Chapter 5

Landau Theory of the Fermi Liquid

5.1 Adiabatic Continuity

The results of the previous lectures, which are based on the physics of non-interacting systems plus lowest orders in perturbation theory, motivates the following description of the behavior of a Fermi liquid at very low temperatures. These assumptions (and results) are usually called the Landau Theory of the Fermi Liquid, or more commonly Fermi Liquid Theory (FLT).[1]

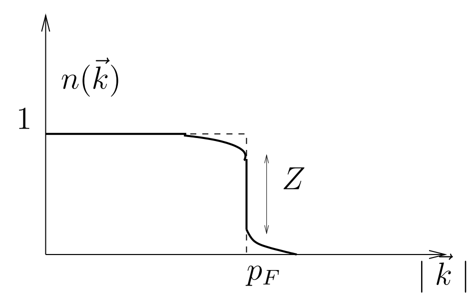


Figure 5.1: Discontinuity in the occupation number at the Fermi surface in a a free and in an interacting system.

The physical picture is the following. In the non-interacting system, the spectrum consists of particles and holes with a certain dispersion (the free

[1] We will follow closely the treatment in Baym and Pethick.

one-particle spectrum) $\varepsilon_0(p)$. These excitations are infinitely long-lived since they cannot decay in the absence of matrix elements (interactions). The ground state is characterized by a distribution function $n_0(\vec{p})$ which at $T = 0$ has the form shown in Fig. (5.1).

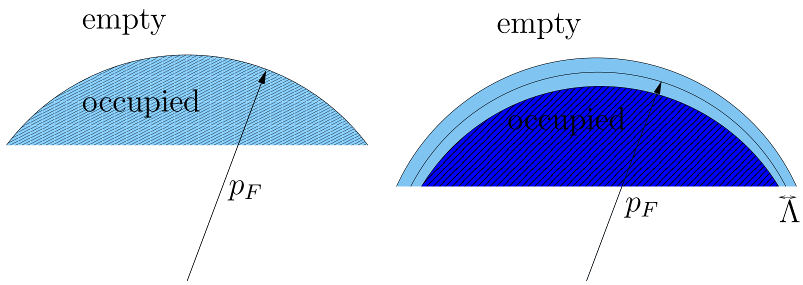
Landau reasoned that what interactions do is to produce (virtual) particle-hole pairs. Thus, the distribution function must change: $n_0(\vec{p}) \to n(\vec{p})$. He further assumed that this change is a smooth function (*i.e.* analytic) of the interaction. In other words, as the interactions are slowly turned on, the non-interacting states are assumed to smoothly and continuously evolve into interacting states, and that this evolution takes place without hitting any singular behavior. Such a singularity would signal an *instability* of the ground state and should be viewed as a *phase transition*. Thus, if there are no phase transitions, there should be a smooth connection between non-interacting and interacting states. In particular the quantum numbers used to label the non-interacting states should also be good quantum numbers in the presence of interactions. The one-electron (particle) state becomes a *quasiparticle* which carries the same *charge* $(-e)$ and spin $(\pm 1/2)$ of the bare electron.

Non-interacting	Interacting
• electron (particle) $\lvert p\rvert > P_F \to$	• quasiparticle $\lvert\vec{p}\rvert > p_F$
• hole ($\lvert p\rvert < P_F$)	• quasihole $\lvert\vec{p}\rvert < p_F$
• charge $\pm e$, spin $\frac{1}{2}$	• charge $\pm e$, spin $\frac{1}{2}$
• always stable	• stable only at low energies ($\omega \to 0$)

The state of the interacting systems can be parameterized by the *actual* distribution function $n(\vec{p})$. Let $\delta n(\vec{p}) \equiv n(\vec{p}) - n_0(\vec{p})$. For the system to be stable $\delta n(p)$ must be non-zero only for $|\vec{p}| \approx p_F$ and the ground state energy is determined by $\delta n(p)$. If $n_0(p) \to n(p) = n_0(p) + \delta n(p)$ with $\delta n/n \ll 1$(for all $\vec{p}$ close to the fermi surface), the total energy is$E = E_0 + \delta E$ and we can expand the excitation energy δE in powers of the change of the distribution function $\delta n(\vec{p})$ as

$$\delta E = \sum_p \varepsilon_p \, \delta n(p) + \cdots \tag{5.1}$$

The excitation energy δE should also tell us how much energy does it cost to add an excitation of momentum $\vec{p}$ close to the Fermi surface. Thus,



(a) Non-interacting system (b) Interacting system

Figure 5.2: Fermi surface in a) a non-interacting system and b) in an interacting system. Only the states in the narrow region $\Lambda \ll p_F$ contribute in the presence of interactions.

the quasiparticle energy $\varepsilon(p)$ should be given by

$$\varepsilon(p) = \frac{\delta E}{\delta n(p)} \tag{5.2}$$

The difference between the energy of the ground state with $N+1$ particles and N particles is (by definition) the chemical potential μ of the system,

$$E(N+1) - E(N) = \mu \tag{5.3}$$

Hence, we see that

$$\mu = \varepsilon(\vec{p}), \qquad \text{with,} \quad |\vec{p}| = p_F \tag{5.4}$$

or, what is the same,

$$\mu = \frac{\partial E}{\partial N} = \varepsilon(p_F) \tag{5.5}$$

The higher order corrections to δE in powers of $\delta n(\vec{p})$ know about the interactions among quasiparticles. This is the Landau expansion.

Is $\varepsilon(p)$ independent of the existence of other quasiparticles? If so $\varepsilon(p)$ would be independent of $\delta n(p)$. In that case $\varepsilon(p) = \varepsilon_0(p)$! This is obviously not true. Thus, to low orders in $\delta n(\vec{p})$ we must have terms of the form

$$E(\delta n) = \sum_{\vec{p}} \varepsilon(p)\delta n(\vec{p}) + \frac{1}{2}\sum_{\vec{p},\vec{p}'} f(\vec{p},\vec{p}')\delta n(\vec{p})\delta n(\vec{p}') + O((\delta n(\vec{p}))^3) \tag{5.6}$$

where we have introduced the *Landau parameters*, the symmetric function $f(\vec{p},\vec{p}') = f(\vec{p}',\vec{p})$.

Near $|\vec{p}| = p_F$ the Fermi velocity is given by $\vec{v}_F(\vec{\vec{p}}) = \partial_{\vec{p}}\varepsilon(\vec{p})$. Hence, we can define an *effective mass*

$$m^* = \frac{p_F}{|\vec{v}_F(p_F)|} \tag{5.7}$$

which is isotropic only if the Fermi surface is isotropic.

Hence we conclude that $\varepsilon(p)$, which is defined by

$$\varepsilon(p) = \frac{\delta E}{\delta n(p)} \tag{5.8}$$

has the form

$$\varepsilon(p) = \varepsilon_0(p) + \sum_{\vec{p}'} f(\vec{p}, \vec{p}')\delta n(\vec{p}') + \cdots \tag{5.9}$$

The correction term gives a measure of the change of the quasiparticle energy due to the presence of other quasiparticles. The function $f(\vec{p}, \vec{p}')$ measures the strength of quasiparticle-quasiparticle interactions. Hence $f(\vec{p}, \vec{p}')$ is an *effective* interaction for excitations arbitrarily close to the Fermi surface.

What about spin effects? If the system is isotropic and there are no magnetic fields present, the quasiparticle with up spin ($\uparrow$) has the same energy as the quasiparticle with down spin ($\downarrow$). Hence $\varepsilon_\uparrow(\vec{p}) = \varepsilon_\downarrow(\vec{p})$ (Note: This relation is changed in the presence of an external magnetic field by the Zeeman effect).

Likewise, the interactions between quasiparticles depends only on the relative orientation of the spins σ and σ'. The Landau interaction term is modified by spin effects as

$$\sum_{\vec{p},\vec{p}'} f(\vec{p}, \vec{p}')\delta n(\vec{p})\ \delta n(\vec{p}') \to \sum_{\vec{p},\vec{p}'} \sum_{\sigma,\sigma'} f_{\sigma\sigma'}(\vec{p}, \vec{p}')\delta n_\sigma(\vec{p})\delta n_\sigma(\vec{p}') \tag{5.10}$$

By symmetry considerations we expect that

$$\begin{aligned} f_{\uparrow\uparrow} =& f_{\downarrow\downarrow} \equiv f^{(S)} + f^{(A)} \\ f_{\uparrow\downarrow} =& f_{\downarrow\uparrow} \equiv f^{(S)} - f^{(A)} \end{aligned} \tag{5.11}$$

Thus we can also write the quasiparticle interaction term as the sum of a symmetric and an antisymmetric (or exchange) term

$$\sum_{\vec{p},\vec{p}'}\sum_{\sigma,\sigma'} f_{\sigma\sigma'}(p,p')\delta n_\sigma(p)\ \delta n_{\sigma'}(p') =$$
$$\text{symmetric} \rightarrow \ \sum_{\vec{p},\vec{p}'} f^{(S)}(p,p')(\delta n_\uparrow(p)+\delta n_\downarrow(p))(\delta n_\uparrow(p')+\delta n_\downarrow(p')) +$$
$$\text{antisymmetric} \rightarrow \ \sum_{\vec{p},\vec{p}'} f^{(A)}(p,p')(\delta n_\uparrow(p)-\delta n_\downarrow(p))(\delta n_\uparrow(p')-\delta n_\downarrow(p')) \tag{5.12}$$

The density of quasiparticle states at Fermi surface, $N(0)$, is

$$N(0) = \frac{1}{V}\sum_{\vec{p},\sigma}\delta(\varepsilon^0_{\vec{p}\sigma}-\mu) = -\frac{1}{V}\sum_{\vec{p}\sigma}\frac{\partial n^0_{\vec{p}\sigma}}{\partial \varepsilon_{\vec{p}\sigma}} \tag{5.13}$$

where we have used the fact that $n^0_{\vec{p},\sigma}$ is the fermi function at $T=0$ (a step function). Hence, we find

$$N(0) = \frac{m^* p_F}{\pi^2\hbar^3} \tag{5.14}$$

where m^* is the effective mass. To avoid confusion, note that $\varepsilon^0_{\vec{p}\sigma}$ represents energy of quasiparticles at the Fermi-surface, while $\varepsilon_0(p)$ represents the energy of the non-interacting system, as indicated previously. In a general scenario, where all of the quasiparticle spins are not quantized along the same axis, the spin polarization of the Fermi liquid is (in the following equations τ represents Pauli matrices, and $\alpha, \overline{\alpha}$ represents the matrix indices)

$$\sigma_i = \sum_p \sum_{\alpha\overline{\alpha}} (\tau_i)_{\alpha\overline{\alpha}}[n(\vec{p})]_{\overline{\alpha}\alpha} \tag{5.15}$$

Equivalently,

$$n(\vec{p}) = \frac{1}{2}\sum_\alpha (n_{\vec{p}})_{\alpha\alpha} \tag{5.16}$$

and

$$\vec{\sigma}(\vec{p}) = \frac{1}{2}\sum_{\alpha\overline{\alpha}} (\vec{\tau})_{\alpha\overline{\alpha}}\cdot[n(\vec{p})]_{\overline{\alpha}\alpha} \tag{5.17}$$

Consequently

$$\delta^2 E = \frac{1}{V^2}\sum_{pp'}\sum_{\alpha\overline{\alpha}\alpha'\overline{\alpha}'} f_{\vec{p}\alpha\overline{\alpha},\vec{p}\,'\alpha'\overline{\alpha}'}(\delta n(p))_{\overline{\alpha}\alpha}(\delta n(p'))_{\overline{\alpha}\alpha} \tag{5.18}$$

where

$$f_{\vec{p}\alpha\overline{\alpha},\vec{p}'\alpha'\overline{\alpha}'} = f^S_{\vec{p}\vec{p}'}\delta_{\alpha\overline{\alpha}}\delta_{\alpha'\overline{\alpha}'} + f^A_{\vec{p}\vec{p}'}\vec{\tau}_{\alpha\overline{\alpha}}\vec{\tau}_{\alpha'\overline{\alpha}'} \tag{5.19}$$

which can also be written more compactly as

$$f_{\vec{p}\vec{p}'} \equiv f^{(S)}_{pp'} + f^{(A)}_{pp'}\vec{\tau}\cdot\vec{\tau} \qquad \text{(no trace)} \tag{5.20}$$

For a rotationally invariant system, the interaction functions $f^{S,A}_{\vec{p},\vec{p}'}$ must depend only on the angle θ defined by

$$\cos\theta(\vec{p},\vec{p}\,') = \frac{\vec{p}\cdot\vec{p}\,'}{p_F^2} \tag{5.21}$$

Hence we should be able to use an expansion of the form

$$f^{S,A}_{\vec{p},\vec{p}\,'} = \sum_{\ell=0}^{\infty} f^{S,A}_{\ell} P_\ell(\cos\theta) \tag{5.22}$$

where

$$f^{S,A}_{l} = \frac{2\ell+1}{2}\int_0^1 dx P_\ell(x)\frac{1}{2}\left(f_{\vec{p}\uparrow,\vec{p}\,'\uparrow} \pm f_{\vec{p}\uparrow,\vec{p}\,'\downarrow}\right) \tag{5.23}$$

which leads to the definition of the *Landau parameters* in terms of angular momentum channels

$$F^{S,A}_\ell \equiv N(0) f^{S,A}_\ell \tag{5.24}$$

5.2 Equilibrium Properties

5.2.1 Specific Heat:

The low temperature specific heat of a Fermi liquid, just as in the case of non-interacting fermions, is linear in T with a coefficient determined by the effective mass m^* of the quasiparticles at p_F. Let's compute the low temperature entropy, or rather the variation of the quasiparticle entropy (per unit volume) as $T \to T + \delta T$

$$S = -\frac{k_B}{V}\sum_{\vec{p}\sigma}[n_\sigma(\vec{p})\ln\ n_\sigma(\vec{p}) + (1-n_\sigma(\vec{p}))\ln(1-n_\sigma(\vec{p}))] \tag{5.25}$$

where $n_\sigma(\vec{p})$ is the Fermi-Dirac distribution

$$n_\sigma(\vec{p}) = \frac{1}{e^{(\varepsilon_\sigma(\vec{p})-\mu)/k_BT}+1} \tag{5.26}$$

and $\varepsilon_\sigma(\vec{p})$ is the quasiparticle excitation energy, *i.e.*,

$$\frac{\delta F}{\delta n_\sigma(\vec{p})} = \varepsilon_\sigma(\vec{p}) \tag{5.27}$$

where $F(\delta n(\vec{p})$ is the free energy.

Thus, the variation of the entropy is

$$\delta S = \frac{1}{TV}\sum_{\vec{p}\sigma}(\varepsilon_\sigma(\vec{p}) - \mu)\delta n_\sigma(\vec{p}) \tag{5.28}$$

where

$$\delta n_\sigma(\vec{p}) = \frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})}\left[-\frac{(\varepsilon_\sigma(\vec{p}) - \mu)}{T}\delta T + \underbrace{\delta\varepsilon_\sigma(\vec{p}) - \delta\mu}\right] \tag{5.29}$$

where the term in braces is due to the quasiparticle interactions. Here, contribution from the first term is

$$\delta S_1 = -\frac{1}{V}\sum_{\vec{p}\sigma}\frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})}(\varepsilon_\sigma(\vec{p}) - \mu)\frac{\delta T}{T^2} \tag{5.30}$$

Since $\frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})}$ is non-zero only within $k_B T$ of the Fermi energy, we find

$$\begin{aligned}\delta S &= -\sum_\sigma \int p^2 \frac{dp}{d\varepsilon}\frac{4\pi}{(2\pi\hbar)^3} d\varepsilon \frac{\partial}{\partial \varepsilon}\left[\frac{1}{e^{(\varepsilon-\mu)/k_B T} + 1}\right]\left(\frac{\varepsilon - \mu}{T}\right)^2 \delta T \\ &\equiv -k_B^2 N(0)\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}\left(\frac{1}{e^x + 1}\right) x^2 \delta T\end{aligned} \tag{5.31}$$

Hence we find that the low temperature contribution of the quasiparticles is (to leading order) given by

$$S_1 = \frac{\pi^2}{3} N(0) k_B^2 T \tag{5.32}$$

and that the specific heat is

$$C_v = T\left(\frac{\partial S}{\partial T}\right)_V = S_1 = \frac{\pi^2}{3}N(0)k_B^2 T = \frac{m^* p_F}{3\hbar^3}k_B^2 T \tag{5.33}$$

We now introduce the Fermi temperature

$$T_F = \frac{p_F^2}{2m^* k_B} \equiv \frac{\varepsilon_F}{k_B} \tag{5.34}$$

as the Fermi energy in temperature units, in terms of which the specific heat becomes

$$C_V = \frac{\pi^2}{2} n k_B \frac{T}{T_F} = \frac{\pi^2}{3} \frac{T}{T_F} C_V^0 \tag{5.35}$$

where $C_V^0 = \frac{3}{2} n k_B$ is the specific heat of a classical ideal gas, and n is the particle density.

Using these results we find that the low temperature correction to the *Free Energy*, $F = E - TS$, is $\delta F \approx -S\delta T$ (to lowest order), *i.e.,*

$$F \approx E_0 - \frac{\pi^2}{4} n k_B \frac{T^2}{T_F} \tag{5.36}$$

where E_0 is the ground state energy. The *chemical potential* μ is (note that T_F is a function of m^* and hence a function of n)

$$\mu(n,T) = -\left(\frac{\partial F}{\partial n}\right)_T = \mu(n,0) - \frac{\pi^2}{4} k_B \left(\frac{1}{3} + \frac{n}{m^*}\frac{\partial m^*}{\partial n}\right)\frac{T^2}{T_F} \tag{5.37}$$

Let us now compute the *compressibility:*

$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{n^2}\frac{\partial n}{\partial \mu} \tag{5.38}$$

where P is the pressure. At $T = 0$,

$$\delta n_\sigma(\vec{p}) = \frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})}(\delta\varepsilon_\sigma(\vec{p}) - \delta\mu) \tag{5.39}$$

The quasiparticle energy $\varepsilon_\sigma(\vec{p})$ depends on μ only through its dependence on $\delta n_{\sigma'}(\vec{p}')$ (*i.e.,* quasiparticle interactions, see Eq. 5.9). As $T \to 0$ both $\frac{\partial n}{\partial \varepsilon}$ and $\delta n_\sigma(p)$ vanish unless all momenta are *at* the Fermi-surface.

$$\delta\varepsilon_\sigma(p) = f_0^S \frac{1}{V}\sum_{\sigma',\vec{p}'} \delta n_{\sigma'}(\vec{p}') \equiv f_0^S \delta n \tag{5.40}$$

where f_0^S is Landau parameter with with $l = 0$. Hence, we have

$$\delta n_\sigma(\vec{p}) = \frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})}(f_0^S \delta n - \delta\mu) \tag{5.41}$$

and

$$\delta n = \frac{1}{V}\sum_{\sigma,\vec{p}} \delta n_\sigma(\vec{p}) = \frac{1}{V}\sum_{\sigma,\vec{p}} \frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})}(f_0^S \delta n - \delta\mu) \tag{5.42}$$

Similarly,

$$\frac{\partial n_\sigma(\vec{p})}{\partial \varepsilon_\sigma(\vec{p})} \longrightarrow -\delta(|p| - p_F) \text{ for } T \to 0 \tag{5.43}$$

Thus,

$$\delta n = -N(0)(f_0^S \delta n - \delta\mu) \tag{5.44}$$

and

$$\delta n[1 + N(0) f_0^S] = N(0)\delta\mu \tag{5.45}$$

using the expression for the (s-wave) symmetric (singlet) Landau parameter,

$$F_0^S = N(0) f_0^S \tag{5.46}$$

we can write

$$\frac{\partial n}{\partial \mu} = \frac{N(0)}{1 + F_0^S} \tag{5.47}$$

which leads to an expression for the *compressibility* κ:

$$\kappa = \frac{1}{n^2}\frac{N(0)}{1 + F_0^S} \tag{5.48}$$

which includes the *Fermi liquid correction* expressed in terms of the Landau parameter F_0^S.

5.3 Spin Susceptibility

We will now determine the (spin) magnetic susceptibility of a Fermi liquid. Thus we need to consider its response to an external magnetic field. Here we will be interested in the effect of the Zeeman coupling, which causes the quasiparticle energy to change by an amount that depends on the spin polarization:

$$-\frac{1}{2}\hbar\gamma\sigma_z H \tag{5.49}$$

where γ is the gyromagnetic ratio, σ_z is the diagonal Pauli matrix, and H is the external (uniform) magnetic field. By taking into account also the change caused to the distribution functions we find

$$\delta\varepsilon_{\vec{p},\sigma} = -\frac{1}{2}\hbar\gamma\sigma_z H + \frac{1}{V}\sum_{\vec{p}',\sigma'} f_{\sigma,\sigma'}(\vec{p},\vec{p}')\delta n_{\vec{p}',\sigma'} \tag{5.50}$$

where, as before,

$$\delta n_{\vec{p},\sigma} = \frac{\partial n_{\vec{p},\sigma}}{\partial \varepsilon_{\vec{p},\sigma}}\left(\delta\varepsilon_{\vec{p},\sigma} - \delta\mu\right) \tag{5.51}$$

The chemical potential is a scalar (and time reversal invariant) quantity and as such it cannot have a linear variation with the magnetic field. Hence the only possible dependence of μ with H must be an even power and (at least) of order H^2. Hence it does not contribute to the magnetic susceptibility (within linear response). We will neglect this contribution. Hence, $\delta n_{\vec{p},\sigma} \propto \delta\varepsilon_{\vec{p},\sigma}$, are independent of the direction of the momentum $\vec{p}$, and have opposite sign for $\uparrow$ and $\downarrow$ quasiparticles. Since $\delta n_{\vec{p},\sigma} \neq 0$ only is $\vec{p}$ is on the Fermi surface (which we will assume to be isotropic), we find

$$\frac{1}{V}\sum_{\vec{p}',\sigma'} f_{\sigma,\sigma'}(\vec{p},\vec{p}')\delta n_{\vec{p}',\sigma'} = 2f_0^A \delta n_\sigma = \sigma_z f_0^A(\delta n_\uparrow - \delta n_\downarrow) \tag{5.52}$$

where δ_σ is the change in the total number of particles (per unit volume) with spin σ. Hence,

$$\delta n_\sigma = \frac{1}{2}N(0)\left(\frac{1}{2}\hbar\gamma\sigma_z H - 2f_0^A\delta n_\sigma\right) \tag{5.53}$$

The net spin polarization is

$$\delta n_\uparrow - \delta n_\downarrow = \frac{\hbar}{2}\gamma\frac{N(0)H}{1+F_0^A} \tag{5.54}$$

and the total magnetization M is

$$M = \gamma\frac{\hbar}{2}\frac{\left(\gamma\frac{\hbar}{2}\right)^2 N(0)}{1+F_0^A}\; H \tag{5.55}$$

We can thus identify the spin susceptibility χ with

$$\chi = \frac{\hbar^2}{4}\frac{\gamma^2 N(0)}{1+F_0^S} \tag{5.56}$$

which is the (Pauli) spin susceptibility of a free Fermi gas with mass m^*, with the Fermi liquid correction.

5.4 Effective mass and Galilean Invariance

In Galilean invariant systems, there is a simple relation between m^*, the bare mass m and the Landau Fermi liquid parameter F_1^S, given by

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(S)} \tag{5.57}$$

To see how this comes about we will consider a Galilean transformation to a frame at speed $\vec{v}$. The Hamiltonian of the system transforms as follows

$$H \to H' = H - \vec{P} \cdot \vec{v} + \frac{1}{2} M \vec{v}^2 \tag{5.58}$$

where $\vec{P}$ is the total momentum operator in the laboratory frame and $M = Nm$ is the total mass of the system. Hence the transformed total energy and total momentum are

$$\begin{aligned} E' &= E - \vec{P} \cdot v + \frac{1}{2} M \vec{v}^2 \\ \vec{P}' &= \vec{P} - M\vec{v} \end{aligned} \tag{5.59}$$

Consider the change in energy due to adding a quasiparticle of momentum $\vec{p}$ in the lab frame. The total mass changes as $M \to M + m$ where m is the *bare* mass. The addition of one quasiparticle involves the addition of one bare particle. In lab frame the momentum increases by $\vec{p}$ and the energy by $\varepsilon_\sigma(\vec{p})$. In the moving frame the momentum increases by

$$\vec{p} - m\vec{v} \tag{5.60}$$

and the energy increases by

$$\varepsilon_{\vec{p}} - \vec{p} \cdot \vec{v} + \frac{1}{2} m \vec{v}^2 \tag{5.61}$$

Therefore the quasiparticle energy in the moving frame is given by

$$\begin{aligned} \varepsilon'_{\vec{p}-m\vec{v}} &= \varepsilon_{\vec{p}} - \vec{p} \cdot \vec{v} + \frac{1}{2} m \vec{v}^2 \\ \Rightarrow \varepsilon'_{\vec{p}} &= \varepsilon_{\vec{p}+m\vec{v}} - \vec{p} \cdot \vec{v} - \frac{1}{2} m \vec{v}^2 \end{aligned} \tag{5.62}$$

which is a consequence of Galilean invariance. Expanding to order $\vec{v}$, and using the definition of the effective mass m^*, we have:

$$\varepsilon'_{\vec{p}-m\vec{v}} \approx \varepsilon_{\vec{p}} + \left(\frac{m-m^*}{m^*}\right)\vec{p}\cdot\vec{v} \tag{5.63}$$

From the moving frame the ground state looks like a Fermi surface centered at $\vec{p} = -m\vec{v}$, hence the occupation numbers change as follows

$$n'_{\vec{p}} = n^0_{\vec{p}+m\vec{v}} = n^0_{\vec{p}} + m\vec{v}\cdot\partial_{\vec{p}} n^0_{\vec{p}} + \cdots \tag{5.64}$$

where $n^0_{\vec{p}+m\vec{v}}$ refers to the lab frame.

The quasiparticle energy in the moving frame is

$$\varepsilon'_{\vec{p}} = \varepsilon_{\vec{p}}\{n'_{\vec{p}'}\} = \varepsilon_{\vec{p}}\{n^0_{\vec{p}+m\vec{v}}\} \tag{5.65}$$

Note that this is valid for one-component systems only. We have

$$\begin{aligned}
\partial_{\vec{p}'} n_{\vec{p}'} &= \partial_{\vec{p}'}\varepsilon_{\vec{p}'}\frac{\partial n^0_{p'}}{\partial\varepsilon_{p'}} \\
\Rightarrow \varepsilon'_p &= \varepsilon_p + \frac{1}{V}\sum_{p'\sigma'} f^S_{pp'} m\vec{v}\cdot\frac{\vec{p}'}{m^*}\frac{\partial n^0_{p'}}{\partial\varepsilon_{p'}} \\
&= \varepsilon_p - \frac{F^S_1}{3}\frac{m}{m^*}\vec{p}\cdot\vec{v}
\end{aligned} \tag{5.66}$$

For $|\vec{p}| = p_F$, the Fermi momentum, we have

$$\begin{aligned}
\frac{m-m^*}{m^*} &= -\frac{m}{m^*}\frac{F^S_1}{3} \\
\Rightarrow \frac{m^*}{m} &= 1 + \frac{1}{3}\,F^{(S)}_1
\end{aligned} \tag{5.67}$$

This implies that the relative deviation of m^* from m is determined by $F^{(S)}_1$.

5.5 Thermodynamic Stability

The ground state should be a minimum of the (Gibbs) free energy which implies that there should be restrictions of the Landau parameters. Consider a distortion of the Fermi Surface characterized by a direction dependent Fermi momentum $p_F(\theta)$.[2]

$$n_\sigma(\vec{p}) = \theta(p_F(\theta) - |\vec{p}|) \tag{5.68}$$

For a stable system, the thermodynamic potential $G = E - \mu n$ must be a minimum. Therefore the change in the (Gibbs) free energy due to the distortion is

$$(E - \mu n) - (E - \mu n)_0 = \frac{1}{V}\sum_{p,\sigma}(\varepsilon_p^0 - \mu)\delta n_\sigma(p) + \frac{1}{2V^2}\sum_{\substack{pp' \\ \sigma\sigma'}} f_{p\sigma,p'\sigma'}\delta n_\sigma(p)\delta n_{\sigma'}(p') \tag{5.69}$$

where

$$\begin{aligned}\delta n_\sigma(p) &= n_\sigma(p) - n_\sigma^0(p) \\ &= \delta p_F \; \delta(p_F - |p|) - \frac{1}{2}(\delta p_F)^2 \frac{\partial}{\partial p}\delta(p_F - p)\end{aligned} \tag{5.70}$$

where the change of the Fermi momentum is $\delta p_F = p_F(\theta) - p_F^0$. The first term in Eq. 5.67 is

$$\frac{1}{V}\varepsilon_{p\sigma}(\varepsilon_p^0 - \mu)\delta n_\sigma(p) = \frac{1}{4}N(0)v_F^2 \sum_\sigma \int_{-1}^{1} d(\cos\theta)\frac{1}{2}(\delta p_F(\theta,\sigma))^2 \tag{5.71}$$

where

$$v_F \delta p_F(\theta) \equiv \sum_{\ell=0}^{\infty} \; v_{\ell,\sigma} P_\ell(\cos\theta) \tag{5.72}$$

The second term in Eq. 5.67 is

$$\frac{1}{8}(N(0)v_F)^2 \sum_{\sigma\sigma'} \int_{-1}^{1} \frac{d\cos\theta}{2} \int_{-1}^{1} \frac{d\cos\theta'}{2} f_{p\sigma,p'\sigma'}\delta p_F(\theta,\sigma)\delta p_F(\theta',\sigma') \tag{5.73}$$

which implies that

$$\begin{aligned}\delta E - \mu \delta n = \sum_{\ell=0}^{\infty} \frac{N(0)}{8(2\ell+1)} &\left[(v_{\ell\uparrow} + v_{\ell\downarrow})^2 \left(1 + \frac{F_\ell^{(S)}}{2\ell+1}\right) \right. \\ &\left. + (v_{\ell\uparrow} - v_{\ell\downarrow})^2 \left(1 + \frac{F_\ell^{(A)}}{2\ell+1}\right) \right]\end{aligned} \tag{5.74}$$

[2]For simplicity we assume that the distorted Fermi surface has azimuthal symmetry.

The Fermi liquid is stable under the deformation if

$$\delta E - \mu \delta n > 0 \tag{5.75}$$

which requires that

$$\begin{aligned} 1 + \frac{F_\ell^{S,A}}{2\ell+1} &\geq 0 \\ F_\ell^{S,A} &\geq -(2\ell+1) \end{aligned} \qquad \text{(Pormeranchuk)} \tag{5.76}$$

Notice that for $\ell = 1$

$$N(0)\left(1 + \frac{1}{3}F_\ell^{(S)}\right) \geq 0 \tag{5.77}$$

is always satisfied for a Galilean invariant one-component system.

What happens if a Pomeranchuk inequality is violated? Clearly if this happens the system will gain energy by distorting the Fermi surface. Thus, the ground state of a system that violates the Pomeranchuk bound has a *broken rotational invariance*. The simplest example of such a state in the *nematic Fermi fluid* in which the symmetry is broken in the quadrupolar ($\ell = 2$ or d-wave) channel.[3]

5.6 Non-Equilibrium Properties

In practice we are also interested in dynamical effects, involving the propagation of excitations. Thus we need to consider systems slightly away from thermal equilibrium and slightly inhomogeneous. We wish to generalize the previous discussion to this case and to define position and time dependent distributions $n_{\sigma\vec{p}}(\vec{r}, t)$. Clearly there is a problem with the uncertainty principle since we cannot define both $\vec{p}$ and $\vec{r}$ with arbitrary precision. At temperature T the momentum fluctuates with a characteristic value $\Delta p \sim \frac{k_B T}{v_F}$. If we wish to define localized quasiparticles a typical length λ, we must have $\lambda \Delta p \gg \hbar$ for the "classical" picture to work, which implies that

$$\lambda \gg \frac{\hbar}{\Delta p} \sim \frac{\hbar v_F}{k_B T} \tag{5.78}$$

[3]To stabilize this state one needs to consider contributions to the Gibbs free energy at orders higher than $(\delta p_F(\theta))^2$.

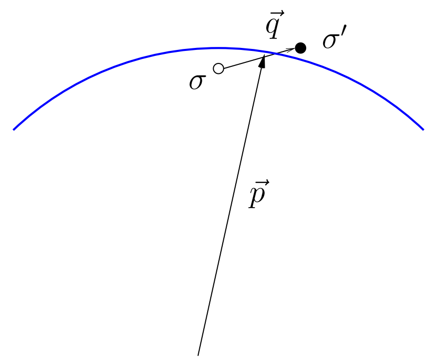


Figure 5.3: A particle-pair with relative momentum $\vec{q}$ and spin polarizations σ and σ', on the FS at $\vec{p}$.

As $T \to 0$ only macroscopic excitations can be described by a classical picture ($\lambda \to \infty$ as $\frac{1}{T}$). In general we will have to use a Wigner distribution function $W(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2; t)$, *i.e.,* the amplitude for removing a particle at $\vec{r}_1$ with spin σ_1 at time t and *at the same time* to add a particle at $\vec{r}_2$ with spin σ_2. The Wigner function is defined as

$$W(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2; t) = \int \frac{d^3p_1}{(2\pi\hbar)^3} \int \frac{d^3p_2}{(2\pi\hbar)^3}\; e^{\frac{i}{\hbar}(\vec{p}_1\cdot\vec{r}_1 - \vec{p}_2\cdot\vec{r}_2)} \langle a^\dagger_{p_2\sigma_2}(t) a_{p_1\sigma_1}(t)\rangle \tag{5.79}$$

Define

$$\begin{aligned} [n_{\vec{p}}(\vec{r}, t)]_{\sigma\sigma'} &= \int d^3\vec{r}' e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}'} W\left(\vec{r} + \frac{\vec{r}'}{2}, \sigma; \vec{r} - \frac{\vec{r}'}{2}, \sigma'; t\right) \\ &\equiv \int \frac{d^3q}{(2\pi\hbar)^3}\; e^{-\frac{i}{\hbar}\vec{q}\cdot\vec{r}} \langle a^\dagger_{\vec{p}+\frac{\vec{q}}{2},\sigma'}(t) a_{\vec{p}-\frac{\vec{q}}{2},\sigma}(t)\rangle \end{aligned} \tag{5.80}$$

where $|FS\rangle$ is the Filled Fermi Sea or the ground state against which the expectation values are evaluated, and $a_{\vec{p}+\frac{\vec{q}}{2},\sigma'}(t) a_{\vec{p}-\frac{\vec{q}}{2},\sigma}(t)|FS\rangle$ is a particle-hole pair with relative momentum $\vec{q}$ localized at $\vec{p}$ on the Fermi Surface.

Clearly a smooth distortion of the Fermi Sea requires a large number of such pairs leading to coherent states of particle-hole pairs. (We'll come back to this later). The quasi-particle density is

$$\sum_\sigma \int \frac{d^3p}{(2\pi\hbar)^3} [n_{\vec{p}}(\vec{r}, t)]_{\sigma\sigma} = \sum_\sigma W(\vec{r}\sigma, \vec{r}\sigma) \tag{5.81}$$

and the number of quasi-particles with momentum $\vec{p}$ is

$$\sum_\sigma \int d^3r [n_p(\vec{r},t)]_{\sigma\sigma} = \sum_\sigma \int d^3r_1 \int d^3r_2 \; e^{-i\vec{p}\cdot\frac{\vec{r}_1-\vec{r}_2}{\hbar}} W(\vec{r}_1\sigma, \vec{r}_2\sigma). \qquad (5.82)$$

For $\lambda \gg \frac{\hbar v_F}{k_B T}$, note that $[n_p(r,t)]_{\sigma\sigma'}$ becomes a classical distribution.

If the system is inhomogeneous, the total energy $\mathcal{E}(t)$ may vary with time. We can still define $\varepsilon_{\sigma\vec{p}}(\vec{r},t)$ as the quasiparticle energy at position $\vec{r}$

$$\delta\mathcal{E}(t) = \int d^3r \delta E(\vec{r},t) = \sum_\sigma \int d^3r \int \frac{d^3p}{(2\pi\hbar)^3} \; \mathcal{E}_{\vec{p}\sigma}(\vec{r},t) \delta n_{p\sigma}(r,t) \qquad (5.83)$$

where $\delta E(\vec{r},t)$ is the local energy density. We have

$$\delta\mathcal{E}_{\vec{p}\sigma}(\vec{r},t) = \sum_{\sigma'} \int d^3r' \int \frac{d^3p'}{(2\pi\hbar)^3} \; f_{\vec{p}\sigma,\vec{p}'\sigma'}(\vec{r},\vec{r}',t) \delta n_{\vec{p}'\sigma'}(\vec{r}',t) \qquad (5.84)$$

If the system is neutral the interactions are typically (assumed to be!) short-ranged and vary only over microscopic scales of the order of $\frac{\hbar}{p_F}$. Therefore, we can replace $f_{\vec{p}\sigma,\vec{p}'\sigma'}(\vec{r},\vec{r}',t)$ by a local form. If there are Coulomb forces

$$f_{\vec{p}\sigma,\vec{p}'\sigma'}(\vec{r},\vec{r}',t) \approx \frac{e^2}{|r-r'|} \; \delta(t-t') + f_{\vec{p}\sigma,\vec{p}'\sigma'} \delta(r-r') \qquad (5.85)$$

However many of these assumptions (concerning the existence and stability of quasiparticles) fails for transverse interactions (mediated by gauge fields) and in one-dimension. We will come back to this problem later.

5.7 Kinetic Equation

We now turn to the problem of the evolution (*i.e.* dynamics) of the quasiparticle disturbances. We will use Landau's "quantum" kinetic theory. We begin by looking at the regime in which $\delta n_{\vec{p}\sigma}(\vec{r},t)$ can be regarded as a *classical* distribution where it should obey a kinetic equation, *i.e.* the Boltzmann equation. As usual this equation is simply the continuity equation for $\delta \vec{n}_{\vec{p},\sigma}(\vec{r},t)$ and embodies the condition of local charge conservation in the fluid. In the absence of collisions between quasiparticles their number must be constant. Hence,

$$\frac{d}{dt} \delta n_{\sigma\vec{p}}(\vec{r},t) = 0 \qquad (5.86)$$

This implies that

$$\frac{\partial}{\partial t}\delta n_{\vec{p}}(\vec{r},t) + \frac{\partial}{\partial \vec{r}} \cdot (\vec{v}_{\vec{p}}\delta n_{\vec{p}}(\vec{r},t)) + \frac{\partial}{\partial \vec{p}} \cdot (\vec{f}_{\vec{p}}(\vec{r},t)\delta n_{\vec{p}}(r,t)) = 0 \tag{5.87}$$

The quasiparticle group velocity in space is

$$\vec{v}_{\vec{p}}(\vec{n}t) = \frac{\partial}{\partial \vec{p}}\, \varepsilon_{\vec{p}}(\vec{r},t) \tag{5.88}$$

The rate of change of quasiparticle momentum (the force) is

$$\vec{f}_{\vec{p}}(\vec{r},t) = -\frac{\partial}{\partial \vec{r}}\, \varepsilon_{\vec{p}}(\vec{r},t) \equiv \frac{d\vec{p}}{dt} \tag{5.89}$$

Hence we obtain *Landau's kinetic equation*

$$\frac{\partial}{\partial t}\delta n_{\vec{p}} - \{\varepsilon_{\vec{p}}, \delta n_{\vec{p}}\}_{\mathrm{PB}} = I[\delta n_{p'}] \tag{5.90}$$

where $I[\delta n_{p'}]$ is the collision integral, and PB denotes the Poisson bracket

$$\{\varepsilon_p, \delta n_p\}_{PB} = \frac{\partial}{\partial \vec{r}}\varepsilon p \cdot \frac{\partial}{\partial \vec{p}}\delta n_p - \frac{\partial \varepsilon_p}{\partial \vec{p}} \cdot \frac{\partial \delta n_{\vec{p}}}{\partial \vec{r}} \tag{5.91}$$

Landau's kinetic equation differs from the Boltzmann equation in that

1. ε_p can be a function of $\vec{r}, t$
2. $\frac{\partial}{\partial \vec{r}}\varepsilon_{\vec{p}}$ includes effective field contributions.

For example, if the system interacts with an external probe of the form of a potential $U(\vec{r},t)$, then the total energy is increased by $\int d\vec{r} \cdot U(\vec{r},t)\delta n(\vec{r},t)$. Therefore

$$\frac{\partial}{\partial \vec{r}}\, \varepsilon_p(\vec{r},t) = \frac{\partial}{\partial \vec{r}}U(\vec{r},t) + \int \frac{d^3p'}{(2\pi\hbar)^3}\, f_{\vec{p}\vec{p}'}\frac{\partial}{\partial \vec{r}}\, \delta n_{\vec{p}'}(r,t). \tag{5.92}$$

The first term on the right hand side in the above equation is present in dilute gases, wheras the second term arises from self-consistency condition as effects of other quasiparticles.

In the quantum case one needs to use Wigner functions, giving rise to the quantum mechanical version of the kinetic equation. In Landau's approach one replaces the Poisson Brackets by $\frac{1}{i\hbar}$ multiplied by commutators, producing the quantum mechanical equations of motion (see Baym and Pethick, pages 19-20). Near equilibrium we can linearize the transport equation, which then becomes the *Landau-Silin Equation*

$$\frac{\partial \delta n_p}{\partial t} + v_{\vec{p}} \cdot \frac{\partial}{\partial \vec{r}} \left(\delta n_p - \frac{\partial n_p^0}{\partial \varepsilon}\ \delta\varepsilon_p \right) = I[n_{p'}] \tag{5.93}$$

where $\vec{v}_p$ and $\frac{\partial n_p^0}{\partial \varepsilon_p}$ are equilibrium functions. The Fourier transformed equation is

$$(\omega - \vec{q} \cdot \vec{v}_p)\delta n_{\vec{p}}(\vec{q}, \omega) - \vec{q} \cdot \vec{v}_p \frac{\partial n_p^0}{\partial \varepsilon_p}\ \delta\varepsilon_p(\vec{q}, \omega) = iI[n_{p'}] \tag{5.94}$$

5.8 Conservation Laws

Let us first define

$$n(r,t) \equiv \sum_\sigma \int \frac{d^3p}{(2\pi\hbar)^3}\ n_{\vec{p}\sigma}(\vec{r}, t) \tag{5.95}$$

By integrating the Landau equation over $\vec{p}$ (and σ) we get

$$\frac{\partial}{\partial t}\left(\sum_\sigma \int_p n_{p\sigma} \right) + \frac{\partial}{\partial \vec{r}} \cdot \left(\int_p \sum_\sigma \vec{v}_p n_{p\sigma} \right) + \int_p \sum_\sigma \frac{\partial}{\partial \vec{p}} (\vec{f}_p n_p) = \sum_\sigma \int_p I_p[n_p] \tag{5.96}$$

The number of quasiparticles is conserved upon collisions, therefore

$$\begin{aligned} \sum_\sigma \int_p I[n_p] &= 0 \\ \int_p \frac{\partial}{\partial p} \cdot [f_p n_p] &= 0 \end{aligned} \tag{5.97}$$

Let us define the current $\vec{j}(r,t) \equiv \sum_\sigma \int_{\vec{p}} \vec{v}_p n_{\vec{p}\sigma}$. The continuity equation, which is just charge conservation, is

$$\frac{\partial n(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{j}(\vec{r}, t) = 0 \tag{5.98}$$

where

$$\vec{j} = \sum_\sigma \int_p \vec{v}_{\vec{p}}\ n_{\vec{p}\sigma} = \sum_\sigma \int_p \nabla_{\vec{p}}\ \varepsilon_{p\sigma}(r,t) n_{\vec{p}\sigma}(r,t) \tag{5.99}$$

On linearizing around equilibrium we get an expression for the current density (superscript zero denotes equilibrium quantities)

$$\vec{j} = \sum_\sigma \int_p (\nabla_p \varepsilon^0_{p\sigma} \ \delta n_{\sigma p} + \nabla_p \delta\varepsilon_p n^0_p) \tag{5.100}$$

Since

$$\delta\overline{n}_{p\sigma} = \delta n_{p\sigma} - \frac{\partial n^0_{p\sigma}}{\partial \varepsilon_{p\sigma}} \delta\varepsilon_{p\sigma} \tag{5.101}$$

which allows us to write the current as

$$\vec{j} = \sum_\sigma \int_p \left[\vec{\nabla}_p \varepsilon^0_{p\sigma} \ \delta n_{p\sigma} - \vec{\nabla}_p n^0_p \ \delta\varepsilon_p \right] \tag{5.102}$$

Using that

$$\vec{\nabla}_p n^0_{p\sigma} = \frac{\partial n^0_{p\sigma}}{\partial \varepsilon_{p\sigma}} \vec{\nabla}_p \varepsilon^0_{p\sigma} \tag{5.103}$$

the current now becomes

$$\vec{j} = \sum_\sigma \int_p (\vec{\nabla}_p \varepsilon^0_{p\sigma}) \delta\overline{n}_{p\sigma} \tag{5.104}$$

Similarly we can write

$$\delta\varepsilon_{p\sigma} = \sum_\sigma \int_{p'} f_{p\sigma,p'\sigma'} \ \delta n_{p'\sigma'}(r,t) \tag{5.105}$$

$$\delta\overline{n}_{p\sigma} = \delta n_{p\sigma} - \frac{\partial n^0_{p\sigma}}{\partial \varepsilon_{p\sigma}} \sum_{\sigma'} \int_{p'} f_{p\sigma,p'\sigma'} \delta n_{p'\sigma'} \tag{5.106}$$

I'll keep only the Fermi surface contribution, and find that the current takes the form

$$\begin{aligned} \vec{j}(r,t) &= \sum_\sigma \int_p (\vec{\nabla}_p \varepsilon^0_p) \delta n_{\vec{p}\vec{\sigma}}(\vec{r},t) \left(1 + \frac{1}{3} \ F^{(S)}_1\right) \\ &= \left(\frac{1 + \frac{1}{3} F^S_1}{m^*}\right) \sum_\sigma \int_p \vec{p} \ \delta n_{\vec{p}\vec{\sigma}}(r,t) \end{aligned} \tag{5.107}$$

Galilean invariance implies $1 + \frac{1}{3} \ F^{(S)}_1 = \frac{m^*}{m}$. Hence we have

$$\vec{j} = \frac{1}{m} \sum_\sigma \int_p \vec{p} \delta n_{\vec{p}\sigma} = \frac{\vec{\mathcal{P}}}{m} \tag{5.108}$$

where the last ratio is momentum density over the bare mass. Otherwise, the general result is

$$\vec{j} = \left(\frac{1 + \frac{1}{3}F_1^S}{m^*}\right)\vec{\mathcal{P}} \tag{5.109}$$

5.8.1 Momentum Conservation

The local momentum density $\vec{g}(\vec{r}, t)$ is given by

$$\vec{g}(r,t) = \sum_\sigma \int_p \vec{p} n_{\vec{p}\sigma}(r,t) \tag{5.110}$$

It obeys the local conservation law

$$\frac{\partial \vec{g}_i}{\partial t} + \nabla_j T_{ij} + \sum_\sigma \int_p \frac{\partial \varepsilon_{p\sigma}}{\partial r_i} n_{p\sigma} = 0 \tag{5.111}$$

where

$$T_{ij} = \sum_\sigma \int_p p_i \frac{\partial \varepsilon_{p\sigma}}{\partial p_j} n_{p\sigma} \tag{5.112}$$

is (almost!) the stress tensor of the Fermi fluid. Using that

$$\frac{\partial \varepsilon_{p\sigma}}{\partial r_i} n_{p\sigma} = \frac{\partial}{\partial r_i}(\varepsilon_{p\sigma} n_{p\sigma}) - \varepsilon_{p\sigma}\frac{\partial n_{p\sigma}}{\partial r_i} \tag{5.113}$$

and

$$\sum_\sigma \int_p \varepsilon_{p\sigma}\frac{\partial n_{p\sigma}}{\partial r_i} = \frac{\partial}{\partial r_i} E - n(\vec{r},t)\nabla_i U(\vec{r},t) \tag{5.114}$$

we can define the stress tensor Π_{ij}

$$\Pi_{ij} = T_{ij} + \delta_{ij}\left(\sum_\sigma \int_p \varepsilon_{p\sigma} n_{p\sigma} - E\right) \tag{5.115}$$

which implies that

$$\frac{\partial g_i(\vec{r},t)}{\partial t} + \nabla_j \Pi_{ij}(\vec{r},t) + n(\vec{r},t)\nabla_i U(\vec{r},t) = 0 \tag{5.116}$$

5.8.2 Energy Conservation

Multiplying by $\varepsilon_{p\sigma}$ and $\sum_\sigma \int_p$ we obtain

$$\sum_\sigma \int_p \varepsilon_{p\sigma} \frac{\partial n_{p\sigma}}{\partial t} + \vec{\nabla} \cdot \sum_\sigma \int_p \left(\nabla_p \varepsilon_{p\sigma}\right) \varepsilon_{p\sigma} n_{p\sigma} = 0 \tag{5.117}$$

We can now define the energy current density

$$\vec{j}_E = \sum_\sigma \int_p (\nabla_p \varepsilon_{p\sigma})(\varepsilon_{p\sigma} - U) n_{p\sigma} \tag{5.118}$$

which obeys the energy conservation equation

$$\frac{\partial}{\partial t}(E - Un) + \vec{\nabla} \cdot \vec{j}_E = -\vec{j} \cdot \vec{\nabla} U \tag{5.119}$$

5.9 Collective modes: Zero sound

We will now look at the solutions of the Landau-Silin kinetic equation, Eq.(5.91), in the limit $T \to 0$ in which the collision integral can be neglected. By linearizing this equation we obtain

$$\left(\frac{\partial}{\partial t} + \vec{v}_p \cdot \vec{\nabla}\right) \delta n_p(r,t) - \frac{\partial n_p^0}{\partial \varepsilon_p} \vec{v}_p \cdot \vec{\nabla} \delta\varepsilon_p(r,t) = 0 \tag{5.120}$$

with

$$\delta\varepsilon_p(r,t) = U(r,t) + \int_{p'} f_{pp'} \delta n_{p'}(r,t) \tag{5.121}$$

We now go to its Fourier transform and assume that the external potential is monocromatic

$$U(r,t) \equiv U e^{i(\vec{q}\cdot\vec{r} - \omega t)} \tag{5.122}$$

and

$$\delta n_p(r,t) = \delta n_p(q,\omega) e^{i(\vec{q}\cdot\vec{r} - \omega t)} \tag{5.123}$$

we obtain

$$(\omega - \vec{q} \cdot \vec{v}_p)\delta n_p + \frac{\partial n_p^0}{\partial \varepsilon_p} \vec{q} \cdot \vec{v}_p (U + \int_{p'} f_{pp'} \delta n_{p'}) = 0 \tag{5.124}$$

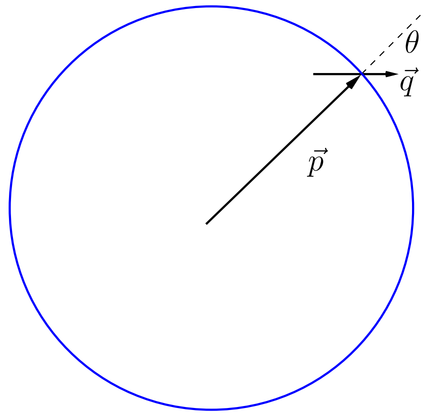


Figure 5.4: A particle-hole fluctuation with momentum propagating with $\vec{q}$ near a point $\vec{p}$ on the Fermi surface.

Let us write δn_p in terms of ν_p defined by

$$\delta n_p \equiv -\frac{\partial n_p^0}{\partial \varepsilon_p}\nu_p \tag{5.125}$$

where we have assumed that only the Fermi surface matters. Within this notation we find that the linearized kinetic equation takes the form

$$\nu_{\vec{p}} + \frac{\vec{q}\cdot\vec{v}_{\vec{p}}}{\omega - \vec{q}\cdot\vec{v}_{\vec{p}}}\int_{\vec{p}'} f(\vec{p},\vec{p}')\frac{\partial n_{\vec{p}'}^0}{\partial \varepsilon_{\vec{p}'}}\nu_{\vec{p}'} = \frac{\vec{q}\cdot\vec{v}_{\vec{p}}}{\omega - \vec{q}\cdot\vec{v}_p}U \tag{5.126}$$

with $\vec{p}$ on the Fermi surface.

We will now make use of the azimuthal symmetry to expand the fluctuation in partial waves

$$\nu_{\vec{p}} = \sum_{\ell=0}^{\infty}\nu_\ell P_\ell(\cos\theta) \tag{5.127}$$

and write

$$\int_{p'} f(\vec{p},\vec{p}')\frac{\partial n_{\vec{p}'}^0}{\partial \varepsilon_{\vec{p}'}}\nu_{\vec{p}'} = -\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}F_\ell^{(S)}P_\ell(\cos\theta)\nu_\ell \tag{5.128}$$

Let us now define the dimensionless parameter s

$$s = \frac{\omega}{qv_F} \tag{5.129}$$

and

$$\Omega_{\ell\ell'}(s) = \Omega_{\ell'\ell}(s) = \frac{1}{2}\int_{-1}^{1} dx P_\ell(x) \frac{x}{x-s} P_{\ell'}(x) \tag{5.130}$$

The Landau-Silin Equation now takes the form

$$\frac{\nu_\ell}{2\ell+1} + \sum_{\ell'=0}^{\infty} \Omega_{\ell\ell'}(s) F_{\ell'}^s \frac{\nu_{\ell'}}{2\ell'+1} = -\Omega_{\ell 0}(s) U \tag{5.131}$$

This equation has solutions of the form $\nu_\ell(s)$.

Let us consider first the s-wave channel, $\ell = 0$. In this channel we find

$$\Omega_{00}(s) = 1 + \frac{s}{2}\ln\left(\frac{s-1}{s+1}\right) = 1 + \frac{s}{2}\ln\left|\frac{s-1}{s+1}\right| + i\frac{\pi}{2}s\,\theta(1-|s|) \tag{5.132}$$

and similar expressions in the other channels.

For example, if we assume that the only non-vanishing Landau parameter is in the s-wave channel, $F_0^s \neq 0$ and $F_\ell^{(s)} = 0 (\ell \geq 1)$, we find the simple equation

$$\nu_0(s) + \Omega_{00}(s) F_0^{(s)} \nu_0(s) = -\Omega_{00}(s) U \tag{5.133}$$

whose solution is

$$\nu_0(s) = -\frac{\Omega_{00}(s) U}{1 + F_0^{(s)} \Omega_{00}} \tag{5.134}$$

The equation for the other angular momentum modes, with $\ell \geq 1$, is

$$\frac{\nu_\ell}{2\ell+1} + \Omega_{\ell 0}(s) F_0^{(s)} \nu_0 = -\Omega_{\ell 0}(s) U \tag{5.135}$$

$$\frac{\nu_\ell}{2\ell+1} - \frac{\Omega_{\ell 0} F_0(s) \Omega_{00}^{(s)} U}{1 + F_0^s \Omega_{00}(s)} = -\Omega_{\ell 0}(s) U \tag{5.136}$$

$$\frac{\nu_\ell}{2\ell+1} = \frac{\Omega_{\ell 0}^{(s)} U [\cancel{F_0^{(s)} \Omega_{00}(s)} - 1 - \cancel{F_0^{(s)} \Omega_{00}(s)}]}{1 + F_0^{(s)} \Omega_{00}(s)} \tag{5.137}$$

Hence,

$$\frac{\nu_\ell}{2\ell+1} = \frac{\Omega_{\ell 0}(s)}{\Omega_{00}(s)} \nu_0(s) \tag{5.138}$$

The equation

$$\nu_0(s) = -\frac{\Omega_{00}(s) U}{1 + F_0^s \Omega_{00}(s)} \tag{5.139}$$

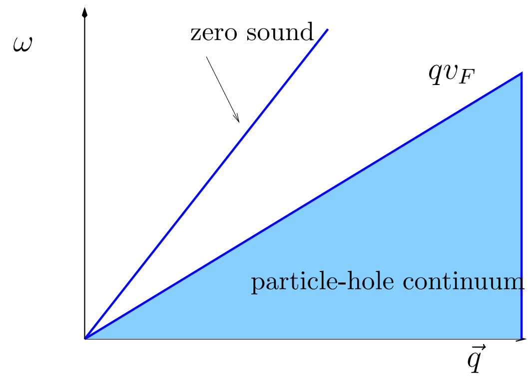


Figure 5.5: Spectrum of collective modes

has poles at the zeros, s_0, of

$$1 + F_0^s \Omega_{00}(s_0) = 0 \tag{5.140}$$

We have the following regimes

- For $0 \leq s < 1$, $\Omega_{00}(s)$ is complex. This solution corresponds to the particle-hole continuum.

- For $1 \leq s < \infty$, $\Omega_{00}(s)$ is a real and monotonically increasing function of s.

In particular, in the latter case we find that if $F_0^{(S)} > 0$, then there is a simple pole with $s_0 > 1$

$$s_0 = \frac{\omega}{|\vec{q}|v_F} = \begin{cases} 1 + e^{-\frac{1}{F_0}}, & \text{for } F_0 \ll 1 \\ \sqrt{\frac{F_0^{(s)}}{3}}, & \text{for } F_0 \gg 1 \end{cases} \tag{5.141}$$

This solution corresponds to an (undamped) sound mode with dispersion $\omega = |\vec{q}|v_F s_0$ and a sound velocity $c_0 = v_F s_0$. This *collective mode* is known as *zero sound*. Notice that the edge of the particle-hole continuum is at $\omega = qv_F$. (See Fig.5.5).

5.10 The Quasiparticle Lifetime

The Landau interactions can in principle give rise to a finite lifetime τ for the quasiparticles. However, if the lifetime remains finite as $\omega \to \mu = E_F \Rightarrow$

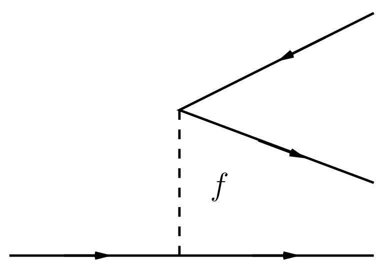


Figure 5.6: A process that leads to a finite quasiparticle lifetime; f denotes a Landau interaction.

the whole theory breaks down. Thus, stability of the Fermi liquid requires that the lifetime to diverge , $\tau \to \infty$ (at $T = 0$) as $\omega \to \mu$. Similarly, if $T > 0 \Rightarrow$ the lifetime must also diverge as $T \to 0(\omega = \mu)$. In principle the lifetime τ is a function of $\omega - \mu$ and T.

The *rate* of decay is $\frac{1}{\tau} = \Gamma(\omega - \mu, T)$. We can compute this function for $|\omega - \mu| \ll \mu$ and $T \ll \mu(\mu = \varepsilon_F = k_B T_F)$. In terms of the Green function the decay rate shows up as an imaginary part of the self energy, namely

$$\begin{aligned} G(\vec{p}, \omega) &= \frac{1}{G_0^{-1}(\vec{p}, \omega) - \Sigma(\vec{p}, \omega)} = \frac{1}{\omega - (\varepsilon_0(\vec{p}) - \mu) - \Sigma(\vec{p}, \omega)} \\ &= \frac{1}{\omega - (\varepsilon_0(\vec{p}) - \mu + \mathrm{Re}\Sigma(\vec{p}, \omega)) - i\mathrm{Im}\Sigma(\vec{p}, \omega)} \end{aligned} \tag{5.142}$$

where

$$\mathrm{sgn}(\Im\Sigma(p, \omega)) = \mathrm{sgn}(\omega - \mu) \qquad \text{at } |\vec{p}| = p_F \tag{5.143}$$

Hence the rate is given by

$$\Gamma(\vec{p}, \omega) = \mathrm{Im}\Sigma(\vec{p}, \omega) = \frac{1}{\tau} \tag{5.144}$$

and in general it is a function of both $\vec{p}$ and ω.

In terms of diagrams, the lifetime arises because there is a finite amplitude for a process with a quasiparticle at $(\vec{p}, \omega)$ in the initial state and a quasiparticle (with some momentum and frequency) and some particle-hole pairs in the finite state (see Fig.5.6). The amplitude is determined by the Landau interaction parameters f that we defined above.

How does this process enter in the computation of the self-energy? There is a term in perturbation theory of the form shown in Fig. 5.7. Alternately,

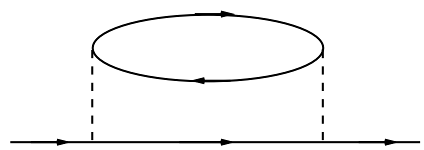


Figure 5.7: Feynman diagram with a contribution to ImΣ.

the imaginary part comes from effects of the collision integral (see Baym and Pethick p. 87) and is given in terms of the t-matrix:

$$\begin{aligned}\frac{1}{\tau_{\vec{p}}} &= \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \\ &\quad 2\pi |\langle p-q, p'+q|t|p,p'\rangle|^2 \delta(\varepsilon(p)+\varepsilon(p')-\varepsilon(p-q)-\varepsilon(p'-q)) \\ &\quad \times [n^0(p')(1-n^0(p'+q))(1-n^0(p-q)) + (1-n^0(p'))n^0(p'+q)n^0(p'-q)]\end{aligned} \tag{5.145}$$

which follows from using Fermi's Golden Rule. The first term in the expression in brackets in Eq.(5.143) represents the rate at which quasiparticles are scattered into new unoccupied states while the second term represents the blocking of such processes due to occupied states.

The t-matrix amplitude $\langle p-q, p'+q|t|p,p'\rangle$ is represented by summing up particle-particle (or particle-hole) ladder diagrams, scattering processes of the type shown in Fig.5.8, given by the solution of the Bethe-salpeter Equation (see Baym and Pethick, p. 77):

$$t_{\vec{p}\vec{p}'}(q,\omega+i\eta) = f_{p,p'} - \sum_{p''\neq p'} f_{p'',p'} \frac{\vec{q}\cdot\vec{\nabla}_{p''} n^0_{p''}}{\omega + i\eta - \vec{q}\cdot\vec{v}_{p''}} t_{p''p'}(q,\omega+i\eta) \tag{5.146}$$

The matrix elements of the t-matrix can also be split into a singlet t^S and triplet t^A channels, and further be expanded in angular momentum components, $t^{S,A}_\ell$:

$$t_{pp'}(q,0) = \sum_\ell t_\ell(q,0) P_\ell(\cos\theta) \tag{5.147}$$

whose coefficients are given by

$$t_\ell^S = \frac{f_\ell^S}{1+\frac{F_\ell^s}{2\ell+1}} \qquad |q| \ll p_F \tag{5.148}$$

$$t_\ell^A = \frac{f_\ell^A}{1+\frac{F_\ell^A}{2\ell+1}} \qquad |q| \ll p_F \tag{5.149}$$

After some algebra on finds that the decay rate at finite temperature T and frequency ε at zero momentum transfer (*i.e.* on the Fermi surface) is given by

$$\frac{1}{\tau_p} \approx \frac{\pi(N(0)t(0))^2}{8\pi v_F(p_F^2/q_c)} \left[(\varepsilon-\mu)^2+(\pi k_B T)^2\right]+\cdots \tag{5.150}$$

($q_c \sim \Lambda$ is a momentum cutoff) which satisfies Landau's assumptions.

The Landau picture we discussed works very well in neutral Fermi fluids (such as the normal phase of ^{3}He) and in most (simple) metals. However we will see that it fails in a number of important situations. In particular it fails in one dimension (for any value of the interaction coupling constants) and also near a *quantum phase transition*. It also fails in systems with strong correlation.

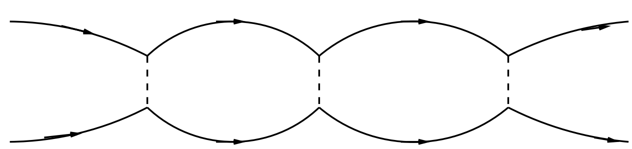


Figure 5.8: Feynman diagram that contributes to $t_{\vec{p}\vec{p}'}(q,\omega+i\eta)$ in the Bethe-Salpeter Equation (5.144).