f(T, h) = f_s(T, h) + f_{reg}(T, h)$$

where f_s is continuous but not analytic.

Since the free energy density must have units of $(\text{volume})^{-1}$ we can conjecture that the singular part f_s must be a homogeneous function $f_s(t, h)$ ($t \equiv \frac{T-T_c}{T_c}$) such that (d : dimension)

$$f_s(\lambda^d t, \lambda^{\delta} h) = \lambda^d f_s(t, h) \quad (1)$$

(Widom scaling). This assumption implies that all the observables have a scaling behavior (near the critical point).

If we set $h=0$ we see that (1) implies that the specific heat $C(t) \sim \frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha} + \text{reg. terms}$

To see this we differentiate twice $f_s(\lambda^p t)$

$$\lambda^{2p} f_s''(\lambda^p t) = \lambda^d f_s''(t)$$

choose $\lambda = t^{-\frac{1}{p}}$

$$\Rightarrow C(t) = \lambda_p^{2p-d} f_s''(1)$$

$$= t^{-\left(2-\frac{d}{p}\right)} f_s''(1)$$

$$\boxed{\alpha = 2 - \frac{d}{p}}$$

Likewise the correlation length $\xi(t)$ has

units of length $\Rightarrow \xi(\lambda^p t) = \lambda^{-1} \xi(t)$

$$\Rightarrow \lambda = t^{-\frac{1}{p}} \Rightarrow \xi(t) = \lambda \xi(1) = t^{\frac{1}{p}} \xi(1)$$

$$\Rightarrow \boxed{\nu = \frac{1}{p}} \Rightarrow \boxed{\alpha = 2 - d\nu} \quad \text{universal scaling relation}$$

Similarly, since $m(t, h) = \frac{\partial f}{\partial h}$

$$\Rightarrow \lambda^g m(\lambda^p t, \lambda^g h) = \lambda^d m(t, h)$$

$$\Rightarrow m(t, h) = \lambda^{g-d} m(\lambda^p t, \lambda^g h)$$

\Rightarrow ① $h=0$ $t < 0 \Rightarrow$ spontaneous magnetization

$$m(t, 0) = \lambda^{g-d} m(\lambda^p t, 0)$$

$$\lambda^p = t^{-1} \Rightarrow m(t, 0) = t^{+\frac{1}{p}(d-g)} m(1, 0)$$

$$\Rightarrow \beta = \frac{1}{p} (d - \delta) = \nu (d - \delta)$$

(2) $t=0$ (@ T_c) and $h \neq 0$

$$\lambda^\delta m(\lambda^\delta h) = \lambda^d m(h)$$

$$\Rightarrow m(h) = \lambda^{\delta-d} m(\lambda^\delta h)$$

$$\lambda = h^{-1/\delta}$$

$$\Rightarrow m(h) = h^{-\frac{1}{\delta}(d-\delta)} m(1)$$

$$\Rightarrow m(h) = \# |h|^{-\frac{1}{\delta}(d-\delta)} \text{sgn}(h) \quad (@ T_c)$$

$$\frac{1}{\delta} = \frac{d-\delta}{\delta}$$

On the other hand

$$X = \int_{\mathbb{R}^d} a(r) \approx \int_{a_0 \leq |r| \leq \xi} d^d r \frac{\#}{r^{d-2+\eta}}$$

$$X_{\text{sing}}(t) \sim \# \int_{a_0}^{\xi(t)} dr \frac{r^{d-1}}{r^{d-2+\eta}} \quad (\text{singular part})$$

$$= \# \xi^{2-\eta} = \# |t|^{-\nu(2-\eta)}$$

$$\Rightarrow \boxed{\gamma = \nu(2-\eta)}$$

and

$$\lambda^{2g} X(\lambda^p t) = \lambda^d X(t)$$

$$X(t) = \lambda^{2g-d} X(\lambda^p t)$$

$$\lambda = t^{-1/p}$$

$$X(t) = t^{-\frac{1}{p}(2g-d)} X(1)$$

$$X(t) = t^{-\delta} X(1)$$

$$\delta = \frac{2g-d}{p} = \nu(2g-d)$$

$$\Rightarrow \nu(2g-d) = \nu(2-\eta)$$

$$g = \frac{d+2-\eta}{2}$$

$$\Rightarrow \delta = \frac{g}{d-g} = \frac{d+2-\eta}{d-2+\eta}$$

and
$$\beta = \frac{\nu}{2} (d-2+\eta)$$

where we recognize the scaling dimension

(or fractal)
$$\Delta = \frac{1}{2} (d-2+\eta)$$

This result can be interpreted as saying

that as $\xi \rightarrow \infty$ (as $T \rightarrow T_c^-$) the magnetization vanishes as

$$m \sim \xi^{-\Delta} \sim t^{\nu\Delta} \Leftrightarrow \beta = \nu\Delta$$

Q: How does scale-invariance arise? To understand this we will discuss the main ideas of the renormalization group (RG).

Let us consider again an Ising Model (d is d -dimensional). The partition function

$$Z = \sum_{[\sigma]} e^{-\beta H[\sigma]}$$

has contributions from all 2^N configurations ($N = \#$ sites). As $T \rightarrow T_c$ we expect that

the important configurations should vary

slowly over short length scales. To construct

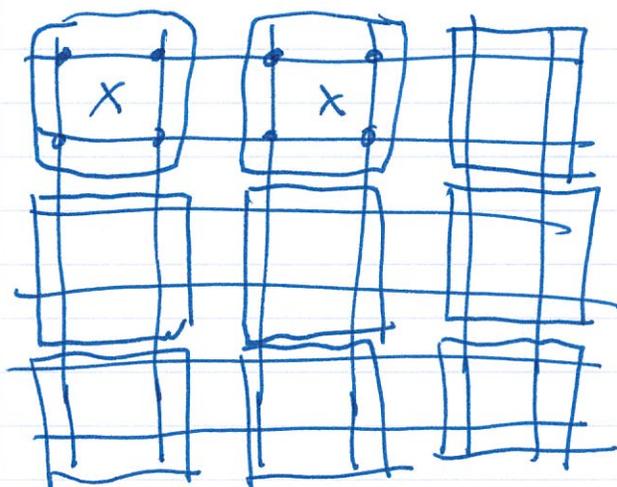
a hydrodynamic theory we will attempt

to coarse-grain the configurations by

performing a sum over configurations that

vary over short scales first.

In order to do that we will define variables that are averaged over short scales, i.e. we will attempt to split the configuration into a fast component and a slow component. Let us imagine dividing the lattice into blocks of linear size b



Let A be a block and $i \in A$ the sites in the block A . We can define ~~our~~ a block spin $\mu(A)$

$$\mu(A) = \frac{\sum_{i \in A} \sigma(i)}{\left\| \sum_{i \in A} \sigma(i) \right\|}$$

which is normalized $\|\mu(A)\|^2 = 1$. This definition is arbitrary and there are many ways of doing this.

Let $H[\sigma]$ be the Hamiltonian defined for a system with lattice spacing a .

We will now write an effective Hamiltonian for the block spin variables $[\mu]$. We define the block spin transformation:

$$T[\mu|\sigma] \quad \left(\text{which in the present choice is } T[\mu|\sigma] = \prod_A \delta\left(\mu(A) - \frac{\sum_{i \in A} \sigma(i)}{\left\| \sum_{i \in A} \sigma(i) \right\|} \right) \right)$$

such that

$$\sum_{[\mu]} T[\mu|\sigma] = 1$$

$$\Rightarrow Z = \sum_{[\sigma]} e^{-H[\sigma]} = \sum_{[\sigma][\mu]} T[\mu|\sigma] e^{-H[\sigma]}$$

$$= \sum_{[\mu]} \left(\sum_{[\sigma]} T[\mu|\sigma] e^{-H[\sigma]} \right)$$

$$= \sum_{[\mu]} e^{-H_{\text{eff}}[\mu]}$$

where

$$e^{-H_{\text{eff}}[\mu]} = \sum_{[\sigma]} T[\mu|\sigma] e^{-H[\sigma]} \Rightarrow Z = \sum_{[\mu]} e^{-H_{\text{eff}}[\mu]}$$

Under this transformation

$$\langle \mu(R) \mu(R') \rangle_{H_{\text{eff}}} = \left\langle \underbrace{\sum_{i \in A(R)} \sigma(i)}_{\parallel \parallel} \underbrace{\sum_{i \in A(R')} \sigma(i)}_{\parallel \parallel} \right\rangle_H$$

\uparrow lattice spacing $b=a$ \uparrow lattice spacing a

If H has correlation length $\xi \Rightarrow$ the effective H_{eff} has $\xi_{\text{eff}} \approx \frac{\xi}{b}$

If we iterate this process n times \Rightarrow

$$\xi = b^n \xi_{\text{eff}} \quad \text{and we stop when } \xi_{\text{eff}} \sim a$$

\Rightarrow as $T \rightarrow T_c$ we will need $n \rightarrow \infty$ since $\xi \rightarrow \infty$.

Let us now assume that we can write the coarse grained H_{eff} as a sum of local operators $\{O_\alpha[\sigma]\}$

$$H = \sum_{\alpha} g_{\alpha} O_{\alpha}[\sigma]$$

\uparrow couplings

\swarrow in terms of the block spins

$$\Rightarrow H_{\text{eff}} = \sum_{\alpha} g_{\alpha}^{\text{eff}}(b) O_{\alpha}[\mu]$$

Let me assume we were clever enough to choose the set $\{O_\alpha(b)\}$ such that after a block transformation (followed by a rescaling to restore the original value of the lattice spacing)

$$\# \quad g_\alpha^{eff}(b) = b^{\gamma_\alpha} g_\alpha \quad \text{where } \gamma_\alpha \text{ are some numbers.}$$

Then as we iterate this process the operators with $\gamma_\alpha < 0$ will be suppressed ~~from~~ from H_{eff} . Similarly the contributions from operators with $\gamma_\alpha > 0$ will increase. We will call the operators with $\gamma_\alpha < 0$ irrelevant and the operators with $\gamma_\alpha > 0$ relevant. Operators with $\gamma_\alpha = 0$ are called marginal and their contribution does not change. Thus, after many actions of this procedure, which is called a renormalization group transformation, the effective Hamiltonian simplifies to one

in which all irrelevant operators (with $Y_a < 0$) are absent. If we change the control parameters of the system so that the relevant operators (with $Y_a > 0$) are fine-tuned to zero, the resulting Hamiltonian ~~is~~ will not change under the action of the RG. This effective Hamiltonian, that we will denote by H^* , is a fixed point of the RG. In

particular at a fixed point of the RG the system becomes invariant under scale transformations: it is a scale-invariant system and as such it looks the same at all scales. ~~Thus~~ At a fixed point the correlation length cannot be finite. Thus we have two choices:

- (a) fixed points with $\xi = 0$ (i.e. the lattice spacing) and (b) fixed points with $\xi = \infty$.

Fixed points with $\xi = 0$ ^{are stable FPs} represent stable phases of matter. At such fixed points all perturbations are irrelevant, up to some critical value of a coupling constant (e.g. T_c). On the other hand fixed points with $\xi = \infty$ control critical points are unstable FPs and represent phase transitions.

To summarize: an RG transformation consists of (a) a block spin transformation that eliminates a finite fraction of degrees of freedom and (b) a rescaling of lengths. RG: it is a group in that two transformations yield another transformation.

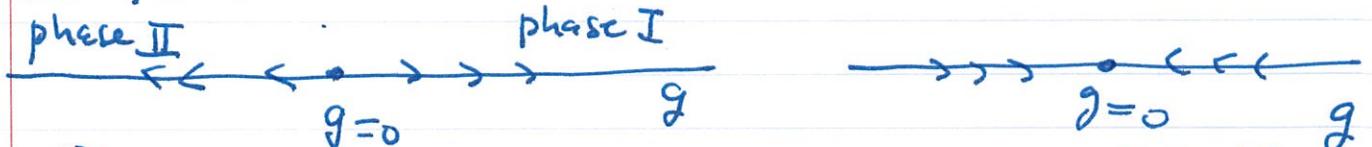
Examples.

(a) Problem with one perturbation

$$H = H^* + \int dx g \mathcal{O}[\psi]$$

$$\Rightarrow H' = H^* + \int dx g b^y \mathcal{O}[\psi]$$

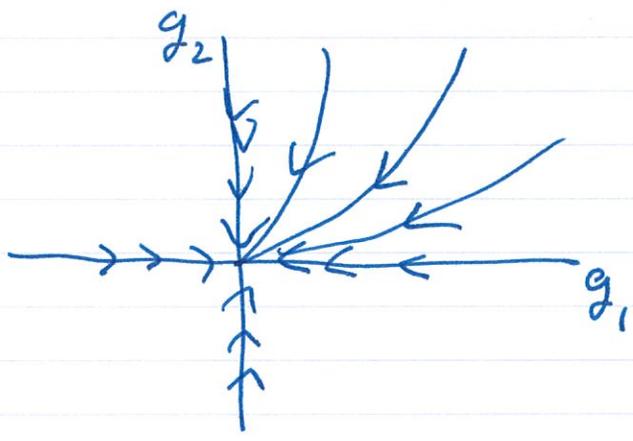
RG flows



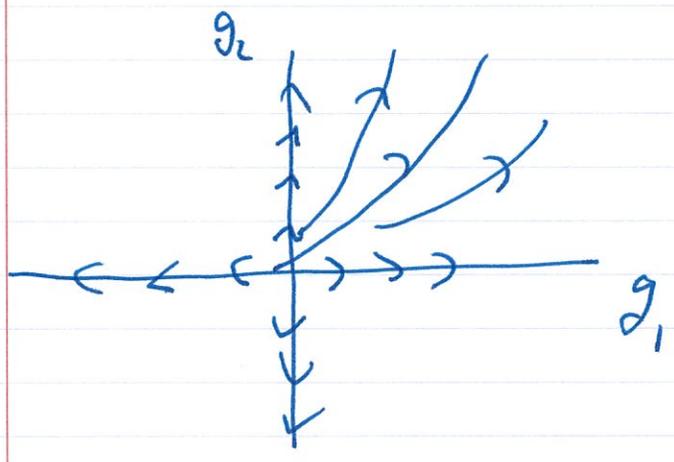
(a) $y > 0$ unstable fixed point

($y < 0$) stable FP.

Other cases: two operators

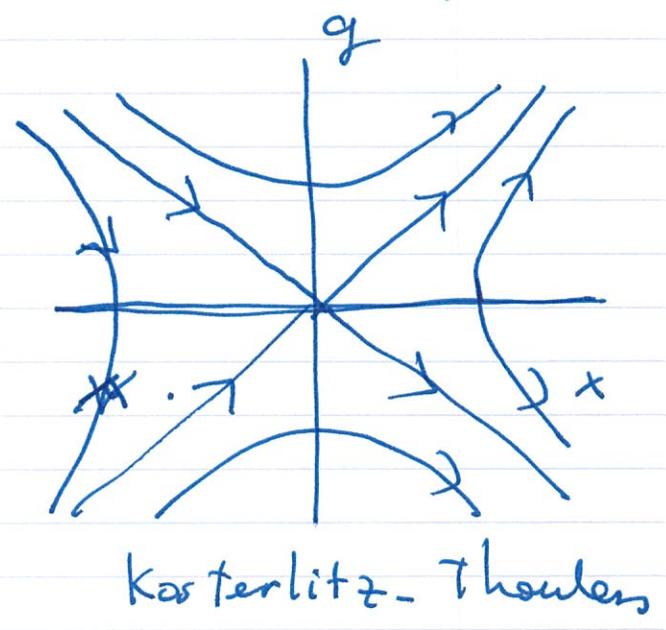
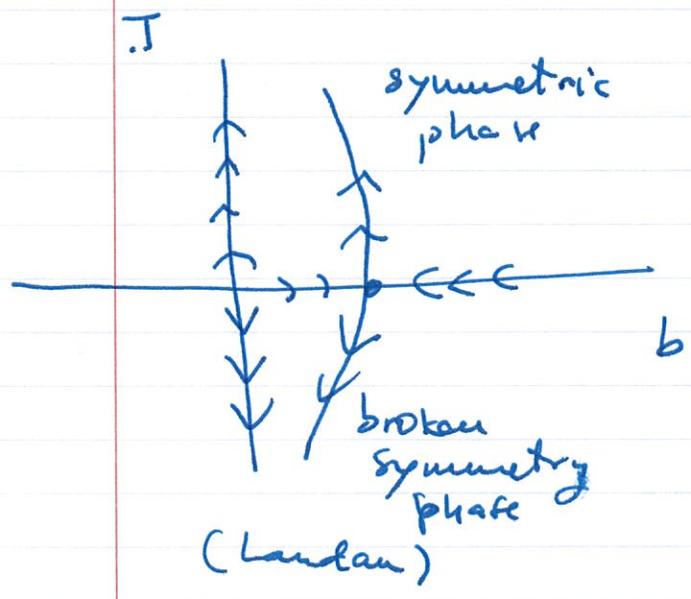


two irrelevant operators (stable)



two relevant operators (unstable)

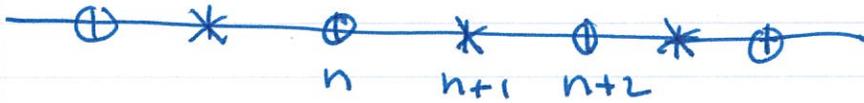
One relevant and one irrelevant ops.



Kosterlitz-Thouless

Simple block spin Transformations

Decimation in 1D Ising



$$Z = \sum_{[\sigma]} e^{\sum_{n=1}^N \beta \sigma_n \sigma_{n+1}} \quad (\text{PBC's})$$

decimation: $\mu_r = \sigma_{2r}$ and integrate out $\{\sigma_{2r+1}\}$

$$\sum_{\sigma_{2r+1} = \pm 1} e^{\beta \sigma_{2r+1} (\sigma_{2r} + \sigma_{2r+2})} = 2 \cosh \beta (\mu_r + \mu_{r+1})$$

$$= e^{\alpha + \beta' \mu_r \mu_{r+1}}$$

$$\begin{cases} e^{\alpha + \beta'} = 2 \cosh 2\beta \\ e^{\alpha - \beta'} = 2 \end{cases}$$

$$\beta' = \frac{1}{2} \ln \cosh(2\beta)$$

$$\alpha = \frac{1}{2} \ln(4 \cosh(2\beta))$$

$$\Rightarrow H' = \text{const.} + \frac{N}{4} \ln(4 \cosh(2\beta)) + \beta' \sum_{r=1}^{N/2} \mu_r \mu_{r+1}$$

In this case the Hamiltonian has the same form \mathbb{E} up to a shift of the constant terms (identity operator) and a renormalization of $\beta = \frac{1}{T} \rightarrow \beta' = \frac{1}{T'}$

$$T' = \frac{2}{\ln \cosh\left(\frac{2}{T}\right)}$$

Does this transformation have a fixed point? A fixed point would satisfy

$$2\beta^* = \ln \cosh(2\beta^*) \quad \left(\text{or } T^* = \frac{2}{\ln \cosh\left(\frac{2}{T^*}\right)} \right)$$

In $d=1$ this can happen only if $T^* = 0, \infty$ ($\beta^* = \infty, 0$).

Near $T=0$ we can expand

$$T' = \frac{2}{\ln\left(\frac{e^{2/T}}{2} \left(1 + e^{-\frac{4}{T}}\right)\right)} =$$

$$\Rightarrow T' = \frac{2}{\frac{2}{T} - \ln 2 + \ln\left(1 + e^{-\frac{4}{T}}\right)}$$

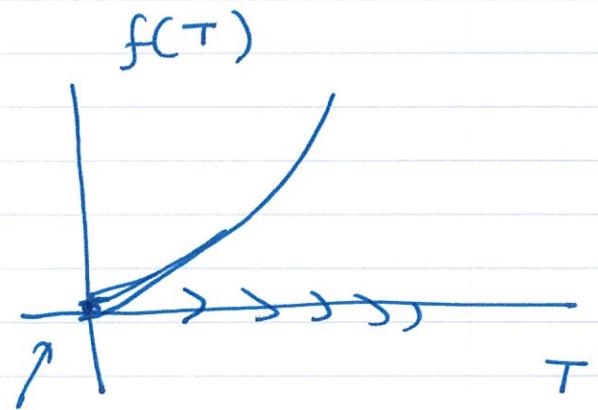
As $T \rightarrow 0$

$$T' \approx \frac{2}{\frac{2}{T} - \ln 2} = \frac{T}{1 - \frac{T}{2} \ln 2} \approx T \left(1 + \frac{T}{2} \ln 2 \right)$$

$$\Rightarrow T' - T = f(T) = \frac{T^2}{2} \ln 2$$

$$\Rightarrow \frac{T' - T}{\ln 2} = \frac{T^2}{2}$$

↑
change
of scale



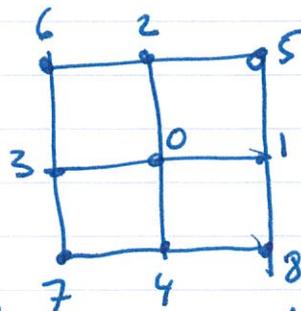
This case is known

as a marginally

$T^* = 0$ is unstable

relevant perturbation.

Can we do this for $d > 1$? For example
on a square lattice we now have 4 neighbors
of each spin



If we attempt to integrate out the spins on

one sublattice (e.g. σ_0) this procedure will generate an interaction between the nearest-neighbor spins $\sigma_1\sigma_2$, $\sigma_3\sigma_4$ and $\sigma_4\sigma_1$.

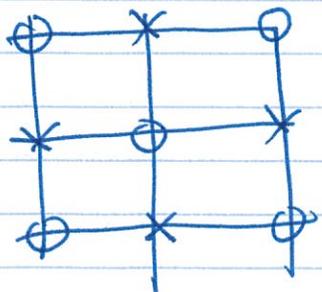
However it will also generate an interaction among the next-nearest-neighbors $\sigma_1\sigma_3$ and $\sigma_2\sigma_4$ as well as a four spin interaction $\sigma_1\sigma_2\sigma_3\sigma_4$. Thus we ~~would~~ would have

to now include these extra interactions ~~to~~ from the beginning. But if we do that we now also get further neighbor (and more spin) interactions. In other terms we will have to truncate this procedure by some approximation (unless we are lucky and these interactions are small).

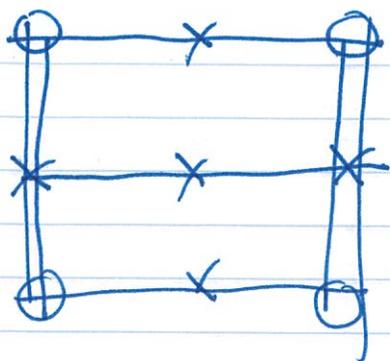
So in general an RG transformation requires that we make some approximation. One possible approach involves doing

Monte Carlo simulations in systems with several interactions of these type on systems of size L and L' and then ask how should we change the couplings so that a set of correlators be the same on both lattices. Thus Monte-Carlo RG is quite efficient and very accurate and it has been used to study the 3D Ising model. However to get a flavor of how these

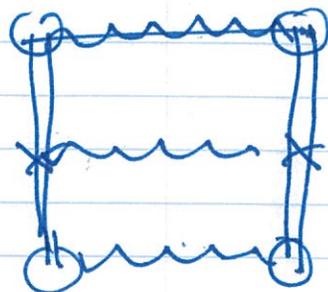
block spins work we can use some convenient approximation. One such approach is the "bond ~~to~~ moving" method of Migdal and Kadanoff. It works as follows. Consider the $d=2$ case (square lattice)



We now move $1/2$ of the vertical bonds (Kadanoff showed this ^{satisfies} is a variational pp.)



after
bond moving



integrate out
the middle
spins

$$m = \beta'_1 = \frac{1}{2} \ln \cosh(2\beta_1)$$

$$\Rightarrow \beta'_2 = 2\beta_2$$

symmetrize

Scale factor
↓
 $\beta_1'' = 2\beta'_1$

$$\beta_2'' = \frac{1}{2} \ln \tanh(2\beta'_2)$$

In d dimensions

$$\beta_1^{(d)} = 2^{d-1} \frac{1}{2} \ln \cosh(2\beta) \quad \text{and}$$

similar expressions for other β 's.

$$\Rightarrow \text{RG: } \beta' = 2^{d-2} \ln \cosh(2\beta)$$

$$T \stackrel{!}{=} \frac{2^{2-d}}{\ln \cosh \frac{2}{T}} \quad (\text{analytic in } d)$$

As $d \rightarrow 1 \Rightarrow T^* \rightarrow 0$
(fixed point)

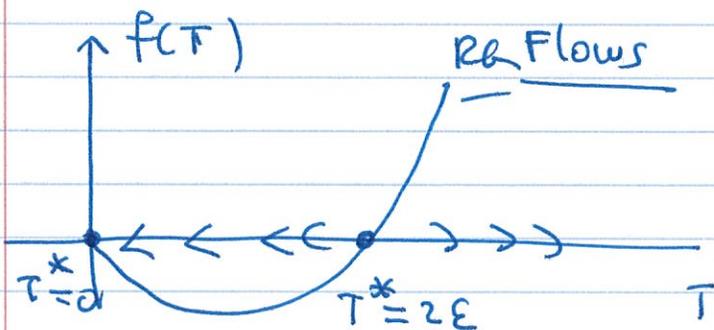
Write $d = 1 + \epsilon$ and $2^{2-d} = 2^{1-\epsilon} \approx 2(1 - \epsilon \ln 2)$

$$T' \cong \frac{2(1 - \epsilon \ln 2)}{\ln \frac{e^{2/T}}{2}} = \frac{2(1 - \epsilon \ln 2)}{\frac{2}{T} - \ln 2}$$

$$T' = \frac{T(1 - \epsilon \ln 2)}{1 - \frac{T}{2} \ln 2} \approx T(1 - \epsilon \ln 2) \left(1 + \frac{T}{2} \ln 2\right)$$

$$\Rightarrow T' \approx T \left(1 - \epsilon \ln 2 + \frac{T}{2} \ln 2\right) + \dots$$

$$f(T) = \frac{T' - T}{T} = \frac{\ln 2}{2} (T^2 - 2\epsilon T + \dots)$$



Two fixed points

$$T^* = 0 \quad \text{and}$$

$$T^* = 2\epsilon$$

(and $T^* = \infty$)

$$\text{slope } f'(0) = -\epsilon \ln 2$$

$$f'(2\epsilon) = +\epsilon \ln 2$$

$$H = \sum_n \beta^* \sigma_n \sigma_{n+1} + \sum_n \delta \beta \sigma_n \sigma_{n+1}$$

$$H' = \sum_n \beta^* \sigma_n \sigma_{n+1} + \sum_n \delta \beta' \sigma_n \sigma_{n+1}$$

with $\delta\beta' = b^y_T \delta\beta$ ($b=2$)

$$\delta\beta = -\frac{\delta T}{T^2}$$

$$\Rightarrow \delta\beta' = b^y_T \delta\beta \Rightarrow \delta T' = T^y_T \delta T$$

$$T' - 2\varepsilon = T - 2\varepsilon + \frac{\ln 2}{2} T (T - 2\varepsilon)$$

$$\delta T' = \delta T [1 + \varepsilon \ln 2] \approx 2^\varepsilon \delta T$$

$$\Rightarrow \boxed{y_T = \varepsilon} \Rightarrow T \text{ is relevant at } T^* = 2\varepsilon$$

But at $T^* = 0$

$$\delta T' = \delta T (1 - \varepsilon \ln 2) \Rightarrow y_T = -\varepsilon \text{ at } T^* = 0$$

$\Rightarrow T$ is irrelevant at $T^* = 0$

($T^* = 0$ is stable)

Correlation length:

$$\xi^1 = \frac{\xi}{2} \quad (\text{after one step})$$

$$\Rightarrow n \text{ steps} \quad \xi_n = \frac{\xi_0}{2^n}$$

After n steps $(\delta T)_n \approx T^*$ (large)

$$(\delta T)_n \approx 2^{n/d_T} \delta T$$

$$\Rightarrow 2^n = \left(\frac{(\delta T)_n}{\delta T} \right)^{1/d_T} = \frac{\sum_n}{\sum_n}$$

since $\sum_n \approx 1$ for $(\delta T)_n \approx T^*$

$$\Rightarrow \xi(\delta T) \approx \xi^* \left(\frac{T^*}{\delta T} \right)^{1/d_T}$$

\Rightarrow as $\delta T \rightarrow 0$ $\xi(\delta T) \rightarrow \infty$ as

$$\xi(\delta T) \sim (\delta T)^{-\nu}$$

with $\nu = \frac{1}{d_T} = \frac{1}{\varepsilon}$ (in this case)

$$\Rightarrow \nu = \frac{1}{d-1} + O(\varepsilon)$$