

$$\epsilon_{||} = 1 + i \frac{\sigma_{||}}{\omega}$$

$$\epsilon_{\perp} = \cancel{1 + i \frac{\sigma_{\perp}}{\omega}} - 1$$

f-Sum Rule

$$\langle \phi | [J_k(x, x_0), J_0(x', x_0)] | G \rangle = \frac{i e^2 n}{m c^2} \partial_k \delta^3(x-x')$$

$$D_{k0}^R(x, x_0; x', x_0) = \frac{e^2 n}{m c^2} \partial_k \delta^3(x-x')$$

$$\partial_k D_{k0}^R = \frac{e^2 n}{m c^2} \nabla^2 \delta^3(x-x')$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} i p_k D_{k0}^R(p, \omega) = - \frac{e^2 n}{m c^2} p^2 = - \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} i \omega D_{00}^R$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} i \omega D_{00}^R(p, \omega) = \frac{e^2 n}{m c^2} p^2$$

Conduction in a non-interacting disordered system

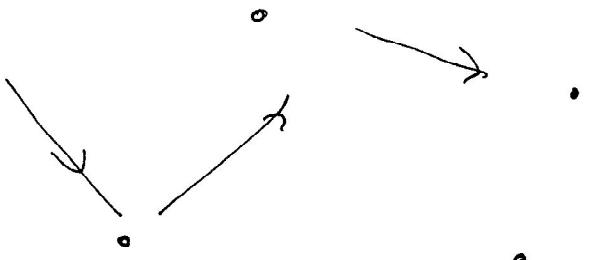
Consider a system at  $T=0$ . For the moment I will neglect the effects of electron-electron and electron-phonon interactions and focus on the effects of impurities. I will assume that the impurities are very heavy so that all the scattering is elastic (we'll later discuss the effects of inelastic scattering) and that the electrons and impurities only couple through the potential

(we'll discuss magnetic impurities later). Scattering can also originate from other imperfections (i.e. amorphousness) etc.

Impurities: We simulate the effects of impurities with an effective local potential (the sum of local contrib.)

$$U(\vec{x}) = \sum_i V(\vec{x} - \vec{R}_i) \quad (\vec{R}_i : \text{location of } i\text{th impurity})$$

and  $V(R)$  is the atomic potential. The location of the impurities is random and  $V(R)$



is short ranged (here I assume screening), this is OK for a metal that may not be OK for a semiconductor). Thus  $V(\vec{x})$  is the sum of random variables and hence it's random in itself. If  $V(R)$

decays sufficiently fast as  $R \rightarrow \infty \Rightarrow$  only the immediate neighbourhood of  $\vec{x}$  matters (i.e. a few neighboring impurities)

$$\Rightarrow \langle U(\vec{x}) \rangle = \left\langle \sum_i V(\vec{x} - \vec{R}_i) \right\rangle = 0 \quad (\text{no bias})$$

over imp. locations

$$\Rightarrow \langle U(\vec{x}) U(\vec{y}) \rangle = w^2 f(\vec{x} - \vec{y}) \quad (w \text{ sets the scale})$$

and  $f(R) \rightarrow 0$  as  $R$  gets large (in practice  $f$  is zero after a few lattice constants)

In addition higher moments are also present  $\langle U(\vec{x}) U(\vec{y}) U(\vec{z}) \rangle$  etc.

However if the scatterer is weak (i.e.  $V$  never gets to be large) we can in practice neglect moments higher than the 2<sup>nd</sup> (see Dornick and Sondheimer) (Also Central Limit Theorem helps) Thus we can approximate  $U(\vec{R})$  by a set of random (unbiased) gaussian variables with variance

$$\langle U(\vec{x}) U(\vec{y}) \rangle = w^2 f(\vec{x} - \vec{y})$$

The function  $f$  will typically decay very rapidly after a few lattice constants and thus, at that scale, can be ~~not~~ replaced by a  $\delta$ -function

$$\langle U(\vec{x}) U(\vec{y}) \rangle = w^2 \delta^{(d)}(\vec{x} - \vec{y})$$

$$\text{since } [\delta] = \frac{1}{\text{Vol.}} \Rightarrow [w]^2 = (\text{Energy})^2 \times \text{Volume}.$$

The only parameters available are: ① scattering strength (i.e. cross sect.)

- ② impurity density
- ③ density of states.

$w^2 \sim$  density of states  $\times$  impurity density  $\times$  typical matrix element  
 $\Rightarrow w^2 \sim n$  (density of scatterers) and measure the strength of the random potential

Lecture 19 (3-12-86)  
 $\Rightarrow$  Schrödinger Eqn. is

$$H = -\frac{\hbar^2}{2m} \nabla^2 + U(x)$$

$$P[U(x)] = \text{const exp} \left\{ - \int d^3x \frac{U^2(x)}{2W^2} \right\}$$

↑  
probab. distribution.

and we have a random Hamiltonian,

we need to find two things : ① New density of states  
 ② conductivity .

I will assume that the scatters are weak and rare  
 (i.e.  $w$  small)

Energy scales

$$K.E. \sim \frac{P_F^2}{2m} \equiv t \gtrless E_F$$

$\uparrow$  eff. mass.

$$U \sim \cancel{t} w$$

The relevant ratio is  $\frac{w}{t} \ll 1$  (weak scattering)

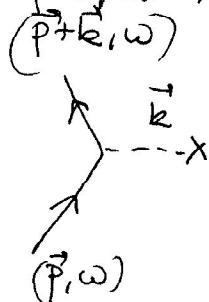
(I will show later that  $\lambda$  (mean free path)  $\sim \frac{1}{w^2}$ )

In practice I will work on the white noise limit

$$\langle U(x) \rangle = 0 \quad \langle U(x) U(y) \rangle = w^2 \delta^4(x-y)$$

How are the electronic properties modified.

Feynman diagram



$\omega$  - indip.

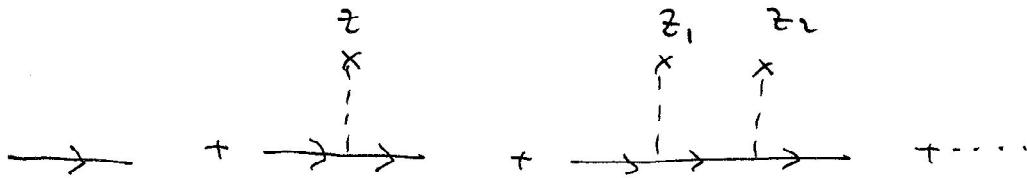
(elastic scattering)

$$H = \int d^3x \psi_{\vec{p}}^+(\vec{p}) (\epsilon(\vec{p}) + \mu) \psi(\vec{p}) + \frac{\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \psi_{\vec{p+k}}^+ \psi(\vec{p}) U(\vec{k})}{U(\vec{k})}$$

(neglect  $\mu$ .)

$$\begin{aligned} \langle U(\vec{p}) U(\vec{k}) \rangle &= \int d^3x \int d^3y \langle U(x) U(y) \rangle e^{i(p \cdot x + k \cdot y)} \\ &= W \int d^3x \int d^3y \delta^3(x-y) e^{i(p \cdot x + k \cdot y)} \\ &= W \int d^3x e^{i(p+q) \cdot x} = \\ &= W (2\pi)^3 \delta^3(\vec{p} + \vec{k}) \end{aligned}$$

Born-Series



$$\begin{aligned} G(x, y; \epsilon) &= G_0(x, y; \epsilon) + \int dz G_0(x, z; \epsilon) G_0(z, y; \epsilon) U(z) \\ &\quad + \int dz_1 \int dz_2 G_0(x, z_1; \epsilon) G_0(z_1, z_2; \epsilon) G_0(z_2, y; \epsilon) U(z_1) U(z_2) \\ &\quad + \dots \end{aligned}$$

In general

$$G(x, y; \epsilon) = \sum_{n=0}^{\infty} G_0(x, z_1; \epsilon) \dots G_0(z_n, y; \epsilon) U(z_1) \dots U(z_n)$$

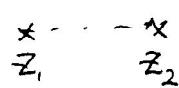
The average Green's Function is

$$\langle G(x, y; \epsilon) \rangle = \sum_{n=0}^{\infty} G_0(x, z_1; \epsilon) \dots G_0(z_n, y; \epsilon) \langle U(z_1) \dots U(z_n) \rangle$$

Lecture 20 (3-14-86)

Diagrams:

$G_0$



$$W^2 \delta(z, -z_c)$$

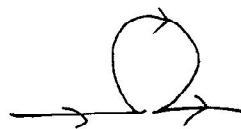
$$\delta f(z_c)$$

$$G = \rightarrow + \rightarrow \overset{\text{---}}{z_1} \overset{\text{---}}{z_2} + \rightarrow \overset{\text{---}}{z_1} \overset{\text{---}}{z_2} \overset{\text{---}}{z_3} + \rightarrow \overset{\text{---}}{z_1} \overset{\text{---}}{z_2} \overset{\text{---}}{z_3} \overset{\text{---}}{z_4} + \dots$$

$$G = G_0(x, y; E) + \int_{z_1, z_2} G_0(x, z_i; E) G_0(z_i, z_j; E) G_0(z_j, y; E) W \delta(z_1 - z_2) +$$

~~Self-energy~~

$$= G_0(x, y; E) + W^2 \int_{z_1} G_0(x, z_1; E) G_0(z_1, z_2; E) G_0(z_2, y; E) +$$



Self-energy: consider only irreducible diagrams

$$G = \rightarrow + \rightarrow \text{shaded circle} \rightarrow + \rightarrow \text{shaded circle} \rightarrow \text{shaded circle} \rightarrow + \dots$$

$$G(x, y) = G_0(x, y) + G_0(x, z_1) \sum_{z_2} (z_1, z_2) G_0(z_2, y) + \\ + G_0(x, z_1) \sum_{z_2} (z_1, z_2) G_0(z_2, z_3) \sum_{z_4} (z_3, z_4) G_0(z_4, y).$$

$$G(x, y) = G_0(x, y) + \int_{z_1, z_2} G_0(x, z_1) \sum (z_1, z_2) G(z_2, y)$$

$\Sigma$ : self-energy = sum of irr. diagrams.

$$\Sigma = \rightarrow \overset{\text{---}}{z_1} \rightarrow + \rightarrow \overset{\text{---}}{z_1} \overset{\text{---}}{z_2} \rightarrow + \rightarrow \overset{\text{---}}{z_1} \overset{\text{---}}{z_2} \overset{\text{---}}{z_3} \rightarrow + \dots$$

To leading order  $\Sigma(p, \omega) = \frac{p-k}{p\omega - \omega k} = \omega^2 \int_{\vec{k}} G_0(\vec{k}, \omega)$

no frequency change: elastic scattering.

Coherent-Potential-Approximation: Sum Rainbow graphs.  
(or ladder)

$$\overrightarrow{\text{---}} + \overrightarrow{\text{---}} + \overrightarrow{\text{---}} + \dots$$

$$\overrightarrow{\cancel{x}} = \overrightarrow{x} + \overrightarrow{\cancel{x}}$$

$$\Sigma = \overbrace{\text{---}}_{\text{full G.f.}} = \omega^2 \int_{\vec{k}} G(\vec{k}, \omega) = \Sigma_0(p, \omega) = \Sigma(\omega)$$

$$G_0(\vec{k}, \omega) = \frac{1}{\omega - E_0(\vec{k}) + \mu + i\epsilon \operatorname{sgn}(|k_f - k_f|)} \approx \frac{1}{\omega - E(\vec{k}) + \mu + i\epsilon \operatorname{sgn}\omega}$$

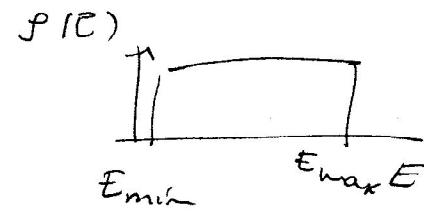
$$G(\vec{k}, \omega) = \frac{1}{\omega - E_0(\vec{k}) + \mu + i\epsilon \operatorname{sgn}\omega - \Sigma(\vec{k}, \omega)}$$

$$\Rightarrow \Sigma(\omega) = \omega^2 \int_{\vec{p}} \frac{1}{\omega - E_0(\vec{p}) + \mu + i\epsilon \operatorname{sgn}\omega - \Sigma(\omega)}$$

$$\int \frac{d^3 p}{(2\pi)^3} f(\vec{p}) = \int_{-\infty}^{+\infty} dE \int_{\vec{p}} P(E) f(E) D.O.S.$$

$$\sum(\omega) = w^2 \int_{-\infty}^{+\infty} dE \frac{f(E)}{\omega - E + \mu + i\beta \operatorname{sgn}\omega - \sum(\omega)}$$

Near  $E_F = \mu$   $\rho(E)$  varies very slowly  $\Rightarrow$  approximate with a constant DOS. within a bandwidth



$$\sum(\omega) \approx w^2 \int_{-\infty}^{+\infty} dE \frac{\rho(E_F)}{\omega - E + \mu + i\beta \operatorname{sgn}\omega - \sum(\omega)}$$

(ignoring band edge effects)

Write  $\sum = \alpha + i\beta \operatorname{sgn}\omega$

$$\alpha + i\beta \operatorname{sgn}\omega = \rho(E_F)w^2 \int_{-\infty}^{+\infty} dE \frac{\omega - E + \mu - \alpha + i\beta \operatorname{sgn}\omega}{(\omega - E + \mu - \alpha)^2 + \beta^2}$$

$$\alpha = \int_{-\infty}^{+\infty} dE \frac{(\omega - E + \mu - \alpha)}{(\omega - E + \mu - \alpha)^2 + \beta^2} \rho(E_F)w^2$$

$$1 \cancel{\beta} = \rho(E_F) w^2 \int_{-\infty}^{+\infty} dE \frac{\cancel{\beta}}{(\omega - E + \mu - \alpha)^2 + \beta^2} \quad \beta \neq 0$$

$$1 = w^2 \rho(E_F) \int_{-\infty}^{+\infty} \frac{dE}{E^2 + \beta^2} = w^2 \rho(E_F) \frac{2\pi i}{2\beta} \xrightarrow{x+i\beta} \xrightarrow{x-i\beta}$$

$\boxed{\beta = \pi w^2 \rho(E_F)}$

$$\alpha \approx -\frac{w_F(E_F)}{2} \ln \frac{E_{\max}}{|\beta|} = -\frac{w_F(E_F)}{2} \ln \left( \frac{E_{\max}}{\pi_B w} \right)$$

→ bandwidth  
shift in  $\mu$ .

$$\Rightarrow \bar{G}(p, \omega) = \frac{1}{\omega - \epsilon(\vec{p}) + \mu + i \frac{1}{2\tau} \text{sgn}\omega}$$

i.e. states with momentum  $\vec{p}$  have a width  $\Gamma = \frac{1}{2\tau}$   
( $\Rightarrow$  rate of decay of  $\vec{p}$ )

$$\frac{1}{2\tau} = \pi g(E_F) w^2$$

$\tau$ : elastic scattering time.

Define mean free path

$$l \sim v_F \tau$$

$$\Gamma_{\text{forward}} = \left| \frac{\partial E}{\partial \vec{p}} \right|_{p_F}$$

$$\Rightarrow \langle G(x, y; \vec{E}) \rangle \sim G_0(x-y, E) e^{-|\vec{x}-\vec{y}|/2l}$$

"phase coherence length"

### Conductivity

We saw before that  $\Pi$  is related to  $D_{ij}$  and thus to  $\sigma$  (conductivity)

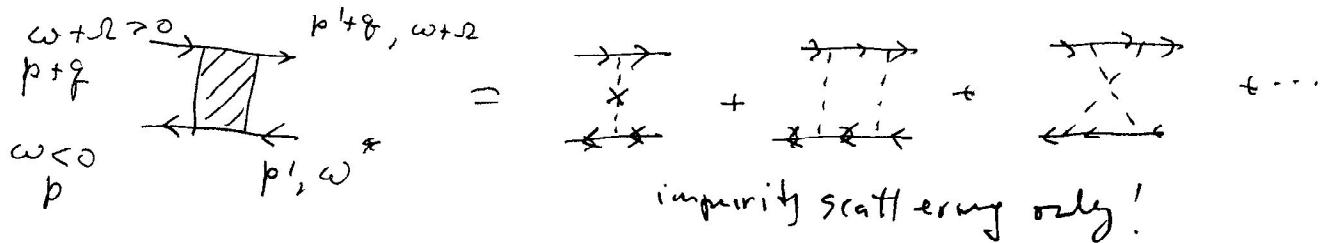
Hence we need to compute a 2-particle Green's Function.

We also saw that when we compute  $\Pi$  only the particle-hole channel contributed



## Lecture 22 (3-19-86)

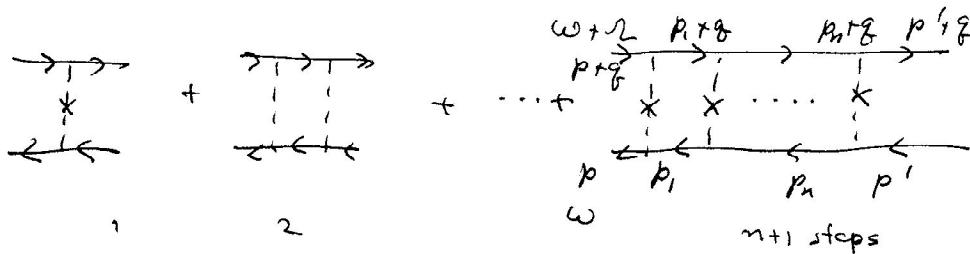
We thus only need to compute a 2-part. G-F with ~~a~~ line with  $\omega > 0$  and another with  $\omega' < 0$ . This is only possible for elastic scattering (i.e., no energy exchange)



Consider the set of ladder diagrams

$$\langle G_s^R G_A^A \rangle$$

$$\sim \langle |G|^2 \rangle$$



$$\begin{aligned} n\text{-th order term} &= \int_{p_1 \dots p_n} G^R(p_1+q, \omega+\omega) G^A(p_1, \omega) \times \dots \times \omega^{n+1} \\ &\quad \dots G^R(p_n+q, \omega+\omega) G^A(p_n, \omega) \times \\ &\quad G^R(p+q, \omega+\omega) G^A(p, \omega) G^R(p+q, \omega+\omega) G^A(p', \omega) \end{aligned}$$

$$= \omega^{n+1} \left[ \int_k G^R(k+q, \omega+\omega) G^A(k, \omega) \right]^n G^R(p+q, \omega+\omega) G^A(p, \omega) G^R(p+q, \omega+\omega) G^A(p', \omega)$$

Write the 2-part connected G-F in terms of a vertex

$$G^{(2)} = G^R(p+q, \omega+\omega) G^A(p, \omega) G^R(p+q, \omega+\omega) G^A(p', \omega) K^{+-}(q, \omega)$$

$$\Rightarrow K^{+-}(q, \omega) = \sum_{n=0}^{\infty} \omega^{n+1} \left[ \int_k G^R(k+q, \omega+\omega) G^A(k, \omega) \right]^n$$

(ladder approx.)  $\Rightarrow K^{+-}(q, \omega) = \frac{\omega^2}{1 - \omega^2 \int_k G^R(k+q, \omega+\omega) G^A(k, \omega)}$

where  $G^{R,A}(k, \omega) = \frac{1}{\omega - E(k) + \mu + i \frac{\epsilon}{2\tau}}$

Let's compute

$$I(q, \omega) = \int_k G^R(k+q, \omega+\omega) G^A(k, \omega)$$

in the limit  $\omega, |\vec{q}| \rightarrow 0$

$$(a) k^{+-}(0, \omega) \quad (q \rightarrow 0)$$

$$\oint_k I(0, \omega) = \int_k G^R(k, \omega+\omega) G^A(k, \omega) =$$

$$\approx g(E_F) \int_{-\infty}^{+\infty} dE \frac{1}{(\omega+\omega + \mu + \frac{i\epsilon}{2\tau})(\omega - E + \mu - \frac{i\epsilon}{2\tau})}$$

$$\begin{aligned} \oint_k &= \int_{-\infty}^{+\infty} dE \frac{1}{[E - (\omega + \omega + \mu + \frac{i\epsilon}{2\tau})][E - (\omega + \mu - \frac{i\epsilon}{2\tau})]} \\ &= \frac{2\pi i g(E_F)}{(\omega + \omega + \mu + \frac{i\epsilon}{2\tau}) - (\omega + \mu - \frac{i\epsilon}{2\tau})} = \frac{2\pi i g(E_F)}{\omega + \frac{i\epsilon}{2\tau}} \end{aligned}$$

If  $\omega\tau \ll 1 \Rightarrow I$  can write

$$I(0, \omega) \approx \frac{2\pi i g(E_F)}{\tau} \frac{2\pi i g(E_F) \tau}{i + \omega \tau} = \frac{2\pi g(E_F) \tau}{1 - i \omega \tau} \approx 2\pi g(E_F) \tau [1 + i \omega \tau]$$

$$\text{But } 2\pi g(E_F) \tau = \frac{1}{W_2}$$

$$\Rightarrow I(0, \omega) \approx \frac{1}{W_2} [1 + i \omega \tau + O(\omega^2)]$$

$$\text{Thus } K^{+-}(0, \tau) = \frac{\omega^2}{1 - \frac{\omega^2}{\omega} \frac{1}{\omega} [1 + i\tau\tau + \dots]}$$

$$K^{+-}(0, \tau) = \frac{\omega^2}{\tau} \left( \frac{1}{-i\tau} \right) \quad \tau > 0$$

For  $\tau < 0$  get similar answer.

$$K^{+-}(0, \tau) = \frac{\omega^2}{\tau} \left( \frac{1}{i|\tau|} \right)$$

$$(b) I(q, \tau) = \int_{\vec{k}} \frac{1}{(\omega + \tau - E(k+q) + i\frac{\epsilon}{2\tau})(\omega - E(k) + \mu - \frac{\epsilon}{2\tau})}$$

$$\underset{\substack{\sim \\ \text{small } q}}{\approx} \frac{1}{\omega^2} (1 + i\tau\tau + O(\tau^2)) - \frac{q^2}{2m^*} \frac{1}{2\pi f(E_F) \omega^2} \frac{2}{d} \frac{E_F}{\pi \omega^2 c_F}$$

↑ dimension.

$$\Rightarrow 1 - \omega I \approx -i|\tau| \tau + \tau D \vec{q}^2 \quad \text{with } D = \frac{\omega^2 c}{d}$$

$$\Rightarrow K^{+-}(q, \tau) \approx \frac{\omega^2}{\tau} \frac{1}{D \vec{q}^2 - i|\tau|} \quad (\text{in the ladder approx.})$$

Diffusion Eqn (Green's Function)

$$[-D \nabla^2 + \partial_t] K(x, t) = \frac{\omega^2}{\tau} \delta(x) \delta(t)$$

$\rightarrow K^{+-}$  is the G.F. for a diffusion eqn and  $D$  is the diffusion constant  $\Rightarrow$  particle-hole excitations propagate diffusively (i.e. randomly with  $\tau$ )

One can now compute  $\Pi$

$$\Pi = \text{Diagram A} = \text{Diagram B} + \text{Diagram C}_1 + \text{Diagram C}_2 + \dots$$

$$\text{II} = (-1) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{\vec{k}} G(k+q, \omega + \omega_0) G(k, \omega) + (-1) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{\vec{k}, \vec{k}'} G(k+q, \omega + \omega_0) G(k, \omega) G(k', q, \omega + \omega_0)$$

(1)  $\sim G(k', \omega) K(q, \omega_0)$

$$\text{where } K^{\frac{+}{-}} \approx \frac{w^2}{c} \frac{1}{Dg^2 - i\Gamma g_1}$$

$K^{\pm\pm} \approx W$  (because  $\frac{1}{|x|} \geq 0$  for particle-particle,  
 i.e.  $K^{\pm\pm}$  are non-diffusive)

Let's separate out contributions in  $p_F$  and  $p_L^h$  pieces.

$$\begin{aligned}
 & +\infty \\
 & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^d} G(k, \omega) G(k+q, \omega+r) = \\
 & \text{cutoff} \nearrow \wedge \infty \\
 & = \int_0^{\infty} d\omega \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^d} G^+(k, \omega) G^+(k+q, \omega+r) + \int_{-\infty}^{-r} d\omega \int_k^{\infty} G^- G^- + \\
 & > 0 \quad \omega + r > 0 \Rightarrow \omega > 0 \quad \omega < 0 \quad \omega + r < 0 \\
 & \qquad \qquad \qquad \omega \in -\mathbb{R}
 \end{aligned}$$

$$+ \int \frac{d\omega}{2\pi} \int_{-\infty}^{\omega} \left( \int \frac{dk}{(2\pi)^d} G^+(k, \omega) G^-(k+q, \omega+\nu) \right) + \int \frac{d\omega}{2\pi} \int_{\omega}^{\infty} G^-(k, \omega) G^+(k+q, \omega+\nu)$$

$\omega > 0 \quad \omega_1, \omega_2 < 0 \quad \text{impossible if } \nu > 0$

~~At first~~ we have integrals of the form.

$$\int_0^{\infty} \frac{d\omega}{2\pi} \int_{\vec{k}} \frac{1}{(\omega - E(k) + \mu + \frac{i}{2\tau}) (\omega + \omega - E(k+q) + \mu + \frac{i}{2\tau})} +$$

$$+ \int_{-\infty}^{-\omega} \frac{d\omega}{2\pi} \int_{\vec{k}} \frac{1}{(\omega - E(k) + \mu - \frac{i}{2\tau}) (\omega + \omega - E(k+q) + \mu - \frac{i}{2\tau})} +$$

$$+ \int_{-\omega}^0 \frac{d\omega}{2\pi} \int_{\vec{k}} \frac{1}{(\omega - E(k) + \mu - \frac{i}{2\tau}) (\omega + \omega - E(k+q) + \mu + \frac{i}{2\tau})}$$

$$\xrightarrow{\omega \rightarrow 0} \approx \tau f(E_F)$$

$$\xrightarrow{\omega \rightarrow 0} \int_{\vec{k}} \left[ \int_0^{\omega} \frac{d\omega}{2\pi} \frac{1}{(\omega - \alpha(k)) (\omega - \alpha(k+q) + \omega)} + \right. \\ \left. \int_{-\infty}^{-\omega} \frac{d\omega}{2\pi} \frac{1}{(\omega - \alpha^*(k)) (\omega + \omega - \alpha^*(k+q))} \right]$$

$$\xrightarrow{\omega \rightarrow \infty} \frac{1}{2\pi} \int_{\vec{k}} \left[ \frac{\ln \frac{\alpha(k)}{\alpha(k+q) - \omega}}{\alpha(k+q) - \alpha(k) - \omega} + \text{odd.} \frac{\ln \frac{\alpha^*(k+q) - \omega}{\alpha^*(k)}}{\alpha^*(k+q) - \alpha^*(k) - \omega} \right]$$

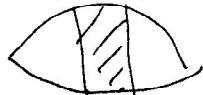
$$\xrightarrow{\omega \rightarrow 0} \frac{1}{2\pi} \int_{\vec{k}} \left[ -\frac{1}{\alpha(k)} + \frac{1}{\omega + \alpha^*(k)} \right]$$

$$\approx \frac{1}{E} \left[ \left( \frac{-f(E_F)}{2\pi} \right) \frac{1}{(E - \mu - \frac{i}{2\tau})} + \frac{f(E_F)}{2\pi} \left( \frac{1}{(E - \mu + \frac{i}{2\tau}) + \omega} \right) \right]$$

$$= \int dE \frac{f(E_F)}{2\pi} \frac{\left(\frac{-i}{\tau} - \omega\right)}{(E - \mu - \frac{i}{2\tau})(E - \mu + \omega + \frac{i}{2\tau})} = -i f(E_F)$$

~~xx~~

The other terms are



The only non-vanishing contribution as  $\omega, \epsilon \rightarrow 0$  comes from the particle-hole channel

$$\begin{aligned} & \text{Diagram: } \text{A loop with two internal lines. The top line has a plus sign (+) and the bottom line has a minus sign (-).} \\ & = \int_{\omega} \int_{\mathbf{k}} G^+ G^- K^{+-} G^+ G^- = \\ & = \int_{\omega} I^2 K^{+-} \approx \left(\frac{1}{W^2}\right)^2 \frac{\omega}{2\pi} \frac{W^2/c}{Dg^2 i \omega} \end{aligned}$$

$$= \frac{\omega}{2\pi c W^2} \frac{1}{Dg^2 i \omega}$$

$$2\pi W^2 c = \frac{1}{f(E_F)}$$

$$= \frac{f(E_F) \omega}{Dg^2 i \omega}$$

Collecting terms I get

$$i\pi = -i f(E_F) + \frac{f(E_F) \omega}{Dg^2 i \omega} = \frac{-i f(E_F)}{Dg^2 i \omega} [Dg^2 i \omega + \omega]$$

$$\Rightarrow \boxed{\Pi = -i f(E_F) \frac{Dg^2}{Dg^2 + \omega}}$$

The retarded  $\Pi$  is

$$\Pi^R(\vec{q}, \omega) = -\frac{e^2 n}{m c^2} \frac{D \vec{q}^2}{D \vec{q}^2 + i\omega} \quad \Rightarrow \quad \text{Re } \Pi^R = \text{Re } \Pi$$

$$\text{Im } \Pi^R = \text{Im } \omega \text{ Re } \Pi$$

Consider the conductivity tensor relative to external field,

$$i\omega \sigma = D - \frac{e^2 n}{mc^2} \mathbb{1}$$

$$D = D_{||} \frac{\vec{P} \times \vec{P}}{P^2} + D_{\perp} \left( \mathbb{1} - \frac{\vec{P} \times \vec{P}}{P^2} \right)$$

$$\sigma = \sigma_{||} \frac{\vec{P} \times \vec{P}}{P^2} + \sigma_{\perp} \left( \mathbb{1} - \frac{\vec{P} \times \vec{P}}{P^2} \right)$$

$$i\omega \sigma_{||} = D_{||} - \frac{e^2 n}{mc^2}$$

$$i\omega \sigma_{\perp} = D_{\perp} - \frac{e^2 n}{mc^2}$$

$$P \rightarrow 0 \Rightarrow D_{||} = D_{\perp} = D$$

$$\sigma_{||} = \sigma_{\perp} = \sigma(\omega)$$

$$D_{00} = \frac{P^2}{\omega^2} \left( D_{||} - \frac{e^2 n}{mc^2} \right)$$

$$D_{00}^R = \Pi^R$$

$$i\omega \sigma_{||} = \frac{\omega^2}{P^2} \Pi^R$$

$$\Rightarrow \sigma_{||} = -\frac{i\omega}{P^2} \Pi^R$$

The Imag. part of  $\Pi^R$   
is the dissipation  $\sigma$

$$\sigma_{||}(\omega) = \omega \lim_{P \rightarrow 0} \frac{1}{P^2} \text{Im } \Pi^R(P, \omega)$$

$$\sigma(\omega) = -\frac{i\omega}{\hbar^2} (-1) g(E_F) \frac{D p^2}{D p^2 + i\omega} \xrightarrow{p \rightarrow 0} \frac{\frac{2e\omega}{\hbar} g(E_F) D}{\omega + i\omega}$$

$$\sigma(\omega) = g(E_F) D \times \left( \frac{e^2}{\hbar} \right)$$

↑  
comes from units.

$$\Rightarrow \sigma = \frac{e^2}{\hbar} g(E_F) D \quad \text{Einstein Relation}$$

If interactions are taken into account we can show that

$g(E_F)$  must be replaced by  $\frac{\partial n}{\partial \mu}$  ( $= g(E_F)$  in the absence of interaction).

### Lecture 23 (3-21-86)

#### Quantum Fluctuations

So much for the Ladder approximation. We've seen that within this approx. there's diffusion in the system. But everything seems to hang from the sum of ladder graphs. I'll show now that the pole in the diffusive propagator  $K^+$  is in fact always there.

The argument is the following

Let me write

$$\begin{aligned} \sum_{\vec{y}} \langle G(\vec{x}, \vec{y}; E_1) G(\vec{y}, \vec{x}; E_2) \rangle &= \\ = \sum_{\vec{y}} \overline{\langle \vec{x} | \frac{1}{E_1 - \hat{H}} | \vec{y} \rangle \langle \vec{y} | \frac{1}{E_2 - \hat{H}} | \vec{x} \rangle} &= \end{aligned}$$

$$\begin{aligned}
 &= \overline{\langle \vec{x} | \frac{1}{(E_1 - \hat{H})} \cdot \frac{1}{E_2 - \hat{H}} | \vec{x} \rangle} = \\
 &= \frac{1}{E_2 - E_1} \left[ \overline{\langle \vec{x} | \frac{1}{E_1 - \hat{H}} | \vec{x} \rangle} - \overline{\langle \vec{x} | \frac{1}{E_2 - \hat{H}} | \vec{x} \rangle} \right]
 \end{aligned}$$

$$E_1 = \mu + \omega + \eta \quad \eta > 0, \omega < 0$$

$$E_2 = \mu + \omega$$

$$\Rightarrow \overline{\langle \vec{x} | \frac{1}{E - \hat{H}} | \vec{x} \rangle} = \langle G(\vec{x}, \vec{x}; E) \rangle = \operatorname{Re} \langle G(\vec{x}, \vec{x}; E) \rangle - \\
 - i\pi \overline{g(\vec{x}; E)} \operatorname{sgn}(E - \mu)$$

and recall that  $\operatorname{Re} \overline{G(\vec{x}, \vec{x}; \mu + \omega + \eta)} - \operatorname{Re} \overline{G(\vec{x}, \vec{x}; \mu + \omega)} \xrightarrow[\eta \rightarrow 0]{} 0$

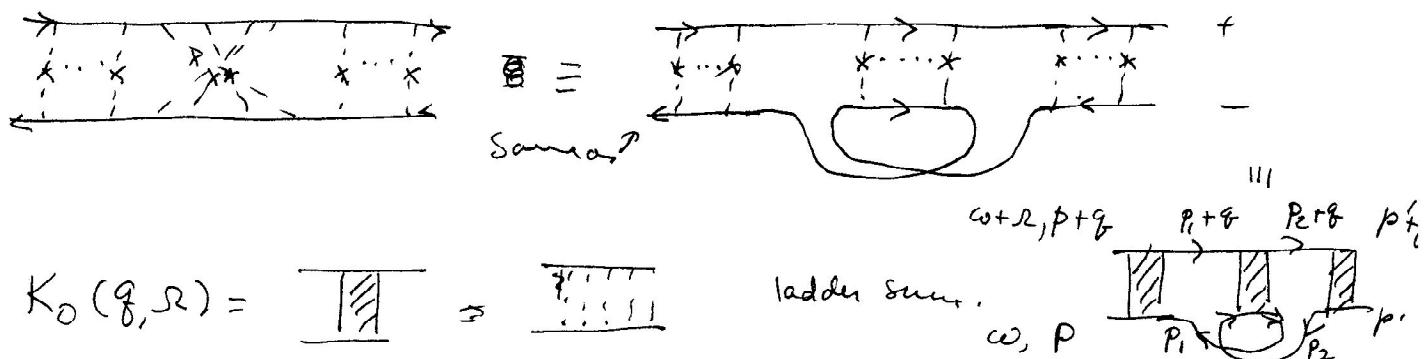
$$\begin{aligned}
 \Rightarrow \lim_{R \rightarrow 0} \sum_{\vec{y}} \overline{G(x, y; E_1)} G(y, x; E_2) &= \frac{1}{(-i\sqrt{2})} - 2\pi i \overline{g(E_F, \vec{x})} \\
 &\quad \mapsto \text{exact D.O.S. at } E_F \\
 &= \frac{2\pi \overline{g(E_F)}}{-i|\sqrt{2}|}
 \end{aligned}$$

Thus the pole is always there. We also saw that if  $\omega$  is small the diffusion constant  $D$  is  $\neq 0$  and finite. This is not the case if  $\omega$  is large and, in fact, never the case for ~~for~~ very thin films (and MOSFETs) and very thin metallic wires. The reason is multiple scattering effects produce destructive interference in the forward direction.

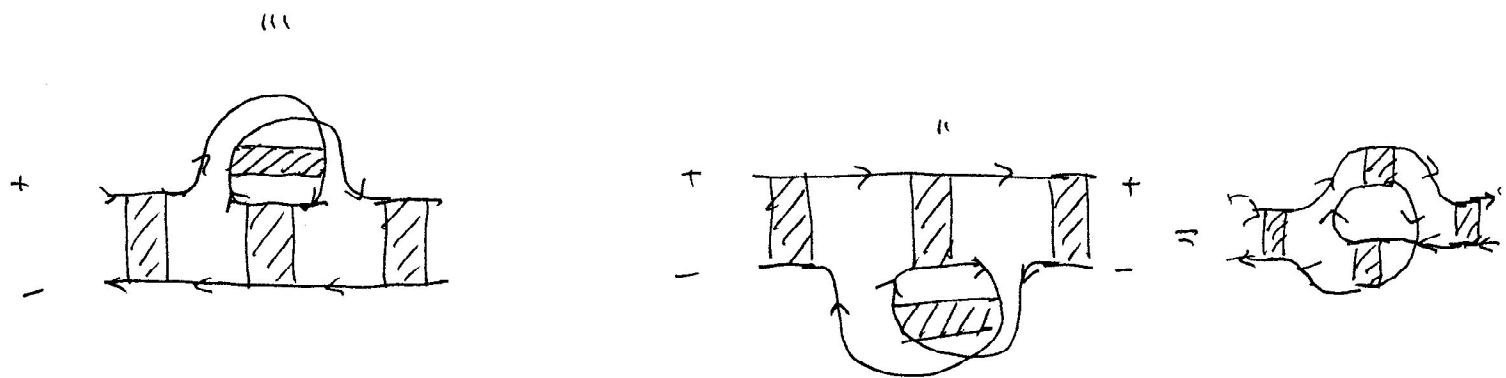
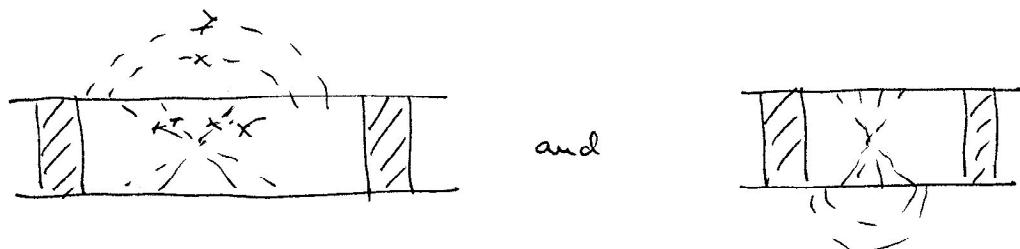
This problem was first addressed by Langer and Neal (1966)

who suggested to look at the first corrections  $\sim \frac{1}{P_F l}$

They considered the maximally crossed diagrams.



There are two more terms



$$\begin{aligned}
 & \int_{P_1 P_2} K_0^{+-}(q, s) G^+(p_1 + q, \omega + s) G^+(p_2 + q, \omega) K_0^{+-}(p_1 + p_2 + q, s) \\
 & \cdot G^+(p_2 + q, \omega + s) G^-(p_2, \omega) K_0^{+-}(q, s)
 \end{aligned}$$

$$= \left[ K_0^{+-}(\vec{q}, \omega) \right]^2 \int_{\vec{p}_1 \vec{p}_2} G^R(p_1 + \vec{q}, \omega + \omega) G^A(p_2, \omega) G^R(p_2 + \vec{q}, \omega + \omega) G^A(p_1, \omega) \\ \times K_0^{+-}(\vec{p}_1 + \vec{p}_2 + \vec{q}, \omega)$$

$\Rightarrow$  Since  $K_0^{+-}(\vec{q}, \omega) = \frac{w^2/c}{D \vec{q}^2 - i|\omega|}$

we see that there is a dangerous pole at  $\vec{Q} = \vec{p}_1 + \vec{p}_2 + \vec{q} = 0$

For  $\vec{q} = 0$  this means  $\vec{p}_1 + \vec{p}_2 = 0$  ("backscattering")

Thus for  $\omega = 0$  and  $\vec{Q} = \vec{p}_1 + \vec{p}_2$  there is a piece of the integral that looks like  $\int \frac{d\vec{Q}}{(\vec{Q} + \vec{q})^2}$

For ~~for d < 2~~  $d \leq 2$  this integral diverges as  $\vec{q} \rightarrow 0$

(in  $d=2$  it's a logarithm). The sum of the three terms is ( $d=2$ )

$$\delta K^{+-}(\vec{q}, \omega) = \frac{w^2/c}{D \vec{q}^2 - i|\omega|} \underbrace{\frac{1}{D \vec{q}^2 - i|\omega|}}_{\text{pole}} \left[ -\frac{1}{2\pi} + \frac{1}{2\pi g(E_F)D} \right] D \vec{q}^2 \ln \frac{|\omega|}{D\Lambda^2}$$

( $\Lambda$  is a max. momentum cutoff). Summing up a geometric series we get

$$\left[ K^{+-}(\vec{q}, \omega) \right]^{-1} = D \vec{q}^2 - i|\omega| + \frac{1}{2\pi} \left[ \frac{1}{2\pi g(E_F)D} \right] D \vec{q}^2 \ln \frac{|\omega|}{D\Lambda^2} + \dots$$

$$\Rightarrow D \left[ 1 + \frac{1}{2\pi} \cdot \frac{1}{2\pi g(E_F)D} \ln \frac{|\omega|}{D\Lambda^2} \right] \vec{q}^2 - i|\omega| + \dots$$

In a three dimensional system the  $\ln \frac{|R|}{DA^2}$  term has to be replaced by  $\left[ \frac{|R|}{DA^2} \right]^{\frac{1}{2}}$ . In a one dimensional system (very thin wire) the term is  $\left( \frac{|R|}{DA^2} \right)^{-\frac{1}{2}}$ .

Thus one gets an effective (or renormalized) diffusion constant

$$D_{\text{eff}} = D \left[ 1 + \frac{1}{2\pi} \left( \frac{1}{2\pi \rho (E_F) D} \right) \ln \left( \frac{|R|}{DA^2} \right) \right] \quad (d=2)$$

The expansion parameter is  $t = \frac{1}{2\pi \rho (E_F) D} = \frac{e^2 / h}{2\pi \sigma} \equiv \frac{e^2}{h \sigma} \equiv r_{\text{resistance}}$

Units:  $[t] = \frac{1}{[\rho][D]} = \frac{1}{\frac{1[\text{kg}]}{[\text{E}][L]^d} \frac{[\text{L}^2]}{[\text{T}]}} = \left[ \frac{1}{[\text{E}]} \right] [L]^{d-2}$  in units of  $\frac{e^2}{h}$

Choose units s.t.  $[\hbar] = 1 \Rightarrow [\text{E}][\text{T}] = 1 \Rightarrow [t] = [L]^{d-2}$

Same as resistance  $L^{d-2}$  ("cross sectional area")

Note that  $t$  (or  $r$ ) are dimensionless in  $d=2$

The expression of  $D_{\text{eff}}$  for a very thin film is such that the correction term is much larger than the classical term (and negative)

at very low frequencies. Thus  $D_{\text{eff}}$  is much reduced

In a bulk (3d) system the situation is quite different. The

correction term  $\sim \left( \frac{|R|}{DA^2} \right)^{\frac{1}{2}} \xrightarrow{|R| \rightarrow 0} 0$ . On the other hand for a wire ( $d=1$ ) it is much worse  $\sim \left( \frac{|R|}{DA^2} \right)^{-\frac{1}{2}}$

Thus for  $d \leq 2$  the corrections dominate  $\Rightarrow$  perturbation theory breaks down. This means that for any amount of disorder the system is far from a weakly disordered metal. In 3d pert. theory is quite all right  $\Rightarrow$  metallic phase is stable. and  $\propto D > 0$ . But in  $d \leq 2$  we find that there's a frequency scale  $\omega_c$  s.t.

$$D \sim 0 \quad \omega_c \sim D_0 \propto e^{-2\pi(2\pi S D_0)} \quad (S D_0 \sim k_F l)$$

Alternatively we can say that there's a length scale beyond which the physics is very different  ~~$\omega_c = D_0 \xi_c^{-2}$~~

We can consider that the system behaves like a metal up to a length scale  $L \lesssim \xi_c \sim l e^{\frac{\pi}{2\pi S D_0}}$ . For systems of linear size  $L \gtrsim \xi_c$  the corrections are overwhelming and ~~can't~~ cannot be regarded as a metal.

Lecture 24 (4-7-86)

### Localization and Scaling Ideas

Thus we see that for a thin film (or a MOSFET) there's a length scale  $\xi_c$  such that for  $L < \xi_c$  the system behaves roughly like a rather poor metal and for  $L \gtrsim \xi_c$  the diffusion constant goes to zero rapidly (i.e. no conduction). The length scale  $\xi_c$  is known as the localization length. It represents the maximum distance ~~for~~ for diffusion.

The localization length  $\xi_c$  is a very sensitive function of  $2\pi \sigma D_0$ .

For  $\sigma D_0 \sim 1$ , the conductivity is  $\sigma = \frac{e^2}{4\pi} 2\pi = \frac{e^2}{h}$ , which is Mott's "minimum metallic conductivity" ( $\sigma \sim 4 \times 10^{-5} \text{ ohms}^{-1}$ )

At that scale  $\sigma D_0 \sim k_F l \sim 1 \Rightarrow k_F \sim \frac{1}{l}$  ( $l = \text{mean-free-path}$ ). We then see that the conductivity depends very strongly on the length scale, being classical for  $L < \xi_c$  and going rapidly to zero for  $L > \xi_c$ . This suggests (Thouless) that it's better to consider the conductance  $G(L)$  of a system of size  $L$

$$G(L) = \sigma(L) L^{d-2}$$

which is extensive.

For Ohm's law to be valid  $\sigma(L)$  should be indep. of  $L$  (for  $L \gg \xi_c \gg l$ ). But in  $d=2$   $G(L) = \sigma(L)$ , i.e. both are dimensionless. Thus, in the absence of fluctuations, a 2d metal must be scale invariant (i.e.  $L$  indep.)

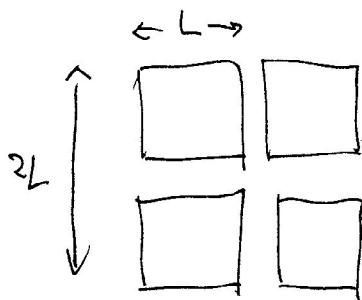
What happens for systems of size  $L \gg \xi_c \gg l$ ?

If the disorder is very strong we expect the states to be localized with exponentially decaying wavefunctions. Consider a system of linear size  $L$ . The conductance  $G(L) \sim e^{-L/\xi(L)}$  since the overlap between localized w.f. is exp-small ( $\xi \sim \text{localization length}$ )

For weak disorder  $G(L) \sim \sigma L^{d-2}$

Suppose we put together  $2^d$  such blocks. The new system has size  $2L$ . The wave functions of the new system can be calculated from those of the subsystems of size  $L$ , in a perturbation expansion in terms of block boundary terms. A typical term of such an expansion has

the form  $\sim \frac{\text{overlap}}{\text{energy denominator}}$



The energy denominator  $\sim$  level spacing  $\sim L^d \sim \frac{1}{S_L L^d}$

where  $S_L$  is the D.O.S. at scale  $L$ .

The overlap can be estimated by noting that if  $L$  blocks of size  $L^d$  are put together in a chain configuration a band of states and ~~overlapping~~ the band width  $\sim$  energy of ~~overlaps~~  $\sim \Delta E$  can be attributed to be the energy uncertainty for a wave packet localized inside  $L$ .

$$\Rightarrow \frac{\text{Overlap}}{\text{E. D.}} \sim \frac{\Delta E}{\frac{1}{S_L L^d}}$$

$$\Delta E_L \sim \frac{\hbar}{T_L}$$

where  $T_L$  is the time it takes for the electron to diffuse across the block

$$T_L \sim \frac{L^2}{D_L}$$

$D_L$  = diffusion constant.

Lecture 25 (4-9-86)

$$\Rightarrow \text{ratio} \sim \frac{\hbar D_L}{L^2} S_L L^d \sim \frac{\text{overlap}}{\text{energy down.}} \sim \frac{k \cdot \text{Energy}}{\text{Width}} \sim \frac{V}{W}$$

Einstein's Relation:  $\sigma_L = \frac{e^2}{h} S_L D_L$

$$\Rightarrow \text{ratio} \sim \hbar^2 \left( \frac{e^2}{h} S_L D_L \right) L^{d-2} = \hbar^2 \sigma_L L^{d-2}$$

$$\Rightarrow \text{ratio} \sim \hbar^2 G(L) \quad (\text{conductance}) \quad \text{as a measure of the coupling}$$

If  $L \gg l \Rightarrow G(2L)$  must be a function of  $G(L)$  only

Define a dimensionless conductance

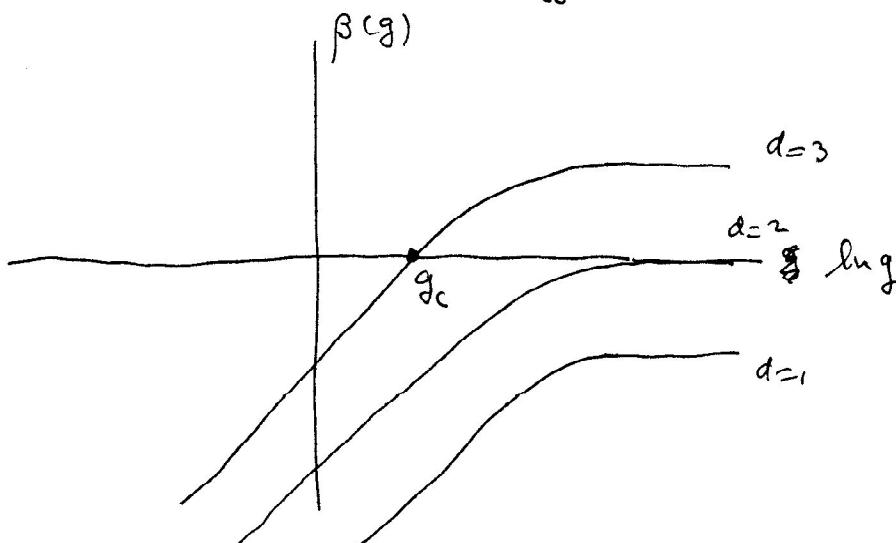
(Abrahams et al.)  $g(L) = G(L) / e^2 / h$

$$\text{and } \beta(g) = \frac{d \ln g(L)}{d \ln L}$$

Asymptotically we know that

$$G(L) \sim \sigma L^{d-2} \quad \begin{array}{l} \text{weak dis.} \\ \sim e^{-L/\xi(L)} \end{array} \quad \begin{array}{l} \text{strong dis.} \\ \text{strong dis.} \end{array}$$

$$\Rightarrow \beta(g) \sim \begin{cases} d-2 + O(g) \\ \ln \frac{g}{g_0} \end{cases}$$



$$\text{In } d=2 \quad D(r) = D_0 \left[ 1 + \frac{1}{2\pi} - \frac{1}{2\pi D_0} \ln \left| \frac{r}{D_0^2} \right| \right]$$

$$g(L) = 2 \rho D$$

$$\ln \left| \frac{r}{D_0^2} \right| \rightarrow \ln \left( \frac{L}{\ell} \right)^{-2} \quad \begin{matrix} \text{for finite size system} \\ (\text{as } r \rightarrow 0) \end{matrix}$$

$$D(L) \approx D_0 \left( 1 - \frac{1}{\pi^2} \frac{1}{g_0} \ln \frac{L}{\ell} \right)$$

$$\Rightarrow g(L) \approx g_0 \left( 1 - \frac{1}{\pi^2 g_0} \ln \frac{L}{\ell} \right)$$

$$\Rightarrow \beta(g) = \frac{\partial \ln g(L)}{\partial \ln L} = -\frac{1}{\pi^2} \frac{1}{g} < 0$$

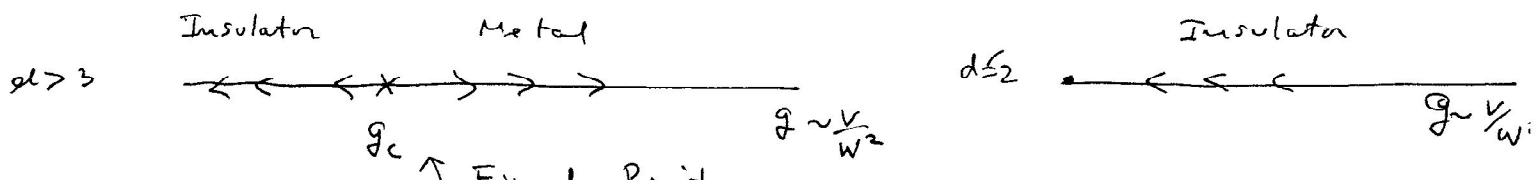
What is the meaning of this function? For  $d=2$

$$\beta(g) < 0 \Rightarrow \frac{\partial \ln g}{\partial \ln L} < 0 \Rightarrow 0 < g \text{ decreases as } L \text{ increases}$$

Thus for any amount of disorder.

In a bulk 3d system there's a critical value of the dimensionless conductance  $g_c$  above which the system scales to large values of  $g$ , i.e. a metal ( $\beta(g) > 0$  for  $g > g_c$ ). Conversely for  $g < g_c$ ,  $\beta(g) < 0$  and  $g_L$  decreases as  $L$  increases

$\Rightarrow g_L \rightarrow 0$  as  $L \rightarrow \infty$ . The system is an insulator.



Behavior of this sort has been observed in MOSFETS and thin film metallic ~~systems~~ systems ( $\text{Ge}_x \text{Au}_{1-x}$ ,  $\text{B}_x \text{Au}_{1-x}$ , etc.). Experimentally the systems are always at finite temperatures. This means that the electrons interact with phonons from the "heat bath" (the lattice). This interaction is inelastic and there is an inelastic scattering time  $\tau_{in}$  associated with it and an inelastic mean-free-path  $l_{in} \sim \sqrt{D\tau_{in}}$ . The inelastic scattering time depends on temperature  $T \Rightarrow \tau_{in}(T)$ . Thus if the frequency  $\omega > \tau_{in}^{-1}$  we expect to be able to ignore inelastic effects. But for  $\omega < \tau_{in}^{-1}$  such effects dominate and  $\sigma$  is in fact  $\neq 0$ . Thus the integrals must be cut off at  $\omega_{min} = \frac{1}{\tau_{in}(T)} = \frac{D_0}{l_{in}^2(T)}$ .

We can show that  $\tau_{in}(T) \sim T^{-p}$  at low temperatures where  $p$  depends on dimension ( $p=2$  in 3D for phonons)

$$\Rightarrow \delta \sigma_{3D}(T) \sim \left[ \frac{\omega_{min}}{D \Lambda^2} \right]^{1/2} \sim T^{p/2} \quad \text{in 3D.}$$

$$\delta \sigma_{2D}(T) \sim \ln \left| \frac{\omega_{min}}{D \Lambda^2} \right| \sim p \ln \left( \frac{T}{T_0} \right)$$

$T$  crossover temperature  $\approx \omega_c$

This logarithmic behavior has been seen by Dolan & Osheroff (thin films) and Mochel (Au/Ge).