Chapter 6

Theory of the Luttinger Liquid

6.1 One-dimensional Fermi systems

We will now consider the case of one-dimensional (1D) Fermi systems where the Landau theory fails. The way it fails is quite instructive as it reveals that in 1D these systems are generally at a (quantum) critical point, and it will also teach as valuable lessons on quantum criticality. It will also turn out that the problem of 1D Fermi systems is closely related to the problem of magnetism in 1D, quantum spin chains.

One-dimensional (and quasi-one-dimensional) systems of fermions occur in several experimentally accessible systems. The simplest one to visualize is a *quantum wire*. A quantum wire is a system of electrons in a semiconductor, typically a GaAs-AlAs heterostructure built by molecular beam epitaxy (MBE), in which the electronic motion is laterally confined along two directions, but not along the third. An example of such a channel of length L and width d (here shown as a 2D system) is seen in Fig.6.1a. Systems of this type can be made with high degree of purity with very long (elastic) mean free paths, often tens of microns or even longer. The resulting electronic system is a one-dimensional electron gas (1DEG). In addition to quantum wires, 1DEGs also arise naturally in carbon nanotubes, where they are typically multicomponent (with the number of components being determined by the diameter of the nanotube.)

Other one-dimensional Fermi systems include the edge states of *two-dimensional* electron gases (2DEG) in large magnetic fields in the regime in which the *quantum Hall effects* are seen. (We will discuss this problem

later on.) This case is rather special as these edge states can only propagate in one direction, determined by the sign of the perpendicular magnetic field.

Other (quasi) one-dimensional systems occur in organic compounds, such as TTFTCNQ, some Betchgaard salts (commonly called as the "BEDT's" and the "ET's") and TMSF's. There are also quasi-1D chalcogenide materials, *e.g.* $NbSe_3$ (and others) as well as complex oxides. The some of the oxides, *e.g.* $Sr_{14-x}Ca_xCu_{24}O_{41}$ which can be regarded as a set of weakly coupled ladders (as opposed to chains). Quasi-1D Fermi systems are often used to describe complex ordered states in 2D strongly correlated systems. A typical example are the *stripe phases* of the copper oxide high T_c superconductors, such as $La_{2-x}Sr_xCuO_4$, $La_{2-x}Ba_xCuO_4$, and others.

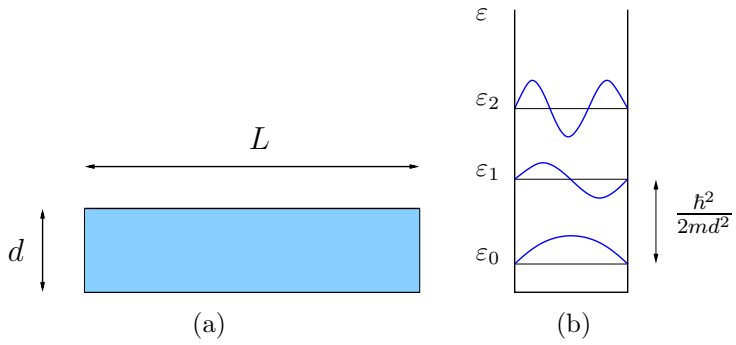

Figure 6.1: a) A long quantum wire of length L and transverse width d ($L \gg d$): a channel for the electron fluid. Only 2D case for simplicity. b) Square well spectrum of the transverse quantum single particle states confined by the finite width d of the wire.

We will consider first the conceptually simpler example of the quantum wire. We will assume that the electron density is such that the Fermi energy lies below the energy of the first excited state. The result is that the single-particle states with momenta in the range $-p_F < p < p_F$ are occupied and the states outside this range are empty. Thus the Fermi "surface" of this system reduces to two Fermi points at $\pm p_F$. We will assume that the wire is long enough, $L \gg d$, so that the single particle states fill up densely the momentum axis, and that the density is high enough so that $\Delta p = 2\pi\hbar/L \ll p_F$. On the other hand, we will assume that the wire is narrow enough so that the next band of (excited) states can effectively be neglected,

$\varepsilon_F \ll \hbar^2/(2md^2)$.* In practice this means that the following inequality holds

$$\frac{L}{d} \gg 1 \gg \frac{d}{\lambda_F} \tag{6.1}$$

where $\lambda_F = \hbar/p_F$ is the Fermi wavelength.

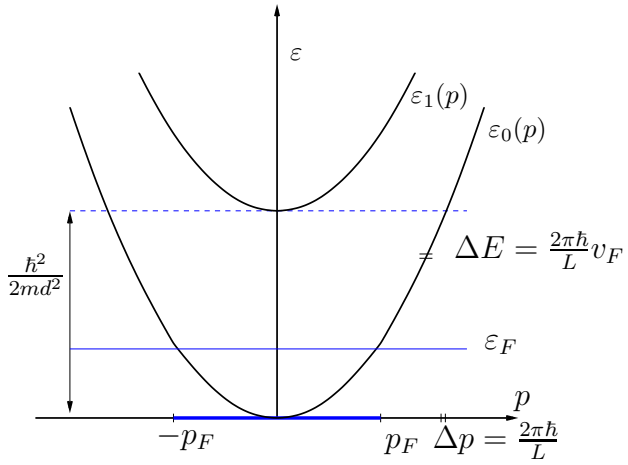

Figure 6.2: Energy-momentum relation of the two lowest bands of propagating non-relativistic free fermions along the length of the quantum wire; ε_F is the Fermi energy, $\pm p_F$ are the two Fermi points, $v_F = p_F/m$ is the Fermi velocity. The filled Fermi sea (occupied states) are shown in blue; ΔE and Δp are the level spacings in a finite wire of length L. We have shifted the minimum of the energy of the lowest band to be at zero.

The Hamiltonian for the 1DEG is $H = H_0 + H_{\rm int}$ where

$$H_0 = \sum_{\sigma=\uparrow,\downarrow} \int_0^L dx\, \psi_\sigma^\dagger(x) \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu \Big) \psi_\sigma(x) \tag{6.2}$$

if the free fermion Hamiltonian, and

$$H_{\rm int} = \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int_0^L dx \int_0^L dx'\, \psi_\sigma^\dagger(x)\psi_\sigma(x)\, U(x-x')\, \psi_{\sigma'}^\dagger(x')\psi_{\sigma'}(x') \tag{6.3}$$

*If the Fermi energy $\varepsilon_F \gtrsim \hbar^2/(2md^2)$ the two lowest bands will be partially occupied and there will be four Fermi points.

where $U(x-x')$ is the interaction potential, which can be Coulomb or short ranged, depending of the physical situation. In what follows for simplicity we will use periodic boundary conditions, that require

$$\psi_\sigma(x+L) = \psi_\sigma(x) \tag{6.4}$$

which amount to wrap the system ('compactification') on a circle. Sometimes we may want to use the more physical open boundary conditions.

In many cases we will be interested in lattice systems. So, consider a one dimensional chain of N sites (atoms) and lattice spacing a, and total length $L = Na$. The lattice Hamiltonian is

$$\begin{aligned} H = \quad & \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} t\Big(\psi_\sigma^\dagger(j)\psi_\sigma(j+1) + \text{h.c.}\Big) \\ & + \sum_{j=1}^{N} \Big(U n_\uparrow(j) n_\downarrow(j) + V n(j) n(j+1)\Big) \end{aligned} \tag{6.5}$$

where $n_\sigma(j) = \psi_\sigma^\dagger(j)\psi_\sigma(j)$ is the fermion occupation number with spin σ at site j, and $n(j) = \sum_\sigma n_\sigma(j)$ is the total occupation number (i.e. the charge) at site j. This Hamiltonian is known as the extended Hubbard model. Here U is the on-site interaction and V is the nearest neighbor repulsion. This model describes a system of electrons with hopping only between nearest neighboring sites; t, the hopping amplitude, is the local kinetic energy. This system has only one band of single particle states with dispersion relation

$$\varepsilon(p) = 2t\cos(pa) \tag{6.6}$$

In the thermodynamic limit, $N \to \infty$, the momenta p lie on the first Brillouin Zone, $-\frac{\pi}{a} < p \leq \frac{\pi}{a}$. In general we will be interested in a system either at fixed chemical potential μ or at fixed density $n = N_e/N$. The effective model for interacting systems that we will discuss will describe equally (with minor changes) the low energy physics of both continuum and lattice systems.

What is special about one dimension?

- In the Landau theory of the Fermi liquid we considered the low energy states and we saw that they can be described in terms of *particle-hole pairs*. In dimensions $D > 1$ the momentum $\delta\vec{q}$ of the pair is not necessarily parallel to the Fermi wave vector $\mathbf{p}_F$ of the location of the

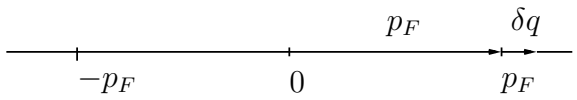

Figure 6.3: One-dimensional kinematics: the momentum of a particle-hole pair of momentum q is always parallel (or anti-parallel) to the Fermi wave vector p_F.

Fermi surface where the pair is excited. However in 1D δq *must* be either parallel or anti-parallel to the Fermi momentum p_F as the FS has collapsed to just two (or more) points.

- This kinematic restriction implies that particle-hole pairs form effectively long lived bound states, the collective modes, since the particle and the hole move with the same speed (the Fermi velocity). We will see that this implies that the low energy effective theory is a theory of *bosons*. This is the main reason why the non-perturbative theory of 1D fermions, bosonization, works.

- Another insight can be gleaned by looking at the density correlators, whose singularities are the collective modes. In $D > 1$ the retarded density-density correlation function $D^R(\mathbf{q}, \omega)$ of a free fermion is

$$D^R(\mathbf{q}, \omega) = \int \frac{d^D p}{(2\pi)^D} \frac{n_{\mathbf{p}} - n_{\mathbf{p}+\mathbf{q}}}{\omega - \varepsilon(\mathbf{p}+\mathbf{q}) + \varepsilon(\mathbf{p}) + i\eta} \tag{6.7}$$

For low momenta $|\mathbf{q}| \ll p_F$ and at low energies $\omega \ll E_F$, $D^R(\mathbf{q}, \omega)$ can be written as an integral on the Fermi surface

$$\begin{aligned} D^R(\mathbf{q}, \omega) &\simeq \int \frac{d^D p}{(2\pi)^D} \frac{\mathbf{q} \cdot \widehat{\mathbf{p}}_F}{\omega - (\mathbf{q} \cdot \widehat{\mathbf{p}}_F) v_F + i\eta} \, \delta\left(|\mathbf{p}| - p_F\right) \\ &= \frac{p_F^{D-1}}{(2\pi)^D} \oint_{FS} d\widehat{\mathbf{p}}_F \frac{\mathbf{q} \cdot \widehat{\mathbf{p}}_F}{\omega - (\mathbf{q} \cdot \widehat{\mathbf{p}}_F) v_F + i\eta} \end{aligned} \tag{6.8}$$

For $D > 1$ the angular integration is a function of $\mathbf{q}$ and ω which has branch cuts. For instance, in 3D we saw that the result is

$$D^R(\mathbf{q}, \omega) \sim 1 + \frac{\omega}{2 q v_F} \ln \left| \frac{\omega - q v_F}{\omega + q v_F} \right| + \dots \tag{6.9}$$

We saw before that the branch cuts mean that the collective modes (zero sound) eventually becomes Landau damped.

- However, in 1D there is no such angular integration (the FS is just two points!) and the result is

$$D^R(q,\omega) \sim \frac{q}{2\pi}\Big(\frac{1}{\omega - qv_F + i\eta} - \frac{1}{\omega + qv_F + i\eta}\Big) \tag{6.10}$$

  This expression contains two singularities, two *poles*, representing bosonic states that move to the right (the first term) or to the left (the second term). It is easy to check that this result is consistent with f-sum rule.

Furthermore, these results suggests that a theory of free fermions in 1D must be, in some sense, equivalent to a theory of a Bose field whose excitations obey the dispersion relation $\omega = pv_F$. In other terms, the bosons are density fluctuations, which in this case are just sound waves.

6.2 Dirac fermions and the Luttinger Model

We will now proceed to construct an effective low energy theory by following a procedure similar to what led to the Landau theory of the Fermi liquid. The result, however, will be quite different in 1D.

To this end we will first look at the free fermion system and focus on the low energy excitations. In 1D instead of a Fermi surface we have (at least) two Fermi points at $\pm p_F$. The low energy fermionic states have thus momenta $p \sim \pm p_F$ and single particle energy close to ε_F:

$$\varepsilon(p) \simeq \varepsilon_F + (|p| - p_F)v_F + \dots \tag{6.11}$$

We are interested in the electronic states near the Fermi energy. Thus, consider the fermion operator $\psi_\sigma(x)$, whose Fourier expansion is (we will set $\hbar = 1$ from now on) in the thermodynamic limit ($L \to \infty$)

$$\psi_\sigma(x) = \int \frac{dp}{2\pi}\, \psi_\sigma(p)\, e^{ipx} \tag{6.12}$$

Only its Fourier components near $\pm p_F$ describe low energy states. This suggests that we restrict ourselves to the modes of the momentum expansion in a neighborhood of $\pm p_F$ of width 2Λ, and that we write

$$\psi_\sigma(x) \simeq \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi}\ e^{i(p+p_F)x}\psi_\sigma(p+p_F) + \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi}\ e^{i(p-p_F)x}\psi_\sigma(p-p_F) \tag{6.13}$$

and that we define *right* and *left* moving fields $\psi_{\sigma,R}(x)$ and $\psi_{\sigma,L}(x)$ such that

$$\psi_\sigma(x) \simeq \; e^{ip_F x}\; \psi_{\sigma,R}(x) + e^{-ip_F x}\; \psi_{\sigma,L}(x) \tag{6.14}$$

Thus we have split off the rapidly oscillating piece of the field and we focus on the slowly varying parts, $\psi_{\sigma,R}(x)$ and $\psi_{\sigma,L}(x)$, whose Fourier transforms are

$$\psi_{\sigma,R}(p) = \psi_\sigma(p + p_F), \qquad \text{and} \qquad \psi_{\sigma,L}(p) = \psi_\sigma(p - p_F) \tag{6.15}$$

respectively.

The free fermion Hamiltonian

$$H_0 = \sum_\sigma \int \frac{dp}{2\pi}\; \varepsilon(p)\; \psi^\dagger_\sigma(p)\psi_\sigma(p) \tag{6.16}$$

becomes

$$H_0 = \sum_\sigma \int_{-\Lambda}^{\Lambda} \frac{dp}{2\pi}\; p v_F \left(\psi^\dagger_{\sigma,R}(p)\psi_{\sigma,R}(p) - \psi^\dagger_{\sigma,L}(p)\psi_{\sigma,L}(p)\right) \tag{6.17}$$

where we have linearized the dispersion $\varepsilon(p)$ near the Fermi momenta $\pm p_F$. Let us define the two-component spinor

$$\psi_\sigma(x) = \begin{pmatrix} \psi_{\sigma,R}(x) \\ \psi_{\sigma,L}(x) \end{pmatrix} \tag{6.18}$$

in terms of which the free fermion Hamiltonian is

$$H_0 = \sum_\sigma \int \frac{dp}{2\pi}\, \psi^\dagger_\sigma(p)\sigma_3 p v_F \psi_\sigma(p) = \sum_\sigma \int dx\; \psi^\dagger_\sigma(x)\sigma_3 i v_F \partial_x \psi_\sigma(x) \tag{6.19}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.20}$$

In this form the effective low energy Hamiltonian reduces to the (massless) Dirac Hamiltonian in 1D. In Fig.6.2 we show the dispersion in the spinor notation.

Most of the interaction terms we discussed above can be expressed in terms of the local densities of right and left moving fermions

$$j_{R,\sigma}(x) = \psi^\dagger_{R,\sigma}(x)\psi_{R,\sigma}(x), \qquad j_{L,\sigma}(x) = \psi^\dagger_{L,\sigma}(x)\psi_{L,\sigma}(x) \tag{6.21}$$

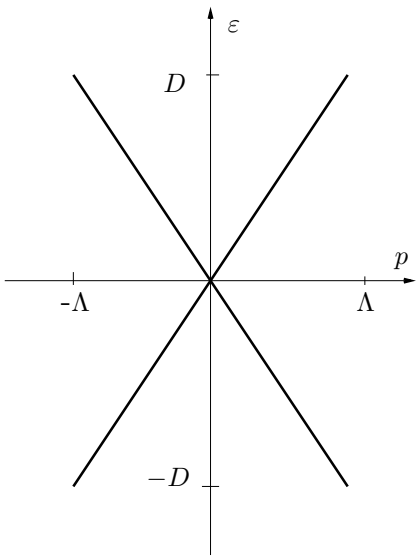

Figure 6.4: The Dirac dispersion; the slope is the Fermi velocity v_F. The momentum cutoff is Λ and the energy cutoff is $D = v_F\Lambda$.

from which we can write the slowly varying part of the charge density operator $j_0(x)$ (i.e. with Fourier component with wave vectors close to $p = 0$) and the charge current density $j_1(x)$ as

$$j_0(x) = j_R(x) + j_L(x), \qquad j_1(x) = j_R(x) - j_L(x) \tag{6.22}$$

which is a 2-vector of the form

$$j_\mu(x) = (j_0, j_1) \tag{6.23}$$

with $\mu = 0, 1$ (not to be confused with the chemical potential!). Thus, the coupling to a slowly varying external electromagnetic field $A_\mu(x) = (A_0, A_1)$ is represented by a term of the form

$$H_{\rm em} = \int dx \Big(- eA_0(x)j_0(x) + \frac{e}{c}A_1(x)j_1(x)\Big) \tag{6.24}$$

The actual particle density operator of the microscopic system,

$$\rho(x) = \sum_\sigma \psi^\dagger_\sigma(x)\psi_\sigma(x) \tag{6.25}$$

can be written in the form (using the decomposition of the Fermi field in right and left movers)

$$\rho(x) = \rho_0 + j_0(x) + \sum_\sigma \Big(e^{2ip_F x} \psi^\dagger_{R,\sigma}(x)\psi_{L,\sigma}(x) + e^{-2ip_F x}\psi^\dagger_{L,\sigma}(x)\psi_{R,\sigma}(x) \Big) + \dots \tag{6.26}$$

where $\rho_0 = \frac{N_e}{L} = \frac{2p_F}{\pi}$ is the average total density of electrons (including spin), and where $\dots$ represent terms that oscillate more rapidly for large p_F. The significance of the oscillatory terms can be seen by adding a coupling to a periodic potential $V(x)$ (with wave vector $2p_F$) of the form*

$$V(x) = V_0 \cos(2p_F x) \tag{6.27}$$

to the Hamiltonian, leading to a new term†

$$\begin{aligned} H_{\rm pot} &= \int dx V(x)\rho(x) \\ &= -e \int \frac{dp}{2\pi} \int \frac{dq}{2\pi} V(q)\psi^\dagger(p+q)\psi(p) \\ &= \int \frac{dp}{2\pi} (-eV_0) \Big(\psi^\dagger_R(p)\psi_L(p) + \psi^\dagger_L(p)\psi_R(p) \Big) \\ &= \int dx\ (-eV_0) \Big(\psi^\dagger_R(x)\psi_L(x) + \psi^\dagger_L(x)\psi_R(x) \Big) \end{aligned} \tag{6.28}$$

In other terms, a periodic potential of wave vector $K = 2p_F$ causes backscattering: it scatters a right moving fermion into a left moving fermion (and viceversa). Similarly, a periodic potential of wave vector $K \ll 2p_F$ scatters right movers into right movers (and left movers into left movers).

Thus, in the case of a free fermion coupled to a periodic potential $V(x)$ with wave vector $K = 2p_F$, the potential induces backscattering that mixed the two Fermi points at $\pm p_F$. This leads to the existence of an energy gap at the Fermi energy. In terms of the Dirac Hamiltonian, the periodic potential $V(x)$ leads to the Hamiltonian

$$H = \int dx\ \psi^\dagger(x) \Big(i v_F \sigma_3 \partial_x + eV_0 \sigma_1) \psi(x) \tag{6.29}$$

*For simplicity here we consider only potentials with wave vectors commensurate with the fermion density, $K = 2p_F$. More general cases can also be considered and lead to interesting physical effects.

†We will drop the spin indices from now on.

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.30}$$

where eV_0 is the energy gap Δ, and is usually denoted by $\Delta = mv_F^2$ (instead of using the speed of light, that is).

In the Dirac theory it is useful to define the matrices $\alpha = \sigma_3$ (in 1D) and $\beta = \sigma_1$, such that the Dirac Hamiltonian reads

$$H = \int dx\ \psi^\dagger(x)\Big(\alpha i v_F \partial_x + \beta\Delta\Big)\psi(x) \tag{6.31}$$

The single-particle spectrum is consists of particles and holes with energy $\varepsilon(p) = \sqrt{v_F^2 p^2 + \Delta^2}$. In the Dirac theory it is customary to define Dirac's γ-matrices. In this 1D case there are just two of them, $\gamma_0 = \beta = \sigma_1$ and $\gamma_1 = \beta\alpha = i\sigma_2$. They satisfy the algebra

$$\{\gamma_0, \gamma_1\} = 0, \qquad \gamma_0^2 = 1, \qquad \gamma_1^2 = -1 \tag{6.32}$$

If we define $\bar\psi = \psi^\dagger\gamma_0$, the fermion mass term is

$$\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R = \psi^\dagger\gamma_0\psi = \bar\psi\psi \tag{6.33}$$

6.3 Order parameters of the 1DEG

Similarly we can also define the matrix $\gamma^5 = \gamma_0\gamma_1 = \sigma_3$, and the bilinear $\bar\psi\gamma^5\psi$

$$\bar\psi\gamma_5\psi = \psi_R^\dagger\psi_L - \psi_L^\dagger\psi_R \tag{6.34}$$

It is straightforward to see that the density $\rho(x)$ can be written as

$$\rho(x) = \rho_0 + j_0(x) + \cos(2p_F x)\ \bar\psi(x)\psi(x) + i\ \sin(2p_F x)\ \bar\psi(x)\gamma^5\psi(x) \tag{6.35}$$

From here we see that if $\langle\bar\psi(x)\psi(x)\rangle \neq 0$ (or $\langle\bar\psi(x)\gamma_5\psi(x)\rangle \neq 0$), then the expectation value of the charge density $\langle\rho(x)\rangle$ has a modulated component (over the background ρ_0). If this were to occur *spontaneously* (i.e. in the absence of an external periodic potential) then the ground state of the system would be a *charge density wave* (CDW). Hence, $\langle\bar\psi\psi\rangle$ and $\langle i\bar\psi\gamma^5\psi\rangle$ play the role of the *order parameters* of the CDW state.*

*We can also see that the expectation value of the density will be *even* (invariant) under inversion, $x \to -x$ (*i.e.* parity), if $\langle\bar\psi(x)\gamma^5\psi(x)\rangle = 0$; conversely, if this expectation value is not zero, the density will not be even under parity, which amounts to a phase shift.

In the absence of the periodic potential the original system is translationally invariant. The periodic potential breaks translation invariance. To see how that works we define the transformation

$$\psi(x) \to e^{i\theta\gamma^5}\psi(x) = \begin{pmatrix} e^{i\theta}\psi_R(x) \\ e^{-i\theta}\psi_L(x) \end{pmatrix} \tag{6.36}$$

which is known as a *chiral transformation*. Under this transformation the two component vector

$$\begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix} \tag{6.37}$$

transforms as a rotation

$$\begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix} \to \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} \bar{\psi}\psi \\ i\bar{\psi}\gamma^5\psi \end{pmatrix} \tag{6.38}$$

under which the density operator becomes

$$\begin{aligned} \rho(x) &\to \rho_0 + j_0(x) + e^{i2(p_F x-\theta)}\psi_R^\dagger(x)\psi_L(x) + e^{-i2(p_F x-\theta)}\psi_L^\dagger(x)\psi_R(x) \\ &= \rho\left(x - \frac{\theta}{p_F}\right) \end{aligned} \tag{6.39}$$

Therefore, a chiral transformation by an angle θ is equivalent to a translation of the charge density by a displacement $d = \frac{\theta}{p_F}$. Notice that transformations by $\theta = n\pi$ have no physical effect as they amount to translations by a distance $n\frac{\pi}{p_F} = 2n\frac{\pi}{K} = n\ell$, i.e. an integer number of periods $\ell = \frac{2\pi}{K}$ of the CDW. Thus only chiral transformations modulo π are observable.

In a similar fashion we can define an operator corresponding to a *spin density wave* (SDW). Indeed, the local magnetization (or spin polarization) density

$$m^a(x) = \psi_\sigma^\dagger(x)\tau^a_{\sigma,\sigma'}\psi_{\sigma'}(x) \tag{6.40}$$

(where τ^a are the three Pauli matrices, acting only on the spin indices σ, σ'), which can be expressed as

$$m^a(x) = j_0^a(x) + e^{2ip_F x}\psi_{R,\sigma}^\dagger(x)\tau^a_{\sigma,\sigma'}\psi_{L,\sigma'}(x) + \text{h.c.} \tag{6.41}$$

where $j_0^a(x)$ is the slowly varying spin density

$$j_0^a(x) = \psi_{R,\sigma}^\dagger(x)\tau^a_{\sigma,\sigma'}\psi_{R,\sigma'}(x) + \psi_{L,\sigma}^\dagger(x)\tau^a_{\sigma,\sigma'}\psi_{L,\sigma'}(x) \tag{6.42}$$

and, similarly, the spin current is

$$j_1^a(x) = \psi^\dagger_{R,\sigma}(x)\tau^a_{\sigma,\sigma'}\psi_{R,\sigma'}(x) - \psi^\dagger_{L,\sigma}(x)\tau^a_{\sigma,\sigma'}\psi_{L,\sigma'}(x) \tag{6.43}$$

The SDW order parameters are

$$N^a(x) = \bar\psi_{s,\sigma}(x)\tau^a_{\sigma,\sigma'}\psi_{s,\sigma'}(x), \qquad N^a_c(x) = i\bar\psi_{s,\sigma}(x)\tau^a_{\sigma,\sigma'}\gamma^5_{s,s'}\psi_{s',\sigma'}(x) \tag{6.44}$$

(where $s, s' = R, L$) and describe modulations of the local spin polarization with wave vector $K = 2p_F$.

Finally let us discuss *pairing* operators. We will see later in this class that they are associated with *superconductivity*. Here we will be interested in pairing operators associated with *uniform* ground states (although modulated states are also possible). Pairing operators that create a pair of quasiparticles with total momentum (close to) zero have the form

$$O_{SP}(x) = \langle\psi^\dagger_{R,\sigma}(x)\psi^\dagger_{L,-\sigma}(x)\rangle, \qquad O_{TP}(x) = \langle\psi^\dagger_{R,\sigma}(x)\psi^\dagger_{L,\sigma}(x)\rangle \tag{6.45}$$

where $O_{SP}(x)$ corresponds to (spin) *single* pairing, and $O_{TP}(x)$ to (spin) *triplet* pairing. Differently from all the operators we discussed so far, the pairing operators do not conserve particle number.

We will see below that all of these order parameters break some (generally continuous) symmetry of the system: translation invariance for th CDW, spin rotations and translation invariance for the SDW, and (global) gauge invariance (associated with particle number conservation) for the superconducting case. There is a theorem, known and the *Mermin-Wagner Theorem*, that states that in a 1D quantum system continuous symmetries cannot be spontaneously broken.† More precisely, this theorem states that correlation functions of order parameters that transform under a continuous global symmetry *cannot decay more slowly than a power law function of distance (or time)*. We will now see that in the case of the Luttinger model the behavior is a exactly power law. We will interpret this as saying that the system is at a (quantum) critical point.

6.4 The Luttinger Model: Bosonization

We will now consider the Luttinger (Tomonaga) model.* We will consider first the case of spinless fermions. The Hamiltonian density $\mathcal{H}$ of the Lut-

†In high energy physics this theorem is often attributed to S. Coleman.

*Also known as the massless Thirring model in high energy physics.

tinger model is

$$\mathcal{H} = \psi^\dagger(x)\Big(\alpha i v_F \partial_x + \beta\Delta\Big)\psi(x) + 2g_2\rho_R(x)\rho_L(x) + g_4\Big(\rho_R(x)^2 + \rho_L(x)^2\Big) \quad (6.46)$$

where $\rho_R(x) \equiv \psi_R^\dagger(x)\psi_R(x)$ and $\rho_L(x) \equiv \psi_L^\dagger(x)\psi_L(x)$ denote the densities of right and left movers, respectively. Here $g_2 = \widetilde{V}(0) - \widetilde{V}(2p_F)$ and $g_4 = \widetilde{V}(0)/2$, where $\widetilde{V}(q)$ is the Fourier transform of the interaction potential. Hence, g_2 measures the strength of the backscattering interactions and g_4 the forward scattering interactions.[†] For a model of spinless fermions on a lattice near half-filling with nearest neighbor interactions with coupling constant V, the coupling constants become $g_2 = 2V$ and $g_4 = V$.

Notice that the Hamiltonian of the Luttinger model has the same form as the Landau theory of the Fermi liquid in which the quasiparticles have only forward scattering interactions, here represented by g_4. Here we have also included backscattering process labeled by g_2, with a wave vector $2p_F$ (*i.e.* across the "Fermi surface"). Due to the kinematical restrictions of a *curved* Fermi surface, in the Landau theory backscattering processes have negligible effects. We will see that in 1D (where there is no curvature) they play a key role.

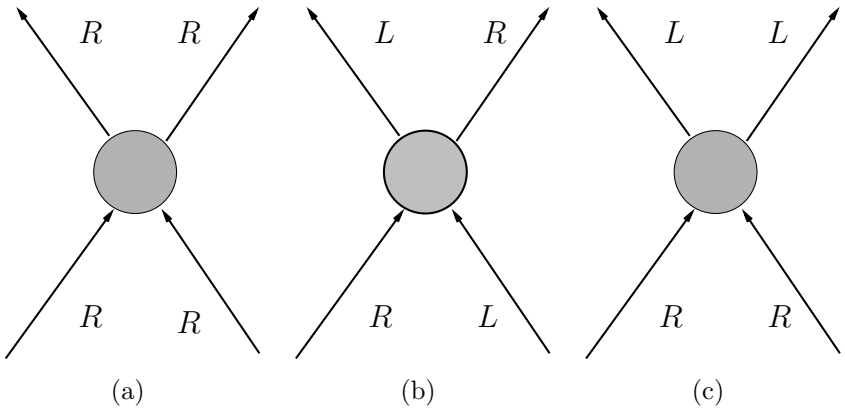

Figure 6.5: a) forward scattering, b) backscattering, c) umklapp scattering

Precisely at half-filling in addition to the back-scattering and forward scattering interactions (Fig.6.5a and 6.5b), an umpklapp interaction must

[†]The notation has a historical origin and it is by now traditional.

also be considered: this is a scattering process in which momentum conservation is conserved up to a reciprocal lattice vector $G = 2\pi$ (Fig.6.5c). An umpklapp process has the form

$$\mathcal{H}_{\rm umklapp} = g_u \lim_{y\to x} \left(\psi_R^\dagger(x)\psi_R^\dagger(y)\psi_L(x)\psi_L(y) + R \leftrightarrow L \right) \tag{6.47}$$

where $g_u = \widetilde{V}(4p_F \simeq 2\pi)$. This coupling cannot be expressed in terms of densities of right and left movers. Since the Hamiltonian of the Luttinger model is written in terms of $\rho_R(x)$ and $\rho_L(x)$, it is invariant under a continuous chiral transformation, *i.e.* it is invariant under an arbitrary continuous translation. An umpklapp term reduces this continuous symmetry to the (discrete) symmetry of lattice displacements.

We will now see that the Luttinger model can be solved exactly by a mapping (or transformation) known as *Bosonization*.

6.5 Bosonization

We are now going to discuss some subtle but very important properties of one-dimensional Fermi systems. To date, these properties are known not to generalize to higher dimensions. Some superficially similar ideas have been recently discussed in the context of "anyon superfluids" (which we will discuss later). The physics is quite different, though.

A very important tool for the understanding of one-dimensional Fermi systems is the *bosonization transformation*. This transformation was first discussed by Bloch and Tomonaga. It was rediscovered (and better understood) by Lieb and Mattis in the 1960's, and by Coleman, Luther, and Mandelstam in the 1970's. Witten solved the non-Abelian version of bosonization in 1984. We will only consider the Abelian case.

Let us consider first a theory of non-interacting (spinless) fermions with Hamiltonian H_0 given by (in units in which the Fermi velocity is $v_F = 1$)

$$H_0 = \int dx \psi^\dagger i\alpha\partial_x\psi \tag{6.48}$$

where $\alpha = \gamma_5$ (defined in the previous section), with canonically quantized

Fermi fields, *i.e.*

$$\begin{aligned}\{\psi^\dagger_\alpha(x),\psi_{\alpha'}(x')\} &= \delta_{\alpha\alpha'}\delta(x-x'),\\ \{\psi_\alpha(x),\psi_{\alpha'}(x')\} &= \{\psi^\dagger_\alpha(x),\psi^\dagger_{\alpha'}(x')\},\end{aligned} \tag{6.49}$$

at equal times.

6.5.1 Anomalous Commutators

Consider now the "vacuum states" $|0\rangle$ and $|G\rangle$, where $|0\rangle$ is the *empty* state and $|G\rangle$ is the *filled Fermi sea* obtained by having occupied all the negative energy one-particle eigenstates of the Hamiltonian H_0. The Hamiltonian H_0 relative to both vacua differs by normal ordering terms. Indeed, for any eigenstate $|F\rangle$ of H_0 one can write

$$H_0 =: H_0 : + E_F |F\rangle\langle F| \tag{6.50}$$

where $: H_0 :$ is the Hamiltonian normal ordered with respect to $|F\rangle$, i.e.

$$: H_0 : |F\rangle = \langle F| : H_0 := 0 \tag{6.51}$$

and E_F is the energy of $|F\rangle$

$$H_0|F\rangle = E_F|F\rangle. \tag{6.52}$$

Clearly, if we choose $|0\rangle$ or $|G\rangle$ as the reference state, E_F will be different.

The currents and densities also need to be normal ordered. This is equivalent to the subtraction of the (infinite) background charge of the reference state, say of the filled Fermi sea. We will see that these apparently "formal" manipulations have a profound effect on the physics.

Let us compute the commutator of the charge density and current operators at equal times $[j_0(x), j_1(x')]$. Relative to the empty state $|0\rangle$, both operators are already normal ordered since a state with no fermions has neither charge nor current, *i.e.*

$$j_0(x)|0\rangle = 0, \qquad j_1(x)|0\rangle = 0. \tag{6.53}$$

Let us consider the right and left components of the current $\rho_{R,L}$ defined by

$$\rho_{R,L} = \frac{1}{2}(j_0 \pm j_1). \tag{6.54}$$

where

$$\rho_R = \psi_R^\dagger \psi_R \tag{6.55}$$

and

$$\rho_L = \psi_L^\dagger \psi_L \tag{6.56}$$

In Fourier components, we find

$$\rho_R(p) = \frac{1}{\sqrt{L}} \sum_k \psi_R^\dagger(k)\psi_R(k+p) \tag{6.57}$$

which annihilates the empty state $|0\rangle$. In fact, for any state $|\phi\rangle$ with a *finite* number of particles, the result is

$$[\rho_{R,L}(p), \rho_{R,L}(p')]|\phi\rangle = 0. \tag{6.58}$$

Consider now the filled Fermi sea, $|G\rangle$. Explicitly we can write

$$|G\rangle = \prod_{p<0} \psi_R^\dagger(p) \prod_{q>0} \psi_L^\dagger(q)|0\rangle. \tag{6.59}$$

In other words, in $|G\rangle$ all right moving states with negative momentum and all left moving states with positive momentum are filled (see Fig.6.6).

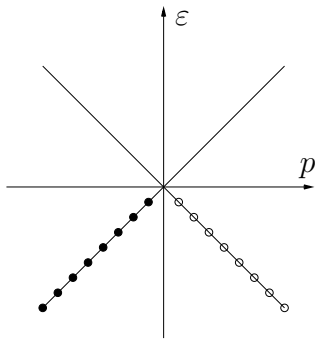

Figure 6.6: Vacuum $|G\rangle$ is obtained by filling the right moving states with negative momentum (filled circles) and filling the left moving states with positive momentum (empty circles).

Let us compute the commutator $[\rho_R(x), \rho_R(x')]$ at equal times. The operator $\rho_R(x)$ is formally equal to a product of fermion operators at the same point. Since we anticipate divergencies, we should "point-split" the product

$$\rho_R(x) = \psi_R^\dagger(x)\psi_R(x) = \lim_{\epsilon \to 0} \psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon) \tag{6.60}$$

and write ρ_R in terms of a normal ordered operator $: \rho_R :$ and a vacuum expectation value

$$\rho_R(x) =: \rho_R(x) : + \lim_{\epsilon \to 0} \langle G|\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon)|G\rangle. \tag{6.61}$$

The singularities are absorbed in the expectation value.

Consider a system on a segment of length ℓ with periodic boundary conditions and expand $\psi_R(x)$ in Fourier series

$$\psi_R(x) = \frac{1}{\sqrt{L}} \sum_{p=-\infty}^{+\infty} \psi_R(p) e^{i\frac{2\pi x p}{\ell}}. \tag{6.62}$$

The vacuum expectation value to be computed is

$$\langle G|\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon)|G\rangle = \frac{1}{\ell} \sum_{p,p'=-\infty}^{+\infty} e^{i\frac{2\pi}{\ell}[(x-\epsilon)p' - (x+\epsilon)p]} \langle G|\psi_R(p)^\dagger \psi_R(p)|G\rangle. \tag{6.63}$$

Using the definition of the filled Fermi sea, we get

$$\langle G|\psi_R^\dagger(p)\psi_R(p')|G\rangle = \delta_{p,p'}\theta(-p), \tag{6.64}$$

$$\langle G|\psi_L^\dagger(p)\psi_L(p')|G\rangle = \delta_{p,p'}\theta(+p). \tag{6.65}$$

Hence

$$\langle G|\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon)|G\rangle = \frac{1}{\ell} \sum_{p=-\infty}^{0} e^{-i\frac{2\pi p}{\ell}(2\epsilon)}. \tag{6.66}$$

This is a conditionally convergent series. In order to make it convergent, we will regulate this series by damping out the contributions due to states deep below the Fermi energy. We can achieve this if we analytically continue ϵ to the upper half of the complex plane (*i.e.* $\epsilon \to \epsilon + i\eta$) to get the convergent expression

$$\begin{aligned}
\langle G|\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon)|G\rangle &= \lim_{\eta\to 0} \frac{1}{\ell} \sum_{p=0}^{\infty} e^{i\frac{4\pi p}{\ell}(\epsilon+i\eta)} \\
&= \lim_{\eta\to 0} \frac{1}{\ell\left(1 - e^{i\frac{4\pi}{\ell}(\epsilon+i\eta)}\right)} \\
&= \lim_{\eta\to 0} \frac{1}{\ell\left(-i\frac{4\pi}{\ell}(\epsilon+i\eta)\right)} \\
&= \frac{i}{4\pi\epsilon}.
\end{aligned} \tag{6.67}$$

Thus, the result is

$$\langle G|\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon)|G\rangle = \frac{i}{4\pi\epsilon} \tag{6.68}$$

Similarly, the expectation value $\langle G|\psi_L^\dagger(x+\epsilon)\psi_L(x-\epsilon)|G\rangle$, is found to be given by

$$\langle G|\psi_L^\dagger(x+\epsilon)\psi_L(x-\epsilon)|G\rangle = \frac{-i}{4\pi\epsilon}. \tag{6.69}$$

The current commutator can now be readily evaluated

$$\begin{aligned}[\rho_R(x),\rho_R(x')] &= \lim_{\epsilon,\epsilon'\to 0}\Big[\psi_R^\dagger(x+\epsilon)\psi_R(x-\epsilon),\psi_R^\dagger(x'+\epsilon')\psi_R(x'-\epsilon')\Big] \\ = \lim_{\epsilon,\epsilon'\to 0}\Big\{&\delta(x'-x+\epsilon'+\epsilon)\psi_R^\dagger(x+\epsilon)\psi_R(x'-\epsilon') \\ &-\delta(x-x'+\epsilon'+\epsilon)\psi_R^\dagger(x'+\epsilon')\psi_R(x-\epsilon)\Big\}.\end{aligned} \tag{6.70}$$

The contributions from normal ordered products cancel (since they are regular). The only non-zero terms are, using Eq.(6.66),

$$[\rho_R(x),\rho_R(x')] = \lim_{\epsilon,\epsilon'\to 0}\left(\frac{i\delta(x'-x+\epsilon'+\epsilon)}{2\pi(x-x'+\epsilon+\epsilon')} - \frac{i\delta(x-x'+\epsilon+\epsilon')}{2\pi(x'-x+\epsilon+\epsilon')}\right). \tag{6.71}$$

Thus, in the limit we find

$$[\rho_R(x),\rho_R(x')] = -\frac{i}{2\pi}\partial_x\delta(x-x') \tag{6.72}$$

and

$$[\rho_L(x),\rho_L(x')] = +\frac{i}{2\pi}\partial_x\delta(x-x'). \tag{6.73}$$

In terms of Lorentz components, we get

$$[j_0(x),j_1(x')] = -\frac{i}{\pi}\partial_x\delta(x-x'), \tag{6.74}$$

whereas

$$[j_0(x),j_0(x')] = [j_1(x),j_1(x')] = 0. \tag{6.75}$$

The commutator $[j_0(x),j_1(x')]$ has a non-vanishing right-hand side which is a c-number. These terms are generally known as *Schwinger terms*. They are pervasive in theories of relativistic fermions. But terms of this sort are also found in non-relativistic systems of fermions at finite densities. In fact, these terms are the key to the derivation of the f-sum rule.

6.5.2 The Bosonization Rules

We thus notice that the equal-time current commutator $[j_0(x), j_1(x')]$ acquires a Schwinger term if the currents and densities are normal ordered relative to the filled Fermi sea. The identity of Eq. (6.74) suggests that there should be a connection between a canonical Fermi field ψ with a filled Fermi sea and canonical Bose field ϕ. Let $\Pi(x)$ be the canonical momentum conjugate to ϕ, i.e. at equal times

$$[\phi(x), \Pi(x')] = i\delta(x - x'). \tag{6.76}$$

If we identify the normal ordered operators

$$j_0(x) = \frac{1}{\sqrt{\pi}}\partial_x\phi(x) \tag{6.77}$$

and

$$j_1(x) = -\frac{1}{\sqrt{\pi}}\partial_t\phi(x) \equiv -\frac{1}{\sqrt{\pi}}\Pi(x), \tag{6.78}$$

we see that Eq. ((6.76) implies

$$\frac{1}{\pi}\left[\partial_x\phi(x), \Pi(x')\right] = -\frac{i}{\pi}\delta'(x - x') \tag{6.79}$$

which is consistent with the Schwinger term. These equations can be written in the more compact form

$$j_\mu = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi \tag{6.80}$$

where $\epsilon_{\mu\nu}$ is the (antisymmetric) Levi-Civita tensor and we are using *from now on* the notation $t \to x_0$, $x \to x_1$ and $x \equiv (x_0, x_1)$. We then arrive at the conclusion that the current commutator with a Schwinger term, Eq. (6.74), is equivalent to the statement that there exists a canonical Bose field ϕ whose *topological current*, Eq.(6.80), coincides with the normal ordered fermion current.

The fermion current j_μ is conserved, *i.e.*

$$\partial_\mu j^\mu = 0 \tag{6.81}$$

which is automatically satisfied by Eq.(6.80). In the case of the *free* theory, the number of left and right movers are separately conserved. This means that not only should j_μ be conserved, but j^5_μ, defined by

$$j^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi \tag{6.82}$$

should also be conserved. Using the identity

$$\gamma_\mu \gamma_5 = \epsilon_{\mu\nu}\gamma^\nu \tag{6.83}$$

we see that j_μ and j^5_μ are in fact related by

$$j^5_\mu = \epsilon_{\mu\nu} j^\nu. \tag{6.84}$$

The divergence of j^5_μ can be computed in terms of the Bose field ϕ as follows

$$\partial_\mu j^{5\mu} = \epsilon^{\mu\nu}\partial_\mu j_\nu = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\epsilon_{\nu\lambda}\partial_\mu\partial^\lambda\phi = \frac{1}{\sqrt{\pi}}\partial^2\phi. \tag{6.85}$$

Thus, the conservation of the *axial* current j^5_μ implies that ϕ should be a free canonical Bose field

$$\partial_\mu j^{5\mu} = 0 \;\Rightarrow\; \partial^2\phi = 0 \tag{6.86}$$

where

$$\partial^2 \equiv \partial_0^2 - \partial_1^2. \tag{6.87}$$

The Lagrangian for these bosons is simply given by*

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\phi\right)^2. \tag{6.88}$$

and the Hamiltonian is

$$\mathcal{H}_0 = \frac{1}{2}\Big(\Pi^2 + (\partial_1\phi)^2\Big) \tag{6.89}$$

Conversely, if ϕ is not free j^5_μ should not be conserved. We will see below that this is indeed what happens in the Thirring-Luttinger model.

Before doing that, let us consider a set of identities, originally derived by Mandelstam. We should expect that these identities should be highly non-local, although they should have local anti-commutation relations. These identities, like all others derived within the Bosonization approach, only make sense within the Operator Product Expansion: the operators so identified give rise to the same leading singular behavior when arbitrary matrix elements are computed. Also, from the Jordan-Wigner analogy, we should expect that the fermion operators, as seen from their representation in terms of bosons, should act like operators which create *solitons.*

*Here we have set, for the moment only, $v_F = 1$.

The free Bose field ϕ can be written in terms of creation and annihilation operators. Let $\phi^+(x)$ ($\phi^-(x)$) denote the piece of $\phi(x)$ which depends on the creation (annihilation) operators only

$$\phi(x) = \phi_+(x) + \phi_-(x) \tag{6.90}$$

where $\phi(x)$ is a Heisenberg operator ($x \equiv (x_0, x_1)$, see Eq.(6.80)). Obviously, ϕ_- annihilates the vacuum of the Bose theory. The operators ϕ_+ and ϕ_- obey the commutation relations

$$[\phi_+(x_0, x_1), \phi_-(x_0', x_1')] = \lim_{\epsilon \to 0} \Delta_+(x_0 - x_0', x_1 - x_1') \tag{6.91}$$

where Δ_+ is given by

$$\Delta_+(x_0 - x_0', x_1 - x_1') = -\frac{1}{4\pi} \ln \left[(c\mu)^2 ((x_1 - x_1')^2 - (x_0 - x_0' + i\epsilon)^2) \right]. \tag{6.92}$$

The arbitrary constant μ has dimensions of mass, *i.e.* $(\text{length})^{-1}$, and it is necessary to make the argument of the logarithm dimensionless. It is customary to do this calculation by adding a small mass μ and to consider the limit $|x_1 - x_1'| \ll \mu^{-1}$. In this case the numerical constant c is related to Catalan's constant.

Consider now the operators $\mathcal{O}_\alpha(x)$ and $\mathcal{Q}_\beta(x)$ defined by

$$\mathcal{O}_\alpha(x) = e^{i\alpha\phi(x)} \tag{6.93}$$

and

$$\mathcal{Q}_\beta(x) = e^{i\beta \int_{-\infty}^{x_1} dx_1' \partial_0 \phi(x_0, x_1')} \equiv e^{i\beta \int_{-\infty}^{x_1} dx_1' \Pi(x_0, x_1')}. \tag{6.94}$$

When acting on a state $|\{\phi(x')\}\rangle$, $\mathcal{O}_\alpha(x)$ simply multiplies the state by $e^{i\alpha\phi(x)}$. The operator $\mathcal{Q}_\beta(x)$ has quite a different effect. Since $\Pi(x)$ and $\phi(x)$ are conjugate pairs, $\mathcal{Q}_\beta(x)$ will shift the value of $\phi(x_0, x_1')$ to $\phi(x_0, x_1') + \beta$ for all $x_1' < x_1$. Thus, $\mathcal{Q}_\beta(x)$ creates a coherent state which we can call a *soliton*

$$\mathcal{Q}_\beta(x) |\{\phi(x_0, x_1')\}\rangle = |\{\phi(x_0, x_1') + \beta\theta(x_1 - x_1')\}\rangle. \tag{6.95}$$

Consider now the operator $\psi_{\alpha,\beta}(x)$ of the form

$$\psi_{\alpha,\beta}(x) = \mathcal{O}_\alpha(x)\mathcal{Q}_\beta(x) = e^{i\alpha\phi(x) + i\beta \int_{-\infty}^{x_1} dx_1' \partial_0 \phi(x_0, x_1')} \tag{6.96}$$

and compute the product $\psi_{\alpha,\beta}(x)\psi_{\alpha,\beta}(x')$ at equal times ($x_0' = x_0$). Using the Baker-Hausdorff formula

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{-[\hat{A},\hat{B}]} = e^{\hat{A}+\hat{B}-\frac{1}{2}[\hat{A},\hat{B}]} \tag{6.97}$$

where $[\hat{A},\hat{B}]$ is a complex valued distribution, we get

$$\psi_{\alpha,\beta}(x)\psi_{\alpha,\beta}(x') = \psi_{\alpha,\beta}(x')\psi_{\alpha,\beta}(x)e^{-i\Phi(x,x')} \tag{6.98}$$

where $\Phi(x,x')$ is given by (all the commutators are understood to be at equal times and $x_0 = x_0'$ but $x_1' \neq x_1$)

$$\begin{aligned} i\Phi(x,x') &= -\alpha^2[\phi(x),\phi(x')] - \beta^2\int_{-\infty}^{x_1} dy_1 \int_{-\infty}^{x_1'} dy_1'[\Pi(y),\Pi(y')] + \\ &\quad -\alpha\beta\int_{-\infty}^{x_1'} dy_1'[\phi(x),\Pi(y')] - \alpha\beta\int_{-\infty}^{x_1} dy_1[\Pi(y),\phi(x')] \\ &= -i\alpha\beta. \end{aligned} \tag{6.99}$$

For the operators $\psi_{\alpha,\beta}(x)$ to have *fermion commutation relations* we need to choose $\alpha\beta = \pm\pi$. It is useful to write left and right components of the Fermi field in the form

$$\psi_R(x) = \left(\frac{c\mu}{2\pi}\right)^{1/2} e^{\frac{\mu}{8\epsilon}} : e^{-i\frac{2\pi}{\beta}\int_{-\infty}^{x_1} dx_1'\Pi(x_0,x_1')+i\frac{\beta}{2}\phi(x)} : \tag{6.100}$$

$$\psi_L(x) = \left(\frac{c\mu}{2\pi}\right)^{1/2} e^{\frac{\mu}{8\epsilon}} : e^{-i\frac{2\pi}{\beta}\int_{-\infty}^{x_1} dx_1'\Pi(x_0,x_1')-\frac{i\beta}{2}\phi(x)} : \ . \tag{6.101}$$

The constant β is arbitrary and it can be chosen by demanding that the currents satisfy the operator identity

$$j_\mu = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi. \tag{6.102}$$

From Eqs.(6.100) and (6.101), it follows that the free fermionic current is identified with the bosonic operator

$$j_\mu = \frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\phi. \tag{6.103}$$

Thus, we must choose $\beta = \sqrt{4\pi}$ for the *free* fermion problem.

The free scalar field operator $\phi(x)$ and the canonical momentum $\Pi(x)$ have the mode expansions

$$\begin{aligned}\phi(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{2|k|}\Big(a(k)\, e^{i(|k|x_0 - kx_1)} + a^\dagger(k)\, e^{-i(|k|x_0 - kx_1)}\Big) \\ \Pi(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{2|k|}\Big(i|k|\, a(k)\, e^{i(|k|x_0 - kx_1)} - i|k|\, a^\dagger(k)\, e^{-i(|k|x_0 - kx_1)}\Big)\end{aligned} \tag{6.104}$$

where the creation and annihilation operators obey standard commutation relations, *i.e.* $[a(k), a^\dagger(k')] = (2\pi)\, 2|k|\, \delta(k - k')$.

The field operator $\phi(x)$ and the canonical momentum $\Pi(x)$ admit a decomposition in terms of right and left moving chiral bosonic fields, $\phi_R(x) \equiv \phi_R(x_0 - x_1)$ and $\phi_L(x) \equiv \phi_L(x_0 + x_1)$, which are given by

$$\begin{aligned}\phi_R(x_0 - x_1) &= \int_0^{\infty} \frac{dk}{2\pi}\, \frac{1}{2k}\Big(a(k)\, e^{ik(x_0 - x_1)} + a^\dagger(k)\, e^{-ik(x_0 - x_1)}\Big) \\ \phi_L(x_0 + x_1) &= \int_{-\infty}^{0} \frac{dk}{2\pi}\, \frac{-1}{2k}\Big(a(k)\, e^{-ik(x_0 + x_1)} + a^\dagger(k)\, e^{ik(x_0 + x_1)}\Big)\end{aligned} \tag{6.105}$$

It is convenient to introduce the *dual* field $\vartheta(x)$, defined by

$$\Pi(x) = \partial_1 \vartheta(x) \tag{6.106}$$

or, equivalently (up to a suitably defined boundary condition),

$$\vartheta(x) \equiv \int_{-\infty}^{x_1} dx_1'\, \Pi(x_0, x_1') \tag{6.107}$$

The field operator $\phi(x)$ and the dual field operator $\vartheta(x)$ obey the *Cauchy-Riemann* equations

$$\partial_0 \phi = \partial_1 \vartheta, \qquad \partial_1 \phi = -\partial_0 \vartheta \tag{6.108}$$

as operator identities. The chiral decomposition reads

$$\begin{aligned}\phi(x_0, x_1) &= \phi_R(x_0 - x_1) + \phi_L(x_0 + x_1) \\ \vartheta(x_0, x_1) &= -\phi_R(x_0 - x_1) + \phi_L(x_0 + x_1)\end{aligned} \tag{6.109}$$

In this subsection we will work primarily with the free fermion problem. In this case the Mandelstam identities, Eq.(6.100) and Eq.(6.101), take the simpler form

$$\begin{aligned} \psi_R(x) &\sim \frac{1}{\sqrt{2\pi a_0}} : e^{i2\sqrt{\pi}\phi_R(x)} : \\ \psi_L(x) &\sim \frac{1}{\sqrt{2\pi a_0}} : e^{-i2\sqrt{\pi}\phi_L(x)} : \end{aligned} \tag{6.110}$$

where a_0 is a short distance cutoff.

It is interesting to consider products of the form $\lim_{y_1\to x_1} \psi_R^\dagger(x)\psi_L(y)$ and $\lim_{y_1\to x_1} \psi_L^\dagger(y)\psi_R(x)$ at equal times. We will use Mandelstam's formulas, Eq.(6.110), to derive an operator product expansion for $\psi_R^\dagger\psi_L$ and $\psi_L^\dagger\psi_R$, both to leading order. We find

$$\lim_{y_1\to x_1} \psi_R^\dagger(x)\psi_L(y) = \frac{1}{2\pi a_0} : e^{-i2\sqrt{\pi}\phi_R(x)} :: e^{-i2\sqrt{\pi}\phi_L(y)} : . \tag{6.111}$$

We can make use of the Baker-Hausdorff formula once again, now in the form

$$: e^{\hat{A}} :: e^{\hat{B}} := e^{[\hat{A}^+,\hat{B}^-]} : e^{\hat{A}+\hat{B}} : \tag{6.112}$$

and write down a *bosonic* expression for $\psi_R^\dagger\psi_L$. The normal ordered operator is, by definition, regular. Thus we can take the limit readily to find

$$\lim_{y\to x} : e^{\hat{A}+\hat{B}} :=: e^{-i\beta\phi(x)} : \ . \tag{6.113}$$

This operator is multiplied by a singular coefficient that compensates for the fact that $\psi_R^\dagger\psi_L$ and $e^{-i\beta\phi}$ have superficially different scaling dimensions. An explicit calculation gives the asymptotic result

$$\lim_{y_1\to x_1} \psi_R^\dagger(x)\psi_L(y) = \lim_{y_1\to x_1} \frac{1}{2\pi a_0} : e^{-i2\sqrt{\pi}\phi(x)} : \tag{6.114}$$

Similarly, one finds the identification

$$\lim_{y_1\to x_1} \psi_L^\dagger(x_0,y_1)\psi_R(x_0,x_1) = \frac{1}{2\pi a_0} : e^{+i2\sqrt{\pi}\phi(x)} : . \tag{6.115}$$

To sum up, the Dirac mass bilinear operator $\bar{\psi}\psi$, *i.e.* the CDW order parameter, is identified with

$$\bar{\psi}(x)\psi(x) \equiv \lim_{y_1\to x_1} \bar{\psi}(x_0,x_1)\psi(x_0,y_1) = \frac{1}{\pi a_0} : \cos(\sqrt{4\pi}\phi(x)) : \tag{6.116}$$

and, similarly,

$$\bar{\psi}(x)i\gamma^5\psi(x) = \frac{1}{\pi a_0} \; : \sin(\sqrt{4\pi}\phi(x)) : . \tag{6.117}$$

As we saw above, at half filling the Umklapp operators play a crucial role at half-filling. These operators enter in the interaction Hamiltonian, through terms of the form

$$\mathcal{H}_u \sim \int dx_1 \left\{ (\psi_R^\dagger \psi_L)^2 + (\psi_L^\dagger \psi_R)^2 \right\}. \tag{6.118}$$

These terms can be bosonized using the Mandelstam identities Eq.(6.110). Indeed, we get the equal-time operator expansion

$$\begin{aligned} \lim_{y_1 \to x_1} (\psi_R^\dagger(x)\psi_L(y))^2 &= \frac{1}{(2\pi a_0)^2} \; : e^{-i\sqrt{4\pi}\phi(x)} :: e^{-i\sqrt{4\pi}\phi(y)} : \\ &= \frac{1}{(2\pi a_0)^2} \; : e^{-4\pi[\phi^+(x),\phi^-(y)]} :: e^{-i2\sqrt{4\pi}\phi(x)} : \\ &= \frac{1}{(2\pi a_0)^2} \; e^{-4\pi\Delta_+(0^+,x_1-y_1)} \; : e^{-i2\sqrt{4\pi}\phi(x)} : \end{aligned} \tag{6.119}$$

where Eqs.(6.97) and (6.91) have been used. In short, the bosonized version of the Umklapp terms is

$$\lim_{y_1 \to x_1} (\psi_R^\dagger(x)\psi_L(y))^2 = \frac{1}{(2\pi a_0)^2} \; : e^{-i4\sqrt{\pi}\phi(x)} : \tag{6.120}$$

and likewise

$$\lim_{y_1 \to x_1} (\psi_L^\dagger(y)\psi_R(x))^2 = \frac{1}{(2\pi a_0)^2} \; : e^{+i4\sqrt{\pi}\phi(x)} : . \tag{6.121}$$

Finally let us consider the bosonized form of the pairing field which for spin-less fermions can only be $\mathcal{O}_P(x) = \psi_R^\dagger(x)\psi_L^\dagger(x)$ (and its adjoint). It is straightforward to see that it maps onto

$$\mathcal{O}_P(x) \sim \psi_R^\dagger(x)\psi_L^\dagger(x) \sim \text{const. } e^{-i2\sqrt{\pi}(\phi_R - \phi_L)} \sim \text{const } e^{i2\sqrt{\pi}\vartheta(x)} \tag{6.122}$$

Hence while *charge fluctuations* (and hence CDW order) are related to the fluctuations of the field ϕ, superconducting fluctuations are related to fluctuations of the *dual field* ϑ, and hence to fluctuations of the *canonical momentum*

Π. In this sense charge and superconducting (phase) fluctuations are *complementary* with each other. Thus, typically (but not always), if one orders and has long range correlations, the other cannot not order and must have short range correlations.

6.6 The Bosonized form of the Luttinger Model

We will now use the identities we just derived to find the bosonized form of the Luttinger model. As we saw the free fermion system maps onto the free boson system (with the same velocity v_F). Hence the free fermion Hamiltonian density (the Dirac Hamiltonian density) becomes

$$\mathcal{H}_0 = \frac{v_F}{2}\Big(\Pi^2 + (\partial_x\phi)^2\Big) \tag{6.123}$$

which, in terms of the field ϕ and the dual ϑ has the symmetric (*self-dual*) form

$$\mathcal{H}_0 = \frac{v_F}{2}\Big((\partial_x\vartheta)^2 + (\partial_x\phi)^2\Big) \tag{6.124}$$

The right and left moving (fermion) densities ρ_R and ρ_L map onto

$$\rho_R = \frac{1}{2\sqrt{\pi}}\Big(\partial_x\phi - \Pi\Big) \equiv \frac{1}{2\sqrt{\pi}}\partial_x(\phi - \vartheta) \tag{6.125}$$

$$\rho_L = \frac{1}{2\sqrt{\pi}}\Big(\partial_x\phi + \Pi\Big) \equiv \frac{1}{2\sqrt{\pi}}\partial_x(\phi + \vartheta) \tag{6.126}$$

In terms of the right and left moving densities the Hamiltonian is*

$$\mathcal{H} = (\pi v_F + g_4)\left(\rho_R^2 + \rho_L^2\right) + 2g_2\rho_R\rho_L \tag{6.127}$$

Hence, the *forward scattering* term of the Luttinger Hamiltonian becomes

$$g_4\Big(\rho_R^2 + \rho_L^2\Big) \to \frac{g_4}{2\pi}\Big(\Pi^2 + (\partial_x\phi)^2\Big) \tag{6.128}$$

Similarly, the backscattering term becomes

$$2g_2\rho_R\rho_L \to \frac{g_2}{2\pi}\Big((\partial_x\phi)^2 - \Pi^2\Big) \tag{6.129}$$

*In high energy physics this is known as the Sugawara form.

Thus, we see that the Hamiltonian of the Luttinger model can be represented by an effective bosonized theory, which includes the total effects of forward and backscattering interactions, and which has the (seemingly) free bosonic Hamiltonian of the form

$$\mathcal{H} \equiv \frac{v}{2}\Big(\frac{1}{K}\Pi^2 + K(\partial_x\phi)^2\Big) \tag{6.130}$$

with an effective velocity v and stiffness K (also known as the *Luttinger parameter*) given by

$$v = \sqrt{\Big(v_F + \frac{g_4}{\pi}\Big)^2 - \Big(\frac{g_2}{\pi}\Big)^2} \tag{6.131}$$

$$K = \sqrt{\frac{v_F + \frac{g_4}{\pi} + \frac{g_2}{\pi}}{v_F + \frac{g_4}{\pi} - \frac{g_2}{\pi}}} \tag{6.132}$$

Thus, we see that the Luttinger model, which describes the *density fluctuations* of a 1D interacting fermion system, is effectively equivalent to a free Bose field with (in addition to the renormalized stiffness K) an effective speed v for the propagation of the bosons (the density fluctuations). We see immediately two effects

- The only effect of the forward scattering interactions, parametrized by the coupling g_4, is only to renormalize the velocity.
- The backscattering interactions, with coupling g_2, renormalize the velocity and the stiffness. Furthermore, for *repulsive* interactions $g_2 > 0$, the stiffness is renormalized upwards, $K > 1$, while for *attractive* interactions, g_2, it is renormalized downwards. We will see that these effects are very important.
- The bosonized form of the Luttinger model has the obvious invariance under $\phi \to \phi + \theta$, where θ is arbitrary. This is the bosonized version of the continuous chiral symmetry of the Luttinger model or, equivalently, the invariance of the original fermionic system under a rigid displacement of the density profile. Due to this invariance the system has long lived long wavelength density (particle-hole) fluctuations that propagate with speed v. In other words, the system has long lived (undamped) sound modes (*i.e.* phonons) much as a 1D quantum elastic solid would.

- We saw that in higher dimensions there are similar collective modes, *zero sound*, which eventually become (Landau) damped. In 1D for a system with a strictly linearized dispersion these modes are never damped.

- This feature of the Luttinger model is, naturally, spoiled by microscopic effects we have ignored, such a band curvature that can be shown to contribute non-quadratic terms to the bosonized Hamiltonian of the form $(\partial_x\phi)^3$ and similar. These non linear terms have two main effects: a) they break the inherent particle-hole symmetry of the Luttinger model, and b) they cause the the boson (the sound modes) to interact with each other and decay, which leads to damping.

At half-filling (obviously on a lattice) we have to consider also the Umklapp term, which becomes

$$\mathcal{H}_u \sim g_u \cos(4\sqrt{\pi}\phi) \tag{6.133}$$

This term formally breaks the continuous $U(1)$ chiral symmetry $\phi \to \phi+\theta$ to a discrete symmetry subgroup $\phi \to \phi + \frac{n\sqrt{\pi}}{4}$, where $n \in \mathbb{Z}$. We will see that when the effects of this operator are important ("relevant") there is a density modulation (a CDW) which is commensurate with the underlying lattice and there is a gap in the fermionic spectrum. In its absence, the fermions remain gapless and the CDW correlations are incommensurate.

Finally, we note that the total charge of the system is

$$Q = -e\int dx j_0(x) = -\frac{e}{\sqrt{\pi}}\int dx \partial_x\phi(x) = -\frac{e}{\sqrt{\pi}}\Delta\phi \tag{6.134}$$

where $\Delta\phi = \phi(+\infty)-\phi(-\infty)$. Hence, in the *charge neutral* sector the system must obey *periodic boundary conditions*, $\Delta\phi = 0$. Conversely, boundary conditions involving the winding of the boson by $\Delta\phi = N\sqrt{\pi}$, where $N \in \mathbb{Z}$, amount to the sector with charge $Q = -Ne$.

We will now summarize our main operator identifications:

$$\boxed{\begin{array}{ccc} j_0 & \to & \frac{1}{\sqrt{\pi}}\partial_x\phi \\ j_1 & \to & -\frac{1}{\sqrt{\pi}}\partial_x\vartheta \\ \psi_R & \to & e^{i2\sqrt{\pi}\phi_R} \\ \psi_L & \to & e^{-i2\sqrt{\pi}\phi_L} \\ \bar{\psi}\psi & \to & \cos(2\sqrt{\pi}\phi) \\ i\bar{\psi}\gamma^5\psi & \to & \sin(2\sqrt{\pi}\phi) \\ \psi_R^\dagger\psi_L^\dagger & \to & e^{i2\sqrt{\pi}\vartheta} \\ \psi_R^\dagger\psi_L^\dagger\psi_R\psi_L & \to & e^{i4\sqrt{\pi}\phi} \end{array}} \tag{6.135}$$

6.7 Spin and the Luttinger model

We will now consider the case of the Luttinger model for spin-1/2 fermions. We will use the same bosonization approach as before. In this context it is known as *Abelian Bosonization* as the $SU(2)$ symmetry of spin is not treated in full. A more correct (and more sophisticated) approach involves *Non-Abelian Bosonization* that we will not do here.

We Hamiltonian density for the Luttinger model for spin-1/2 fermions with both chiralities, denoted below by $s = +1$ (for R)and $s = -1$ (for L), is

$$\begin{aligned} \mathcal{H} \;=\; & -iv_F \sum_{\sigma=\uparrow,\downarrow}\sum_{s=\pm 1} s\psi^\dagger_{s,\sigma}\partial_x\psi_{s,\sigma} \\ & +g_4 \sum_{\sigma,s} \psi^\dagger_{s,\sigma}\psi^\dagger_{s,-\sigma}\psi_{s,-\sigma}\psi_{s,\sigma} \qquad \text{(forward, same branch)} \\ & +g_2 \sum_{\sigma,\sigma'} \psi^\dagger_{1,\sigma}\psi^\dagger_{-1,\sigma'}\psi_{-1,\sigma'}\psi_{1,\sigma} \qquad \text{(forward, both branches)} \\ & +g_{1,\parallel} \sum_{\sigma} \psi^\dagger_{1,\sigma}\psi^\dagger_{-1,\sigma}\psi_{1,\sigma}\psi_{-1,\sigma} \qquad \text{(backscattering, no spin flip)} \\ & +g_{1,\perp} \sum_{\sigma} \psi^\dagger_{1,\sigma}\psi^\dagger_{-1,-\sigma}\psi_{1,-\sigma}\psi_{-1,\sigma} \qquad \text{(backscattering, with spin flip)} \end{aligned} \tag{6.136}$$

For the system to be invariant under $SU(2)$ spin rotations the couplings must satisfy $g_{1,\parallel} = g_{1,\perp} \equiv g_1$.

The umklapp scattering term now is

$$\mathcal{H}_u = g_3\, e^{i(4p_F - G)x}\, \psi^\dagger_{-1,\uparrow}\psi^\dagger_{-1,\downarrow}\psi_{1,\downarrow}\psi_{1,\uparrow} + \text{h.c.} \tag{6.137}$$

where G is a reciprocal lattice vector. As before, we will ignore umklapp processes unless we are at half-filling.

Once again we begin with the free fermion. We then introduce two Bose fields $\phi_\uparrow$ and $\phi_\downarrow$, and their respective canonical momenta, $\Pi_\uparrow$ and $\Pi_\downarrow$. The corresponding free boson Hamiltonian is

$$\mathcal{H}_0 = \frac{v_F}{2}\sum_\sigma \Big(\Pi_\sigma^2 + (\partial_x\phi_\sigma)^2\Big) \tag{6.138}$$

We now define the charge and spin Bose fields ϕ_c and ϕ_s,

$$\phi_c = \frac{1}{\sqrt{2}}\Big(\phi_\uparrow + \phi_\downarrow\Big) \tag{6.139}$$

$$\phi_s = \frac{1}{\sqrt{2}}\Big(\phi_\uparrow + \phi_\downarrow\Big) \tag{6.140}$$

in terms of which $\mathcal{H}_0$ becomes a sum over the charge and spin sectors

$$\mathcal{H}_0 = \frac{v_F}{2}\Big(\Pi_c^2 + (\partial_x\phi_c)^2\Big) + \frac{v_F}{2}\Big(\Pi_s^2 + (\partial_x\phi_s)^2\Big) \tag{6.141}$$

where Π_c and Π_s are the momenta canonically conjugate to ϕ_c and ϕ_s. We will now see that the interactions will lead to a finite renormalization of these parameters, leading to the introduction of a charge and a spin velocity, v_c and v_s, and of the charge and spin Luttinger parameters K_c and K_s.

The charge and spin densities and currents are

$$j_0^c = j_0^\uparrow + j_0^\downarrow = \frac{1}{\sqrt{\pi}}\partial_x(\phi_\uparrow + \phi_\downarrow) = \sqrt{\frac{2}{\pi}}\,\partial_x\phi_c \tag{6.142}$$

$$j_0^s = \frac{1}{2}\left(j_0^\uparrow - j_0^\downarrow\right) = \frac{1}{\sqrt{\pi}}\partial_x(\phi_\uparrow - \phi_\downarrow) = \frac{1}{\sqrt{2\pi}}\,\partial_x\phi_s \tag{6.143}$$

Using the bosonization identities we can write the Luttinger Hamiltonian in the form

$$\begin{aligned}\mathcal{H} = {} & \frac{v_c}{2}\Big(\frac{1}{K_c}\Pi_c^2 + K_c(\partial_x\phi_c)^2\Big) + \frac{v_s}{2}\Big(\frac{1}{K_s}\Pi_s^2 + K_c(\partial_x\phi_s)^2\Big) \\ & + V_c\cos(2\sqrt{2\pi}\phi_c) + V_s\cos(2\sqrt{2\pi}\phi_s)\end{aligned} \tag{6.144}$$

where v_c and v_s are the charge and spin velocities,

$$v_c = \frac{1}{2\pi}\sqrt{\left(2\pi v_F + g_4\right)^2 - \left(g_{1,\parallel} - 2g_2\right)^2} \tag{6.145}$$

$$v_s = \frac{1}{2\pi}\sqrt{\left(2\pi v_F - g_4\right)^2 - \left(g_{1,\parallel}\right)^2} \tag{6.146}$$

K_c and K_s are the charge and spin Luttinger parameters,

$$K_c = \sqrt{\frac{2\pi v_F + g_4 + 2g_2 - g_{1,\parallel}}{2\pi v_F + g_4 - 2g_2 + g_{1,\parallel}}} \tag{6.147}$$

$$K_s = \sqrt{\frac{2\pi v_F - g_4 - g_{1,\parallel}}{2\pi v_F - g_4 + g_{1,\parallel}}} \tag{6.148}$$

The couplings V_c and V_s, due to umklapp and backscattering with spin flip respectively, are given by

$$V_c = \frac{g_3}{2(\pi a)^2}, \qquad V_s = \frac{g_{1,\perp}}{2(\pi a)^2} \tag{6.149}$$

In what follows we will neglect umklapp processes and hence set $V_c = 0$. In the absence of backscattering, $g_1 = 0$, this model is known as the Tomonaga-Luttinger model.

- We now see that this model describes a system with charge and spin bosons, the charge and spin collective modes of the fermionic system. In general the charge and spin velocities are different.
- There is no mixing between charge and spin bosons: *spin-charge separation*
- We also see that for repulsive interactions the charge mode propagates faster than the spin mode, $v_c > v_s$.
- In the same regime, $K_c > 1$ while $K_s < 1$. This will have important consequences.
- The fermion operators, with chirality s and spin σ become

$$\psi_{s,\sigma} = \frac{1}{\sqrt{2\pi a}}\, F_{s,\sigma}\, e^{-i\Phi_{s,\sigma}(x)} \tag{6.150}$$

where

$$\Phi_{s,\sigma} = \sqrt{\frac{\pi}{2}} \left[(\vartheta_c - s\phi_c) + \sigma (\vartheta_s - s\phi_s) \right] \tag{6.151}$$

and $F_{s,\sigma}$ are Klein factors that ensure that fermions with different labels anti-commute with each other,

$$\{F_{s,\sigma}, F_{s'\sigma'}\} = \delta_{s,s'}\delta_{\sigma,\sigma'}, \tag{6.152}$$

and a is a short distance cutoff.

We can now express all the operators we are interested in, in terms of bosons.

- *Charge-Density-Wave*:

$$\mathcal{O}_{CDW} = e^{-i2p_F x} \sum_\sigma \psi^\dagger_{1,\sigma}(x)\psi_{-1,\sigma} \to \frac{1}{\pi a} e^{-i2p_F x} \cos\left(\sqrt{2\pi}\phi_s\right) e^{-i\sqrt{2\pi}\phi_c(x)} \tag{6.153}$$

- *Spin-Density-wave*:

$$\begin{aligned} \mathcal{O}^{(3)}_{SDW} =& e^{-i2p_F x} \sum_{\sigma,\sigma'} \psi^\dagger_{1,\sigma}(x)\tau^3_{\sigma,\sigma'}\psi_{-1,\sigma'} \\ \to& -\frac{1}{\pi a} e^{-i2p_F x}\, 2i \sin\left(\sqrt{2\pi}\phi_s\right) e^{-i\sqrt{2\pi}\phi_c(x)} \qquad (6.154) \\ \mathcal{O}^{(\pm)}_{SDW} =& e^{-i2p_F x} \sum_{\sigma,\sigma'} \psi^\dagger_{1,\sigma}(x)\tau^\pm_{\sigma,\sigma'}\psi_{-1,\sigma'} \\ \to& \frac{1}{\pi a} e^{-i2p_F x}\, e^{-i\sqrt{2\pi}\phi_c}\, e^{\pm i\sqrt{2\pi}\vartheta_s(x)} \qquad (6.155) \end{aligned}$$

- *Singlet Superconductivity*:

$$\mathcal{O}_{SS} = \psi^\dagger_{R,\uparrow}\psi^\dagger_{L,\downarrow} \to e^{i\sqrt{2\pi}\vartheta_c}\, e^{-i\sqrt{2\pi}\phi_s} \tag{6.156}$$

- *Triplet Superconductivity*

$$\mathcal{O}^{(1)}_{TS} = \psi^\dagger_{R,\uparrow}\psi^\dagger_{L,\uparrow} \;\to\; e^{i\sqrt{2\pi}\vartheta_c} e^{i\sqrt{2\pi}\vartheta_s} \tag{6.157}$$

$$\mathcal{O}^{(-1)}_{TS} = \psi^\dagger_{R,\downarrow}\psi^\dagger_{L,\downarrow} \;\to\; e^{i\sqrt{2\pi}\vartheta_c} e^{-i\sqrt{2\pi}\vartheta_s} \tag{6.158}$$

6.8 Correlation functions

We will now compute the correlation functions of the Luttinger model. We will do first the spinless case.

6.8.1 The spinless case

The bosonized Luttinger Hamiltonian density for spinless fermions is

$$\mathcal{H} = (\pi v_F + g_4)\left(\rho_L^2 + \rho_R^2\right) + 2g_2\rho_R\rho_L \tag{6.159}$$

We will diagonalize this Hamiltonian by means of a Bogoliubov transformation (which is canonical):

$$\rho_R = \cosh\lambda\ \tilde{\rho}_R + \sinh\lambda\ \tilde{\rho}_L \tag{6.160}$$

$$\rho_L = \sinh\lambda\ \tilde{\rho}_R + \cosh\lambda\ \tilde{\rho}_L \tag{6.161}$$

where

$$\tilde{\rho}_R = \frac{1}{\sqrt{\pi}}\partial_x\tilde{\phi}_R, \qquad \tilde{\rho}_L = \frac{1}{\sqrt{\pi}}\partial_x\tilde{\phi}_L \tag{6.162}$$

With the choice

$$\tanh 2\lambda = -\frac{g_2}{\pi v_F + g_4} \tag{6.163}$$

the Hamiltonian becomes

$$\mathcal{H} = \pi v\left(\tilde{\rho}_R^2 + \tilde{\rho}_L^2\right) = \frac{v}{2}\left((\partial_x\tilde{\vartheta})^2 + (\partial_x\tilde{\phi})^2\right) \tag{6.164}$$

where, as before,

$$\pi v = \sqrt{(\pi v_F + g_4)^2 - g_2^2} \tag{6.165}$$

and

$$\cosh\lambda = \frac{K+1}{2\sqrt{K}}, \quad \sinh\lambda = \frac{K-1}{2\sqrt{K}} \tag{6.166}$$

and

$$K = \sqrt{\frac{\pi v_F + g_4 + g_2}{\pi v_F + g_4 - g_2}} \tag{6.167}$$

The propagator of the field $\tilde{\phi} = \tilde{\phi}_R + \tilde{\phi}_L$ is*

$$\langle T\left(\tilde{\phi}(x,t)\tilde{\phi}(x',t')\right)\rangle = -\frac{1}{4\pi}\ln\left(\frac{(x-x')^2 - v^2(t-t')^2 + a_0^2 + i\epsilon}{a_0^2}\right) \tag{6.168}$$

*We have regularized the propagator so that it vanishes as $x' \to x$ and $t' \to t$.

from where we get

$$\begin{aligned}\left\langle T\left(\tilde{\phi}_R(x,t)\tilde{\phi}_R(x',t')\right)\right\rangle &= -\frac{1}{4\pi}\ln\Big(\frac{(x-x')-v(t-t')+i\epsilon}{a_0}\Big)\\ \left\langle T\left(\tilde{\phi}_L(x,t)\tilde{\phi}_L(x',t')\right)\right\rangle &= -\frac{1}{4\pi}\ln\Big(\frac{(x-x')+v(t-t')+i\epsilon}{a_0}\Big)\end{aligned} \tag{6.169}$$

Using these expressions we get

$$\begin{aligned}\langle T(\phi_R(x,t)\phi_R(x',t'))\rangle &= \frac{(K+1)^2}{4K}\langle T(\tilde{\phi}_R(x,t)\tilde{\phi}_R(x',t')\rangle + \frac{(K-1)^2}{4K}\langle T(\tilde{\phi}_L(x,t)\tilde{\phi}_L(x',t')\rangle\\ \langle T(\phi_L(x,t)\phi_L(x',t'))\rangle &= \frac{(K-1)^2}{4K}\langle T(\tilde{\phi}_R(x,t)\tilde{\phi}_R(x',t')\rangle + \frac{(K+1)^2}{4K}\langle T(\tilde{\phi}_L(x,t)\tilde{\phi}_L(x',t')\rangle\end{aligned} \tag{6.170}$$

The fermion propagator

The propagator for right moving fermions is

$$\begin{aligned}\langle T(\psi_R(x,t)\psi_R^\dagger(x',t'))\rangle &\sim \frac{1}{\sqrt{2\pi a_0}}\,\langle T(e^{i\sqrt{2\pi}\phi_R(x,t)}e^{-i\sqrt{2\pi}\phi_R(x',t')})\rangle\\ &= \frac{1}{2\pi a_0}\,e^{2\pi\langle T(\phi_R(x,t)\phi_R(x',t'))\rangle}\\ = \frac{1}{2\pi a_0}\,\Big(\frac{a_0}{(x-x')-v(t-t')+i\epsilon}\Big)^{\frac{(K+1)^2}{4K}}&\,\Big(\frac{a_0}{(x-x')+v(t-t')+i\epsilon}\Big)^{\frac{(K-1)^2}{4K}}\end{aligned} \tag{6.171}$$

and for left moving fermions

$$\begin{aligned}\langle T(\psi_L(x,t)\psi_L^\dagger(x',t'))\rangle &\sim \frac{1}{\sqrt{2\pi a_0}}\,\langle T(e^{-i\sqrt{2\pi}\phi_L(x,t)}e^{i\sqrt{2\pi}\phi_L(x',t')})\rangle\\ &= \frac{1}{2\pi a_0}\,e^{2\pi\langle T(\phi_L(x,t)-\phi_L(x',t'))\rangle}\\ = \frac{1}{2\pi a_0}\,\Big(\frac{a_0}{(x-x')+v(t-t')+i\epsilon}\Big)^{\frac{(K+1)^2}{4K}}&\,\Big(\frac{a_0}{(x-x')-v(t-t')+i\epsilon}\Big)^{\frac{(K-1)^2}{4K}}\end{aligned} \tag{6.172}$$

In the free fermion case, $K = 1$, this propagator just becomes

$$\langle T(\psi_R(x,t)\psi_R^\dagger(x',t'))\rangle = \frac{1}{2\pi a_0}\,\frac{a_0}{(x-x') - v(t-t') + i\epsilon} \tag{6.173}$$

(and a similar expression for left movers). We see that while in th free fermion case the propagator has a simple pole (and hence a finite fermion residue $Z = 1$, as soon as the interactions are turned on the pole becomes a branch cut.

Order parameters

1. The propagator of the CDW order parameter is found to be

$$\frac{1}{(2\pi a_0)^2}\langle T(e^{i2\sqrt{\pi}\phi(x)}e^{-i2\sqrt{\pi}\phi(x)})\rangle \sim \frac{1}{(2\pi a_0)^2}\left(\frac{a_0^2}{(x-x')^2 - v^2(t-t')^2 + i\epsilon}\right)^{\frac{1}{K}} \tag{6.174}$$

2. The propagator for the superconducting order parameter is, instead,

$$\frac{1}{(2\pi a_0)^2}\langle T(e^{i2\sqrt{\pi}\vartheta(x)}e^{-i2\sqrt{\pi}\vartheta(x)})\rangle \sim \frac{1}{(2\pi a_0)^2}\left(\frac{a_0^2}{(x-x')^2 - v^2(t-t')^2 + i\epsilon}\right)^{K} \tag{6.175}$$

6.8.2 The spin 1/2 case

The behavior of the correlation functions for the case of spin-1/2 fermions can be computed similarly. Since the Hamiltonian of the Luttinger model decomposes into a sum of terms for the charge and spin sectors, respectively, we will find that the correlation functions *factorize* into a contribution from the charge sector and a contribution from the spin sector. We will not do all possible cases but just the most interesting ones.

Since $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s$, the propagators factorize. In other terms, the system behaves as if the electrons have *fractionalized* into two independent excitations: a) a spinless *holon* with charge $-e$, and b) a spin-1/2 charge neutral *spinon*. This feature is known as *spin-charge separation*. It is a robust feature of these 1D systems in the low energy limit.

We will follow the same approach as in the spinless case, although we will do it less explicitly. Here too we define the densities of right and left moving

fermions with either spin polarization,

$$\rho_{R,\sigma} = \frac{1}{\sqrt{\pi}}\partial_x \phi_{R,\sigma}, \qquad \rho_{L,\sigma} = \frac{1}{\sqrt{\pi}}\partial_x \phi_{L,\sigma} \tag{6.176}$$

and write the Luttinger Hamiltonian in terms of these densities. It reduces to

$$\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s \tag{6.177}$$

where

$$\mathcal{H}_c = \frac{1}{2}\left(\pi v_F + g_4\right)\left(\rho_{c,R}^2 + \rho_{c,L}^2\right) + \frac{1}{2}\left(2g_2 - g_{1,\parallel}\right)\rho_{c,R}\,\rho_{c,L} \tag{6.178}$$

$$\mathcal{H}_s = \frac{1}{2}\left(\pi v_F - g_4\right)\left(\rho_{s,R}^2 + \rho_{s,L}^2\right) - \frac{1}{2}\,g_{1,\parallel}\,\rho_{s,R}\,\rho_{s,L} \tag{6.179}$$

We now perform Bogoliubov transformations (separately) for charge and spin, whose parameters λ_c and λ_s are

$$\tanh 2\lambda_c = -\frac{2g_2 - g_{1,\parallel}}{\pi v_F + g_4} \tag{6.180}$$

$$\tanh 2\lambda_s = +\frac{g_{1,\parallel}}{\pi v_F - g_4} \tag{6.181}$$

The Luttinger parameters K_c and K_s are

$$K_c = e^{2\lambda_c} = \sqrt{\frac{\pi v_F - g_4 + 2g_2 - g_{1,\parallel}}{\pi v_F + g_4 - 2g_2 + g_{1,\parallel}}} \tag{6.182}$$

$$K_s = e^{2\lambda_s} = \sqrt{\frac{\pi v_F - g_4 - g_{1,\parallel}}{\pi v_F - g_4 - g_{1,\parallel}}} \tag{6.183}$$

and

$$\pi v_c = \sqrt{\left(\pi v_F + g_4\right)^2 - \left(2g_2 - g_{1,\parallel}\right)^2} \tag{6.184}$$

$$\pi v_s = \sqrt{\left(\pi v_F - g_4\right)^2 - \left(g_{1,\parallel}\right)^2} \tag{6.185}$$

The transformed densities and bosons are denoted by

$$\tilde{\rho}_{c,R} = \frac{1}{\sqrt{\pi}}\partial_x \tilde{\phi}_{c,R}, \qquad \tilde{\rho}_{c,L} = \frac{1}{\sqrt{\pi}}\partial_x \tilde{\phi}_{c,L} \tag{6.186}$$

$$\tilde{\rho}_{s,R} = \frac{1}{\sqrt{\pi}}\partial_x \tilde{\phi}_{s,R}, \qquad \tilde{\rho}_{s,L} = \frac{1}{\sqrt{\pi}}\partial_x \tilde{\phi}_{s,L} \tag{6.187}$$

The fermion propagator

The operators for right and left moving fermions with spin σ now take the form

$$\psi_{R,\sigma} \sim \frac{1}{\sqrt{2\pi a_0}}\, e^{i\sqrt{2\pi}\phi_{R,c}}\, e^{i\sigma\sqrt{2\pi}\phi_{R,s}} \tag{6.188}$$

$$\psi_{L,\sigma} \sim \frac{1}{\sqrt{2\pi a_0}}\, e^{-i\sqrt{2\pi}\phi_{L,c}}\, e^{-i\sigma\sqrt{2\pi}\phi_{L,s}} \tag{6.189}$$

After some algebra we find

$$\begin{aligned}\langle T\psi_{R,\uparrow}(x,t)\psi^\dagger_{R,\uparrow}(0,0)\rangle &= \langle T\psi_{R,\downarrow}(x,t)\psi^\dagger_{R,\downarrow}(0,0)\rangle \\ &= \frac{a_0^{\gamma_c+\gamma_s}}{2\pi}\Big(a_0+i(v_ct-x)\Big)^{-1/2}\Big(a_0+i(v_st-x)\Big)^{-1/2} \\ &\times\Big(x^2+[(a_0+iv_ct)^2]\Big)^{-\gamma_c/2}\Big(x^2+[(a_0+iv_st)^2]\Big)^{-\gamma_s/2}\end{aligned} \tag{6.190}$$

and

$$\begin{aligned}\langle T\psi_{L,\uparrow}(x,t)\psi^\dagger_{L,\uparrow}(0,0)\rangle &= \langle T\psi_{L,\downarrow}(x,t)\psi^\dagger_{L,\downarrow}(0,0)\rangle \\ &= \frac{a_0^{\gamma_c+\gamma_s}}{2\pi}\Big(a_0+i(v_ct+x)\Big)^{-1/2}\Big(a_0+i(v_st+x)\Big)^{-1/2} \\ &\times\Big(x^2+[(a_0+iv_ct)^2]\Big)^{-\gamma_c/2}\Big(x^2+[(a_0+iv_st)^2]\Big)^{-\gamma_s/2}\end{aligned} \tag{6.191}$$

where

$$\gamma_{c,s} = \frac{1}{4}\Big(K_{c,s}+\frac{1}{K_{c,s}}\Big)-\frac{1}{2} \tag{6.192}$$

Order parameters

1. The CDW correlator is

$$\begin{aligned}&\langle T\mathcal{O}_{CDW}(x,t)\mathcal{O}^\dagger_{CDW}(0,0)\rangle = \\ &\frac{1}{(\pi a_0)^2}\langle T\cos(\sqrt{2\pi}\phi_s(x,t))\cos(\sqrt{2\pi}\phi_s(0,0))\rangle\langle Te^{-i\sqrt{2\pi}\phi_c(x,t)}e^{i\sqrt{2\pi}\phi_c(0,0)}\rangle \\ &= \frac{2}{(\pi a_0)^2}\,\Big(\frac{a_0^2}{x^2-v_c^2t^2+a_0^2+i\epsilon}\Big)^{\frac{1}{2K_c}}\,\Big(\frac{a_0^2}{x^2-v_s^2t^2+a_0^2+i\epsilon}\Big)^{\frac{1}{2K_s}}\end{aligned} \tag{6.193}$$

2. The (transverse) SDW correlator is

$$\begin{aligned}&\langle T\mathcal{O}^{(\pm)}_{SDW}(x,t)\mathcal{O}^{(\pm)}_{SDW}{}^{\dagger}(0,0)\rangle = \\ &\frac{1}{(2\pi a_0)^2}\langle Te^{\pm i\sqrt{2\pi}\vartheta_s(x,t)}e^{\mp i\sqrt{2\pi}\vartheta_s(0,0)}\rangle\langle Te^{-i\sqrt{2\pi}\phi_c(x,t)}e^{i\sqrt{2\pi}\phi_c(0,0)}\rangle \\ &= \frac{1}{(2\pi a_0)^2}\left(\frac{a_0^2}{x^2-v_c^2t^2+a_0^2+i\epsilon}\right)^{\frac{1}{2K_c}}\left(\frac{a_0^2}{x^2-v_s^2t^2+a_0^2+i\epsilon}\right)^{\frac{K_s}{2}}\end{aligned} \tag{6.194}$$

3. The singlet superconductor correlator is

$$\begin{aligned}&\langle T\mathcal{O}_{SS}(x,t)\mathcal{O}^{\dagger}_{SS}(0,0)\rangle = \\ &\frac{1}{(2\pi a_0)^2}\langle Te^{i\sqrt{2\pi}\vartheta_c(x,t)}e^{-i\sqrt{2\pi}\vartheta_c(0,0)}\rangle\langle Te^{-i\sqrt{2\pi}\phi_s(x,t)}e^{i\sqrt{2\pi}\phi_s(0,0)}\rangle \\ &= \frac{2}{(\pi a_0)^2}\left(\frac{a_0^2}{x^2-v_c^2t^2+a_0^2+i\epsilon}\right)^{\frac{K_c}{2}}\left(\frac{a_0^2}{x^2-v_s^2t^2+a_0^2+i\epsilon}\right)^{\frac{1}{2K_s}}\end{aligned} \tag{6.195}$$

6.9 Susceptibilities of the Luttinger Model

6.9.1 The fermion spectral function and ARPES

The fermion (electron) spectral function $\mathcal{A}_{s,\sigma}(p,\omega)$ (where $s = R, L$ and $\sigma =\uparrow,\downarrow$) is defined by

$$\begin{aligned}\mathcal{A}_{s,\sigma}(p,\omega) &\equiv -\frac{1}{\pi}\text{Im } G^{ret}_{s,\sigma}(p,\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dte^{-i(px-\omega t)}\left(G(x,t)+G(-x,-t)\right)\end{aligned} \tag{6.196}$$

where

$$G^{ret}_{s,\sigma}(x,t) = -i\theta(t)\left\langle\left\{\psi_{s,\sigma}(x,t),\psi_{s,\sigma}(0,0)\right\}\right\rangle \tag{6.197}$$

is the fermion retarded Green function, $G(x,t) = G_{R,\uparrow}(x,t)$ (since the system is invariant under parity and spin reversal) is the time-ordered propagator we derived before, and p is measured form the Fermi point at p_F. The detailed

form of the spectral function for the general case is complicated. Explicit expressions are given in the book of A. Gogolin *et al.*[*] Here we will just quote the main results and analyze its consequences.

For a free fermion system the Luttinger parameters $K_c = K_s = 1$ and hence $\gamma_c = \gamma_s = 0$. Similarly, the charge and spin velocities are equal in that case, $v_c = v_s = v_F$. Hence, in the free fermion case, we see that the spectral function $\mathcal{A}_{s,\sigma}(p,\omega)$ reduces to the sum of two poles (resulting from the poles in the propagator), for right and left movers respectively, each with a quasi-particle residue $Z = 1$.

The situation changes dramatically for the interacting case no matter how weak the interactions are. For simplicity we will only discuss the case in which the system has a full $SU(2)$ spin invariance, in which case $K_s = 1$ and $\gamma_s = 0$. We see that instead of poles, the fermion propagator has branch cuts, whose tips are located at $\omega = \pm p v_{c,s}$ ($\pm$ here stands for right and left movers). An analysis if the integral shows that close to these singularities the spectral function has the behavior[†]

$$\mathcal{A}(p, \omega \simeq p v_c) \sim \theta(\omega - p v_c)\ (\omega - p v_c)^{(\gamma_c - 1)/2} \tag{6.198}$$

$$\mathcal{A}(p, \omega \simeq -p v_c) \sim \theta(-\omega - p v_c)\ (-\omega - p v_c)^{\gamma_c} \tag{6.199}$$

$$\mathcal{A}(p, \omega \simeq p v_s) \sim \theta(\omega - p v_s)\ (\omega - p v_s)^{\gamma_c - 1/2} \tag{6.200}$$

where p is the momentum of the incoming fermion (measured from p_F.) Thus, the free fermion poles are replaced in the interaction system by power-law singularities. These results show clearly the spin-charge separation: an injected electron has decomposed into (soliton-like) excitations, holons and spinons, that disperse at a characteristic (and different) speed.

In an angle-resolved photoemission (ARPES) experiment high-energy photons impinge on the surface of a system. If the photons energ is high enough (typically the photons are X-rays from synchrotron radiation of a particle accelerator), there is a finite amplitude for an electron to be ejected from the system (a photo-electron), leaving a hole behind. In an ARPES experiment the energy and momentum (including the direction) of the photo-electron are measured. It turns out that the intensity of the emitted photo-electrons is proportional to the *spectral function* of the hole left behind at a known momentum and energy. Although it is not technically possible to do an ARPES

[*] A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems*; see Chapter 19. They use the notation $\rho_{s,\sigma}(p,\omega)$ for the spectral function.

[†] For the $SU(2)$ symmetric case, $\gamma_s = 0$ and the is no singularity at $\omega \sim -p v_s$.

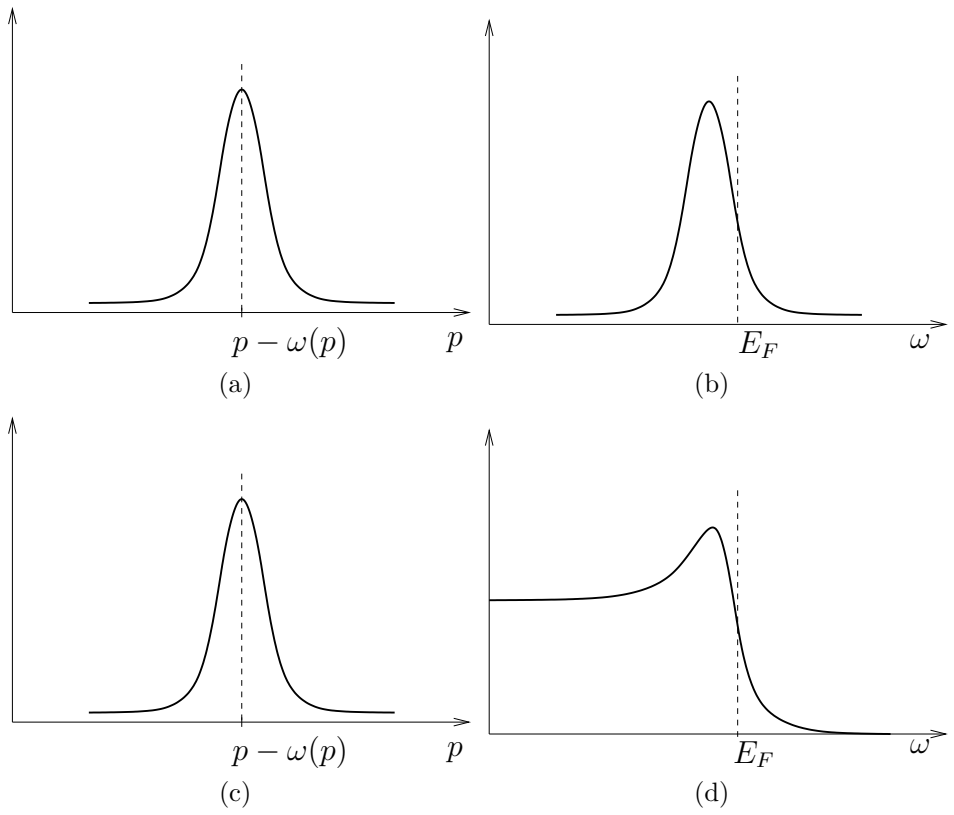

Figure 6.7: ARPES spectra: a) MDC in a Fermi liquid, b) EDC in a Fermi liquid, c) MDC in a Luttinger liquid, d) EDC in a Luttinger liquid.

experiment in a literally one-dimensional system, it is possible to do it quasi-1D systems, arrays of weakly coupled 1DEGs. Experiments of this type are done in systems of this type, such as the blue bronzes, although their degree of quasi-one-dimensionality is not strong enough to see the effects we discuss here.

The data from ARPES experiments is usually presented in terms of cuts of the spectral function: a) as *energy distribution curves* (EDC's) in which the spectral function at fixed momentum is plotted as a function of energy, and b) as *momentum distribution curves* (MDC's) in which the spectral function at fixed energy is plotted as a function of momentum (see Fig.6.7 a-d.) Even if an ARPES experiment could be done in a Luttinger liquid, it is important

to include the effects of thermal fluctuations since all experiments are done at finite temperature. One important effects is that the singularities of the spectral functions will be rounded at finite temperature. For example, the singularity of the EDC near the charge right moving branch for $p = 0$ (near p_F), which diverges as $\omega^{(\gamma_c-1)/2}$ as $\omega \to 0$ at $T = 0$, saturates at finite temperature T with maximum $\sim T^{(\gamma_c-1)/2}$ (which will grow bigger as T is lowered.) The same holds for the EDC at $\omega = 0$ as a function of momentum p (at p_F), which will saturate at a value $\sim (T/v_c)^{(\gamma_c-1)/2}$. A more detailed study of the spectral function at finite temperature T (which must be done numerically) shows that the EDCs are much broader than the MDCs and look like what is shown in Fig.6.7.[‡]

6.9.2 The tunneling density of states and STM

In a scanning tunneling microscopy (STM) experiment, a (very sharp) metallic tip (typically made of a simple metal such as gold) is placed near a very flat (and clean) surface of en electronic system. There a finite voltage difference V between the tip and the system and, depending on its sign, electrons will tunnel from the tip to the system or viceversa. An STM instrument is operated by scanning the system (*i.e.* by displacing the tip) while keeping the tip at a fixed distance from the surface and at a fixed voltage difference. If the tip is sharp enough the intensity of the measured tunneling current (of electrons), which reflects the local changes of the electronic structure, can be used to map the local environment with an atomic precision.

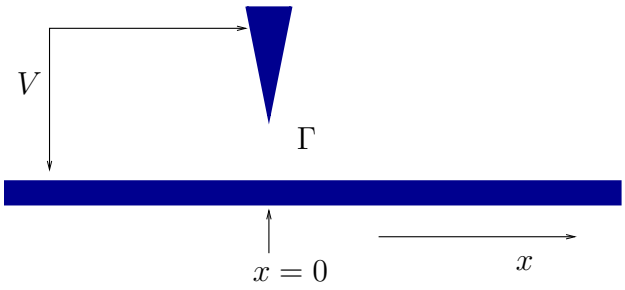

Figure 6.8: Sketch of an STM setup.

[‡]Similar behaviors are seen in ARPES experiments in high temperature superconductors.

We will now see that the local differential conductance measured in STM has direct information on the local density of states. To see how this works let us consider a simple model of the operation of the STM. Let H_{tip} be the Hamiltonian which describes the electronic states in the tip. Let us denote by $\psi_{tip}(t)$ the be fermionic operator that removes an electron from the tip in some one-particle state (which we will not need to know) close to the tip Fermi energy.

The tunneling process from the tip to the 1DEG at a point $x = 0$ is described by a term in the Hamiltonian of the form

$$\mathcal{H}_{tunnel} = \delta(x)\Gamma \sum_{\sigma} \psi^{\dagger}_{\sigma}(0)\psi_{tip} + \text{h.c.} \equiv \delta(x)\Gamma \sum_{\sigma} \Big(\psi^{\dagger}_{R,\sigma}(0) + \psi^{\dagger}_{L,\sigma}(0)\Big)\psi_{tip} + \text{h.c.} \tag{6.201}$$

The tunneling current operator J at the point contact is given by

$$J = ie\Gamma \sum_{\sigma} \Big[\Big(\psi^{\dagger}_{R,\sigma}(0) + \psi^{\dagger}_{L,\sigma}(0)\Big)\psi_{tip} - \text{h.c.}\Big] \tag{6.202}$$

We will assume that the energy of this state is higher than the Fermi energy in the Luttinger liquid by an amount equal to eV, where V is the voltage difference. We will assume that this is a rather uninteresting metal well described by a Fermi liquid with a density of one-particle states $\rho_{tip}(E)$ which is essentially constant for the range of voltages V used. Hence we can use the approximation $\rho_{tip}(E_F + eV) \simeq \rho_{tip}(E_F)$. A fixed voltage V is equivalent to a difference of the chemical potentials of eV between the tip and the 1DEG. The same physics can be described by assigning the following phase factor to the tunneling matrix element Γ

$$\Gamma \to \Gamma e^{i\frac{e}{\hbar}Vt} \tag{6.203}$$

both in the tunneling Hamiltonian and in the definition of a tunneling current.[§]

We now use perturbation theory in powers of the tunneling matrix element Γ to find expectation value of the current operator, which will be denoted by I. To the lowest possible order in Γ, I is given by

$$I = 2\pi\frac{e}{\hbar}|\Gamma|^2 \int_{-eV}^{0} dE\ \rho_{LL}(E,T)\ \rho_{tip}(E + eV, T) \tag{6.204}$$

[§]This is equivalent to a time-dependent gauge transformation.

and the differential tunneling conductance $G(V,T)$ is

$$G(V,T) = \frac{dI}{dV} \simeq \frac{2\pi e}{\hbar}|\Gamma|^2 \rho_{tip}(0)\rho_{LL}(E,T) \tag{6.205}$$

Here $\rho_{LL}(E,T)$ is the one-particle local density of states of the Luttinger liquid

$$\begin{aligned}\rho_{LL}(E,T) &= -\frac{1}{\pi}\mathrm{Im}G^{ret}_{LL}(x=0,\omega=E,T) \\ &= -\frac{4}{\pi}\mathrm{Im}G^{ret}_{R,\uparrow}(x=0,E,T) = 4\int_{-\infty}^{\infty}\frac{dp}{2\pi}\mathcal{A}_{R,\uparrow}(p,E,T)\end{aligned} \tag{6.206}$$

where $\mathcal{A}_{R,\uparrow}(p,E,T)$ is the spectral function defined above, and the factor of 4 arises since right and left movers (with both spin orientations) contribute equally at equal positions (denoted by $x=0$). Alternatively,

$$\rho_{LL}(E,T) = -\frac{4}{\pi}\mathrm{Im}\int_{-\infty}^{\infty} dt e^{-i\frac{E}{\hbar}t}\, G^{ret}_{R,\uparrow}(x=0,t,T) \tag{6.207}$$

At $T=0$, by computing this Fourier transform one finds that $\rho_{LL}(E)$ has a power-law behavior

$$\rho_{LL}(E) \propto E^{2(\gamma_c+\gamma_s)} \tag{6.208}$$

Hence, the differential tunneling conductance essentially measures the local density of states of the Luttinger liquid.[¶] Therefore, at $T=0$, the differential tunneling conductance behaves as

$$G_{LL}(V) \propto V^{2(\gamma_c+\gamma_s)} \tag{6.209}$$

whereas for $T>0$ one finds a saturation for $V \ll T$:

$$G_{LL}(V,T) \propto T^{2(\gamma_c+\gamma_s)} \tag{6.210}$$

The crossover between the $T>0$, $V\to 0$ Ohmic behavior and the $T\to 0$, $V>0$ Luttinger behavior occurs for $eV \sim k_B T$.

In contrast, for a free fermion (and for a Landau Fermi liquid)

$$G_{FL}(V) = \text{const} \tag{6.211}$$

[¶]This is actually a general result

since in this case $\gamma_c = \gamma_s = 0$.

Therefore, for a free fermion we find that the point contact is Ohmic, $I \propto V$. For a Luttinger liquid there is instead a *power-law suppression* of the tunneling differential conductance for $T \ll V$ (see Fig.6.9), and Ohmic behavior for $T \gg V$(with a conductance that scales as a power of T.) These behaviors reflect the fact that there are no stable electron-like quasiparticles in the Luttinger liquid: the electron states are *orthogonal* to the states in the spectrum of the Luttinger liquid leading to a vanishing of the quasiparticle residue and to characteristic power-law behaviors in many quantities. This fact is known as the *orthogonality catastrophe*.

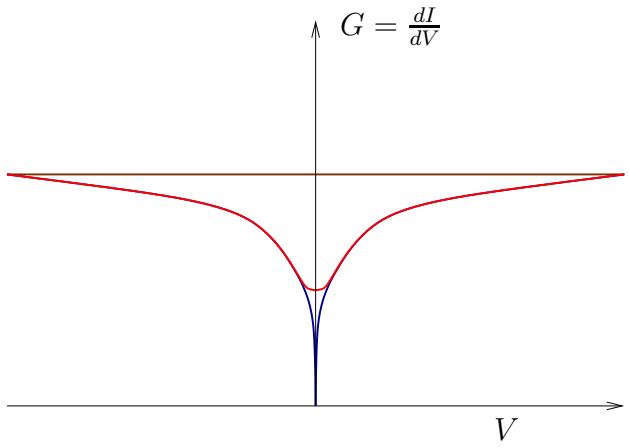

Figure 6.9: Differential tunneling conductance $G = \frac{dI}{dV}$ as a function of bias voltage V, in a Fermi liquid (brown), and in a Luttinger liquid at $T = 0$ (dark blue) and at $T > 0$ (red).

6.9.3 The fermion momentum distribution function

We will now discuss the fermion momentum distribution functions at zero temperature, $T = 0$. Since we have right and left movers, with both spin orientations, in principle we have four such functions. However, the Luttinger liquid state is invariant under global spin flips, $\uparrow \leftrightarrow \downarrow$, and under parity, $R \leftrightarrow L$. Thus all four momentum distributions are equal to each other. Let us

compute, say, $n_{R,\uparrow}(p)$ which is given by the *equal time* correlator

$$\begin{aligned} n_{R,\uparrow}(p) &= \lim_{t'\to t+0^+} \langle \psi^\dagger_{R,\uparrow}(p,t)\psi_{R,\uparrow}(p,t')\rangle \\ &= \lim_{L\to\infty} \frac{1}{L}\int_{-L/2}^{+L/2} dx \int_{-L/2}^{+L/2} dx' e^{-ip(x-x')} \lim_{t'\to t+0^+} \langle T\psi^\dagger_{R,\uparrow}(x,t)\psi_{R,\uparrow}(x',t')\rangle \\ &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{A}_{R,\uparrow}(p,\omega) \end{aligned} \tag{6.212}$$

(here T means time-ordering!).

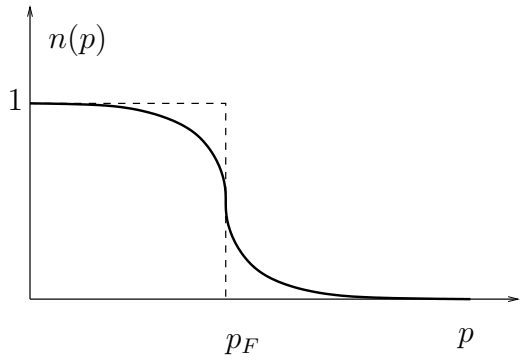

Figure 6.10: The fermion momentum distribution function in a Luttinger liquid.

A lengthy computation of the Fourier transforms leads to the result at $T = 0$ (here p is measured from p_F)

$$n_{R,\uparrow}(p) \sim \text{const} + \#\ |p|^{2(\gamma_c+\gamma_s)}\ \text{sign}(p) \tag{6.213}$$

At finite temperature $T > 0$ this singularity is rounded by thermal fluctuations which dominate for momenta $|p| \lesssim k_B T/v_c$, which lead to a smooth momentum dependence in this regime. This is why Luttinger behavior is difficult to detect in the momentum distribution function.

Thus, instead of a jump (or discontinuity) of Z (the quasiparticle residue) at p_F (the Fermi liquid result), in a Luttinger liquid there is no jump (since $Z = 0$!). Instead we find that the momentum distribution function has a weak singularity at p_F. This is what replaces the "Fermi surface" in a Luttinger

liquid. we will show below that this happens since the Luttinger liquid is a (quantum) critical system and the fermions have an *anomalous dimension* given by $2(\gamma_c + \gamma_s)$.

6.9.4 Dynamical susceptibilities at Finite Temperature

Finally, we will discuss the behavior of dynamical susceptibilities at finite $T > 0$. In the preceding sections we gave explicit expressions for the correlators (time-ordered) of various physical quantities (order parameters and currents) at $T = 0$ in real space and time. Here we will need the dynamical susceptibilities, which are the associated retarded (instead of time-ordered) correlators at finite $T > 0$ in real momentum and frequency.

We saw earlier in the class that we can determine all of these properties from the temperature correlators, *i.e.* in imaginary time τ, restricted the the interval $0 \leq \tau < 1/T$ (with $k_B = 1$). We accomplish this by first making the analytic continuation

$$vt \to -ivt, \tag{6.214}$$

which implies to introduce the complex coordinates

$$x - vt \to z = x + ivt \quad \text{and} \quad x + vt \to \bar{z} = x - ivt \tag{6.215}$$

Next we perform the *conformal mapping* from the complex plane labelled by the coordinates z to the cylinder, labelled by the coordinates $w = x + iv\tau$ (see Fig.6.11)

$$x + ivt \to e^{2\pi \frac{T}{v}(x+iv\tau)} \tag{6.216}$$

Thus, the long axis of this cylinder is space (labelled by $-\infty \leq x \leq \infty$), and the circumference is the imaginary time τ, $0 \leq \tau \leq 1/T$. Under the conformal mapping the boson propagator (in imaginary time t) tuns out to transform as

$$\langle \phi(x,t)\phi(x',t')\rangle \to \frac{1}{2\pi} \ln \left| \frac{\pi T}{\sinh\left(\pi \frac{T}{v}(w - w')\right)} \right| \tag{6.217}$$

where $w = x + iv\tau$.

The computation of the correlators of the observables we are interested in is the same than at $T = 0$, except that the boson propagator changes as

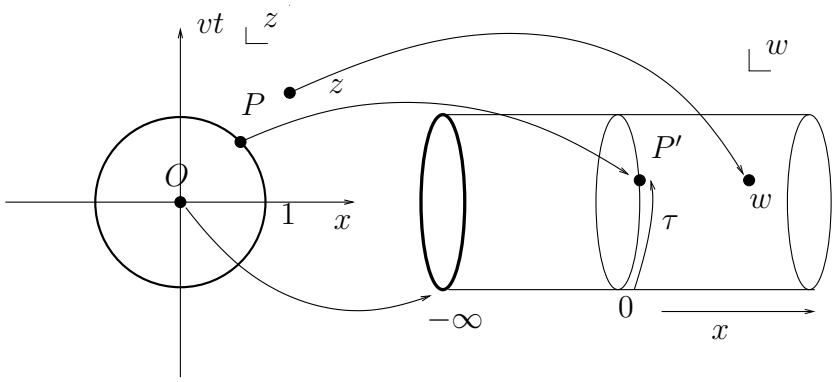

Figure 6.11: The conformal mapping $z = e^{2\pi T w/v}$ which maps the complex plane $z = x + ivt$ to the cylinder $w = x + iv\tau$. Under this mapping the origin O on the plane maps onto $-\infty$ on the cylinder.

shown above.[‖] Thus, to compute the temperature propagators we perform (a) the analytic continuation followed by (b) the conformal mapping. This leads to the following identification for the power-law factors in the correlators[**]

$$\begin{aligned}\left(\frac{1}{(x-x')\mp v(t-t')+i\epsilon}\right)^{\gamma} &\rightarrow \left(\frac{1}{(x-x')\pm iv(t-t')}\right)^{\gamma} \\ &\rightarrow \left(\frac{\frac{\pi T}{v}}{\sinh\left(\frac{\pi T}{v}\left[(x-x')\pm iv(\tau-\tau')\right]\right)}\right)^{\gamma}\end{aligned} \tag{6.218}$$

where γ is an exponent. Notice that the temperature changes the behavior of the boson propagator on distances long compared with the thermal wavelength v/T to

$$\langle\phi(x,t)\phi(x',t')\rangle \rightarrow \ln\left(\frac{\pi T}{2}\right) - \frac{\pi T}{v}\left|x-x'\right| + \ldots \tag{6.219}$$

[‖]The form of the boson propagator on the cylinder insures that the correlators are translation invariant and periodic (or anti-periodic for fermions). This result can also be derived by an explicit calculation of the propagator (without using conformal mappings).

[**]To restore proper units we must set $T/v \rightarrow k_B T/\hbar v$. Here we use $k_B = \hbar = 1$.

The long distance behavior of the boson propagator at finite T changes the behavior of the other correlators as well. In this regime they exhibit exponential decay of correlations over distances long compared with the thermal wavelength.

Contrary to what we did in perturbation theory, where the correlators are given in momentum and frequency space, the bosonization approach yields the exact correlators in real space and time. Thus to compute spectral functions and other quantities of interest we now must perform Fourier transforms on the analytic continuation of these expressions (some of which have a somewhat involved analytic structure). But the expressions we have are not perturbative, they are exact!

The CDW susceptibility

The thermal CDW correlation function, *i.e.* the CDW propagator in imaginary time at finite temperature T, is

$$\begin{aligned} D_{CDW}(x,\tau;T) &= \left\langle T_\tau \left(\mathcal{O}_{CDW}(x,\tau)\mathcal{O}^\dagger_{CDW}(0,0) \right) \right\rangle \\ &\sim \left[\frac{\left(\frac{\pi T}{v_c}\right)^2}{\sinh\left(\frac{\pi T}{v_c}(x+iv_c\tau)\right)\sinh\left(\frac{\pi T}{v_c}(x-iv_c\tau)\right)} \right]^{\frac{1}{2K_c}} \\ &\times \left[\frac{\left(\frac{\pi T}{v_s}\right)^2}{\sinh\left(\frac{\pi T}{v_s}(x+iv_s\tau)\right)\sinh\left(\frac{\pi T}{v_s}(x-iv_s\tau)\right)} \right]^{\frac{1}{2K_s}} \end{aligned} \tag{6.220}$$

The CDW dynamical susceptibility at finite temperature $\chi_{CDW}(p,\omega;T)$ is the Fourier transform of this expression in x and τ (after an analytic continuation to real time t). The Fourier transform has a complex analytic structure due to the branch cuts and to the difference in the charge and spin velocities. Of direct physical interest is the imaginary time of the dynamical susceptibility, $\chi''_{CDW}(p,\omega;T)$ at finite temperature which is measured by inelastic X-ray scattering.[††] Although the general form of $\chi''_{CDW}(p,\omega;T)$ can be determined

[††]Up to a Bose factor $\chi''_{CDW}(p,\omega)$ is proportional to the inelastic cross section.

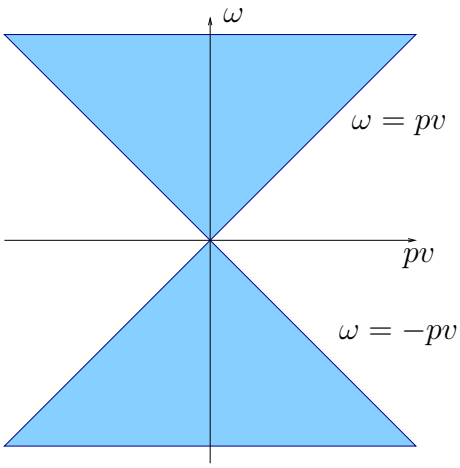

Figure 6.12:

numerically, a simple expression (which captures the main physics) can be obtained by setting $v_c = v_s = v$:

$$\chi''_{CDW}(\omega, p > 0; T) \sim -\frac{\sin(\pi\gamma)}{T^{2(1-\gamma)}} \mathrm{Im} \left\{ f\left(\frac{\omega - pv}{4\pi T}\right) f\left(\frac{\omega + pv}{4\pi T}\right) \right\} \quad (6.221)$$

Here p is measured from $2p_F$, and

$$\gamma = \frac{1}{2}\left(\frac{1}{K_c} + \frac{1}{K_s}\right) \quad (6.222)$$

The complex function $f(x)$ is given by

$$f(x) = \frac{\Gamma(\frac{\gamma}{2} - ix)}{\Gamma(1 - \frac{\gamma}{2} - ix)} \quad (6.223)$$

where $\Gamma(z)$ is the Euler Gamma function. At very low temperatures, $|\omega \pm pv| \ll T$, $\chi''_{CDW}(p, \omega; T)$ converges to the $T = 0$ result:

$$\chi''_{CDW}(p, \omega; T = 0) \propto \left| \frac{\omega^2 - p^2 v^2}{4\pi^2} \right|^{-\frac{1}{2}\left(1 - \frac{1}{K_c}\right)} \left(\theta(\omega - pv) + \theta(-(\omega + pv))\right) \quad (6.224)$$

where I have set $K_s = 1$ (we will see below that this is required by the $SU(2)$ spin rotational invariance). Since $K_c > 1$ (for repulsive interactions)

we see that the spectral function is largest near $\omega = \pm pv$ ("on-shell") where it diverges as a power law. Notice that this is a one-sided singularity as the spectral function vanishes on the other side of the mass shell condition (the edge of the shaded area in Fig.6.12.) This divergence is cutoff at finite temperature T, where it takes the maximum value determined by T.

The same behavior is found for the full *static susceptibility* at the ordering wave vector (which here this means $p = 0$). Although it can be determined from the general expression for $\chi''_{CDW}(p, \omega; T)$, it is instructive to determine it more directly by the following simple argument. At finite temperature, the long distance behavior of the correlators in real space on distances long compared to the thermal wave length v/T is an exponential decay. This behavior effectively cuts off the infrared singularities in many quantities such as the static susceptibility at the ordering wavevector $Q = 2p_F$, $\chi_{CDW}(\omega = 0, p = 0)$ (again, the momentum p is measured from the ordering wavevector). This quantity can be computed directly from a Fourier transform of the thermal CDW correlation function in imaginary time at $T = 0$ with a long distance cutoff of v/T.

$$\begin{aligned} \chi_{CDW}(0,0;T) &\sim \int_{|x|<v/T} dx \int_{|t|<1/T} dt\ D_{CDW}(x,t;T=0) \\ &\sim 2\pi \text{const.} \int_a^{v/T} \frac{r dr}{r^{2\gamma}} \\ &\propto T^{-\left(1-\frac{1}{K_c}\right)} \end{aligned} \tag{6.225}$$

where we have set $K_s = 1$. Thus, for $K_c > 1$ (repulsive interactions) the CDW susceptibility at the ordering wave vector $Q = 2p_F$ diverges as $T \to 0$. This means that the $T = 0$ system is *almost ordered*. We will see below that this behavior means that it is actually *critical at* $T = 0$, and we will identify the exponent γ with the *scaling dimension* (or *dimension* in short) of the CDW order parameter.

A similar analysis can be used to find the *structure factor* $S(p; T)$ (once again measured from $Q = 2p_F$) at temperature T, the Fourier transform *in space* of the *equal-time* correlation function,

$$S(p;T) = \int_{-\infty}^{\infty} dx\ e^{-ipx}\ D(x, t=0; T) \tag{6.226}$$

$S(p;T)$ is measured by X-ray diffraction experiments. An analysis of this Fourier transform (similar to what we did above) leads to the following result

$$S(p;T) \propto 2K_c a^{-1/K_c} - \frac{K_c^2 \Gamma(\frac{1}{K_c})}{K_c - 1} \times \begin{cases} \cos(\frac{\pi}{2K_c})|p|^{1/K_c}, & \text{for } \frac{T}{v} \ll |p| \\ \left(\frac{T}{v}\right)^{1/K_c}, & \text{for } \frac{T}{v} \gg |p| \end{cases} \tag{6.227}$$

Hence, in contrast with the static susceptibility, the structure factor at $T = 0$, $S(p, T = 0)$, does not diverge as $T \to 0$, and instead has a weak singularity (a *cusp*) at the ordering wavevector.

Finally, we note that for spin-rotational invariant systems (for which $K_s \to 1$) there are logarithmic corrections to these results due to corrections to scaling effects (we will discuss them briefly below). We can put all of this together in terms of the equal-time density correlation function (including the logarithmic correction):

$$\langle \rho(x)\rho(0)\rangle = \frac{1}{K_c(\pi x)^2} + \text{const.}\frac{\cos(2p_F x)}{|x|^{1+\frac{1}{K_c}}} \left|\ln|x|\right|^{-3/2} \ldots \tag{6.228}$$

and the $2p_F$ CDW susceptibility (also including the logarithmic correction)

$$\chi_{CDW}(T) \sim \frac{|\ln T|^{-3/2}}{T^{1-\frac{1}{K_c}}} \tag{6.229}$$

The SDW susceptibility

The correlation function of the SDW order parameter can be analyzed in a similar fashion. The dynamical susceptibility is measured by inelastic neutron scattering. Here I will only quote the main results.

The equal-time transverse spin correlation function turn out to be (setting $K_s = 1$)

$$\langle \vec{S}(x) \cdot \vec{S}(0)\rangle = \frac{1}{(\pi x)^2} + \text{const.}\frac{\cos(2p_F x)}{|x|^{1+\frac{1}{K_c}}} \sqrt{\left|\ln|x|\right|} + \ldots \tag{6.230}$$

and the SDW transverse susceptibility (at the ordering wavevector $2p_F$) is

$$\chi^{\perp}_{SDW}(T) \sim \frac{\sqrt{|\ln T|}}{T^{1-\frac{1}{K_c}}} \tag{6.231}$$

So, up to logarithmic corrections, it has the same behavior as the CDW correlators and susceptibilities. This is a special property for the spin-rotational invariant system.

The Superconducting susceptibility

Finally, we quote the results for the singlet superconductor equal-time correlation function

$$\langle \mathcal{O}^{\dagger}_{SS}(x)\mathcal{O}_{SS}(0)\rangle \sim \frac{\text{const.}}{|x|^{1+K_c}} \left|\ln|x|\right|^{-3/2} \tag{6.232}$$

and finite temperature susceptibility

$$\chi_{SS}(T) \sim T^{K_c-1} \left|\ln T\right|^{-3/2} \tag{6.233}$$

Hence, the superconducting susceptibility does not diverge as $T \to 0$ for $K_c > 1$ (repulsive interactions) but it does for $K_c < 1$ (attractive interactions). The different behavior of the superconducting and CDW susceptibilities follows directly from duality.