COURSE 7

INTERACTIONS AND LOCALIZATION IN A DISORDERED ELECTRON GAS

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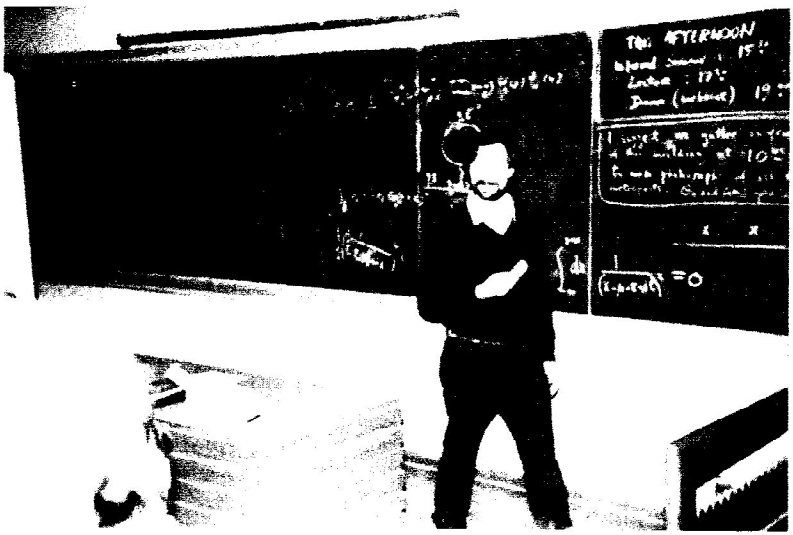
THIS AFTERNOON

Contents

1. Introduction

The electronic properties of an amorphous metal are very complex. In general the local electrostatic potential felt by the electrons can be approximated by a random variable. The basic assumption is that one is looking at a scale in which the system is homogeneous. In such a system the properties of the one-particle states depend on the relative size of the width of the distribution of electrostatic potentials w to the matrix element t for electronic hopping a typical microscopic distance a_0. In 1958 Andrson [1] considered such a problem. His line of thought was (I think!) as follows. In the limit where $w \ll t$ we have a weakly disordered metal. It is clear that if the Fermi energy E_F is sufficiently far away from the edges, the one-particle state with energy E_F should be extended

$$\text{(i.e. } |\psi(x)|^2 \underset{x\to\infty}{\to} \text{constant)}$$

and the system should conduct. On the other hand if $w \gg t$, the kinetic energy term is a perturbation and the electron is trapped in the deep valleys of the random electrostatic potential $V(x)$. To any finite order p in a perturbative expansion in t/w, the electron will be able to propagate up to a distance of the order $pa_0/2$. If this expansion has a finite radius of convergence then *all* the eigenstates of the system should be localized. Of course his analysis was more involved than this but the argument is essentially the one given before. He then concluded that there should be a critical value of w/t beyond which all states should be localized. For $w/t \ll 1$ we have $\int |\psi|^2 = \infty$ while

$$\int |\psi|^2 < \infty \quad \text{for} \quad \frac{w}{t} \gg 1.$$

The nature of the transition at $(w/t)_{\text{critical}}$ remained a mystery for very many years. Mott proposed that the transition at the mobility edge E_c (the energy that separates extended from localized states for $w/t < (w/t)_{\text{critical}}$) should be first order and the conductivity σ would vanish *discontinuously* at E_c [2, 3]. The other alternative is that the transition is continuous and that the conductivity should vanish continuously as one crosses the mobility edge. The transition is second order. In such case

there should exist a length scale with diverges at the mobility edge. In fact for $w/t \gg 1$, or alternatively deep in the tail of the density of states (see the lectures of G. Parisi in these Proceedings), the *amplitude* of the wave function of a localized state decays exponentially with the distance with a typical scale ξ, the localization length. If the transition is second order then ξ should diverge at the mobility edge and a continuum, universal, description of the transition should be possible. It has been argued by Anderson himself, by Thouless [4], Wegner [5] and by Abrahams et al. [6] that the last alternative is the correct one.

2. Localization in non-interacting systems

It is not my purpose to give a comprehensive review on localization theory. There are a number of excellent reviews in the literature. See for instance, the review by Thouless [7] and his lectures in Les Houches, 1978 [4]. There are a number of very instructive lectures in the Les Houches Winter Study Institute, February 1980 [8]. See also the paper by Mc Kane and Stone [9].

Instead I will describe the basic ideas of the scaling theory of localization and discuss how should these ideas be modified in order to include interaction effects in a consistent fashion.

Why is localization theory difficult? The main reason is that this is a theory in which several scales are present and this renders straightforward perturbation theory around a perfect metal useless. We have the situation shown in the diagram of fig. 1.

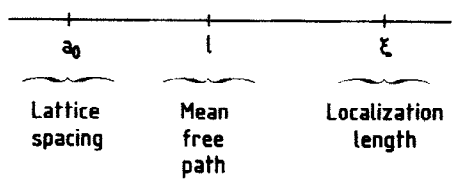

Fig. 1. Scales in a disordered electronic system.

The first scale, a_0, is simply the interatomic spacing. The second scale, l (the mean free path), is the scale at which the electron looses memory of the *phase* of its wave function. Finally there is the localization length that controls the decay of the *amplitude* of the wave function. The problem with perturbation theory around a pure metallic state is that the only thing it can do is to provide for a mean free path. In terms of Feynman diagrams we have, for the one-particle Green's function, (G.F.), the diagrammatic expansion shown in fig. 2.

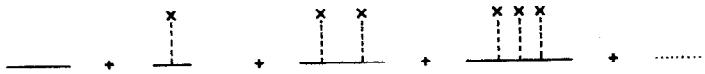

Fig. 2. The one-particle Green's function.

This is nothing but the Born series. Here the crosses represent the action of an impurity or of the random potential $V(x)$. The random potential $V(x)$ is generally assumed to be a set of random gaussian variables independently distributed:

$$\bar{V}(x) = 0, \quad \overline{V(x)V(y)} = W^2\delta(x-y). \tag{1}$$

The averaged G.F. is then calculated by tying up to crosses (fig. 3).

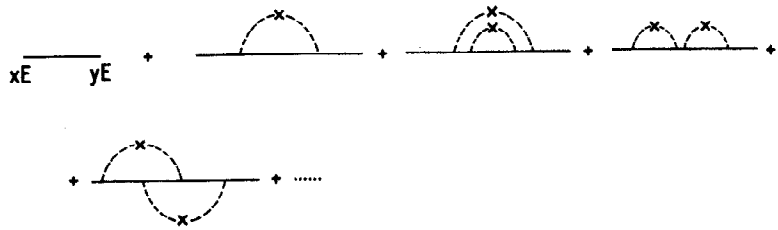

Fig. 3. The averaged one-particle Green's function.

Note that since the scattering is elastic the energy label of the line does not change. Each solid line is a bare electronic propagator

$$G(x-y,E) = \langle x| \frac{1}{-\Delta + V(x) - E} |y\rangle. \tag{2}$$

Each broken line represents an impurity average $\overline{V(x)V(y)}$. Thus the leading contribution to the self-energy is given by the one-particle irreducible graph of fig. 4.

Fig. 4. Leading contribution to the self-energy.

The main effect of this graph is to give a *finite* imaginary part to the self energy of the electron. There is a momentum independent, generally divergent, contribution to the real part but it can be cancelled by a suitable redefinition of the chemical potential. I will drop from now on all momentum independent real contributions to the self energy Σ.

2.1. The N-orbital model

One can make this calculation self-consistent by summing up the set of rainbow graphs of fig. 5.

Fig. 5. Summing rainbow graphs.

This is precisely what the Coherent Potential Approximation (CPA), mentioned in Parisi's lectures (this Volume), does. This approximation can be made exact by considering the N-orbital model introduced by Wegner [10] and studied by Oppermann and Wegner [11] in the $1/N$ expansion. In this model there are N species of electrons, labelled by an index $\alpha = 1, \ldots, N$. The random potential $V(x)$ is now $V_{\alpha\beta}(x)$ with the following properties

$$\bar{V}_{\alpha\beta}(x) = 0,$$

$$\overline{V_{\alpha\beta}(x)V_{\gamma\delta}(y)} = M\delta(x-y)(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \quad (V \text{ real}). \tag{3}$$

It is easy to see that all the rainbow graphs are of the same order in N. Hence $N \to \infty$ reproduces the CPA theory [10].

The Hamiltonian of the system, in the formulation of second quantization, is (including interaction terms)

$$H = \int \mathrm{d}^d x\, \psi_\alpha^+(x)(-\tfrac{1}{2}\nabla^2)\psi_\alpha(x) + \frac{V_{\alpha\beta}(x)}{\sqrt{N}}\, \psi_\alpha^+(x)\psi_\beta(x)$$

$$+ \int \mathrm{d}^d x \int \mathrm{d}^d y\, \frac{g}{2N}\, \psi_\alpha^+(x)\psi_\beta^+(y)U(x-y)\psi_\beta(y)\psi_\alpha(x),$$

$$\{\psi_\alpha(x), \psi_{\alpha'}^+(x')\} = \delta_{\alpha\alpha'}\delta^{(d)}(x-x'). \tag{4}$$

In this lecture we consider first $g = 0$. In the next lectures we will discuss the new features that appear for $g \neq 0$ when U is a short-ranged instantaneous interaction.

2.1.1. The $N = \infty$ limit

In this model it is useful to consider the two electron-electron scatterings (induced by impurity averaging), indicated in eq. (3), as conceptually different. This is shown in fig. 6. With these definitions the graphs of fig. 5 can be written in the way shown in fig. 7. Clearly an insertion with a

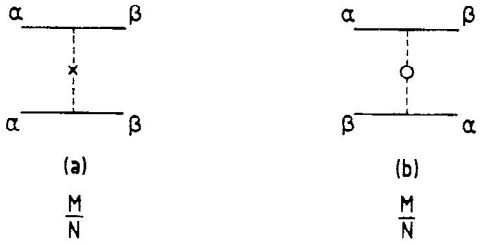

Fig. 6. (a) and (b) represent both processes indicated in eq. (3).

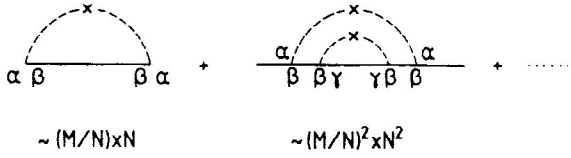

Fig. 7. The rainbow graphs again.

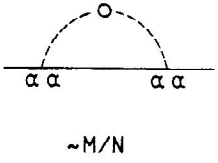

Fig. 8. The insertion of a circle.

circle is down by one power of N^{-1} as shown in fig. 8. Thus the CPA theory follows at $N = \infty$ and we can write for the averaged one-particle GF

$$G_\infty(x, y; E) = G_0(x, y; E) + M \int \mathrm{d}z\, G_0(x, z; E) G_\infty(z, z; E) G_\infty(z, y; E). \tag{5}$$

In momentum space we find

$$\Sigma(E) = M \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{G_0^{-1}(\boldsymbol{p}, E) - \Sigma(E)}, \tag{6}$$

where $G_\infty^{-1}(\boldsymbol{p}, E) = G_0^{-1}(\boldsymbol{p}, E) - \Sigma(E)$.

If we neglect the contribution to the real part (something we can do if we are far away from band edges) one immediately finds that eq. (6) implies

$$\Sigma = \pi M \rho_\infty(E_\mathrm{F}) = 1/2\tau \quad (E > 0), \tag{7}$$

where $\rho_\sigma(E_F)$ is the density of states of the system at the Fermi surface.

We can then write the one-particle averaged GF in the form

$$G_\sigma^{-1}(\boldsymbol{p}, \omega) = (\omega - E_0(\boldsymbol{p}) + \mu) + \mathrm{i}\frac{\operatorname{sgn}(\omega)}{2\tau}, \tag{8}$$

where τ is the life-time (mean free time) and it is equal to

$$\tau^{-1} = 2\pi\rho_\sigma(E_F)M.$$

μ is the chemical potential and

$$E_0(\boldsymbol{p}) = p^2/2m \quad (m = 1).$$

The presence of a finite mean free time τ implies that the GF decays exponentially with the separation [12]. Thus the electron loses memory of the phase of its wave function for distances bigger than $v_F\tau = l$ (mean free path).

Are there any localization effects in this theory? We can check that this is *not* the case by considering the two-particle Green's function $K_{\alpha\beta\beta\alpha}(xx; yy, E+\mathrm{i}\varepsilon, E+\omega-\mathrm{i}\varepsilon)$. The interest of this two-particle Green's function stems from the fact that it gives the conductivity, through the Kubo formula (see Abrahams' lectures in ref. 8). Another motivation for the study of K is that being the square of a GF (when $g = 0$!) the phase averaging effects are expected to drop out. If the system is a conductor we do not expect an exponential decay in K. Note that the $\mathrm{i}\varepsilon$ convention implies that $E < 0$ and $E+\omega > 0$.

If we denote the connected part of K by $\hat{K}$, one finds that the conductivity σ is related to $\hat{K}$ through [10]

$$\sigma(\omega, E) = \frac{e^2}{4\pi^2 V}\sum_{\alpha,\beta}\frac{\hat{c}}{\hat{c}q^2}\,\hat{K}_{\alpha\beta\beta\alpha}(q, E, E+\omega). \tag{9}$$

At $N = \infty$ only ladders of impurity averages contribute to $K_{\alpha\beta\beta\alpha}$ (fig. 9). Summing up all these graphs one finds

$$K^{+-}_{\alpha\beta\beta\alpha}(q, E+\omega, E) = \frac{M}{N\tau}\,\frac{1}{Dq^2 - \mathrm{i}|\omega|}, \quad (E+\omega > 0,\ E < 0) \tag{10}$$

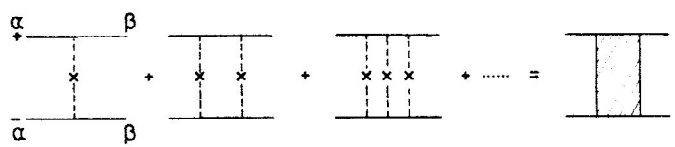

Fig. 9. The diagrams for the two-particle Green's function at $N = \infty$.

where D is the diffusion constant at $N = \infty$

$$D = v_F^2 \tau \, d. \tag{11}$$

Formula (10), together with the Einstein equation

$$\sigma = \frac{e^2}{\hbar} N(E) D, \tag{12}$$

implies a finite conductivity. We then conclude that at least away from band edges all states are extended.

It is instructive to repeat the same calculation for $E > 0$ and $E + \omega > 0$ (the same result applies when both are negative). In this case the result is quite different

$$K^{--}_{\alpha\beta\beta\alpha}(q, E+\omega, E) = \frac{M}{N} + \text{finite terms.} \quad (E > 0, E+\omega > 0). \tag{13}$$

The finite terms are set to zero in our approximation. The graphs that contribute to K^{--} are the same ones that contribute to K^{-+}. The energy range is different.

Eq. (10) suggests the presence of a Goldstone mode in the theory (Wegner, in ref. 8). In fact using eqs. (7) and (10) one can write (at $q = 0$)

$$K^{--}_{\alpha\beta\beta\alpha}(0, E+\omega, E) \sim \text{constant} \, \frac{\rho_\alpha(E_F)}{-i|\omega|} \tag{14}$$

and K^{--} blows up as $|\omega| \to 0$ so far as $\rho_\alpha(E_F)$ is finite. Wegner went further and tried to draw a parallel with the behavior of ferromagnetic systems with a continuous symmetry group in its broken phase. In his picture K^{--} behaves like the transverse susceptibility, $\rho_\alpha(E_F)$ as the order parameter and $-i|\omega|$ is the symmetry breaking field. Using a replica method he found that the Goldstone-like behavior is a consequence of the Ward identity [8, 10]

$$\sum_{r',\beta} K_{\alpha\beta\beta\alpha}(r, r', Z | r', r, Z') = \frac{G_{\alpha\alpha}(r, r, Z') - G_{\alpha\alpha}(r, r, Z)}{Z - Z'}, \tag{15}$$

where $\operatorname{Im} Z > 0$ and $\operatorname{Im} Z' < 0$.

To go beyond these results one has to consider the next order in N^{-1}. This task was done by Oppermann and Wegner. The results however seem to have a wider applicability than that. Many authors have considered a loop expansion around a weakly disordered metal and

found results which are consistent with the N^{-1} expansion if the theory is renormalized. Why renormalization? It turns out that the theory without interactions contains logarithmic infrared divergences (to order N^{-1}) in 2 dimensions. These infrared divergent contributions are present only in the K^{+-} mode but not in the one-particle GF or in the K^{++} mode. These "simplifying facts" will turn out to be a property only of the non-interacting theory.

2.1.2. N^{-1} *corrections*

The infrared divergent contributions to $K^{+-}_{\alpha\beta\beta\alpha}$, to leading order in N^{-1}, are shown in fig. 10. We denote the sum shown in fig. 9 by a shaded box.

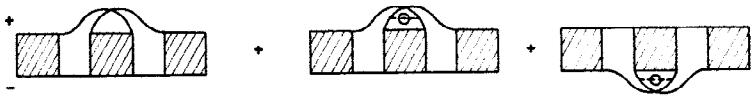

Fig. 10. N^{-1} corrections to the diffusive mode.

The result is (in two dimensions)

$$\delta K^{+-} = \frac{1}{N}\,\frac{M}{\tau(Dq^2 - i|\omega|)}\,\frac{1}{Dq^2 - i|\omega|}\left[-\frac{1}{2\pi}\,\frac{1}{2\pi\rho_\infty D}\,Dq^2 \ln\frac{|\omega|}{D\Lambda^2}\right], \tag{16}$$

where Λ is a cutoff in momentum transfer. This result seems to indicate that if one renormalizes the diffusion constant D (or rather $1/2\pi\rho D$) the theory can be made finite. A little dimensional analysis reveals that the variable $x = 1/2\pi\rho D$ is dimensionless in $d = 2$ and scales like

$$[x] = L^{d-2}. \tag{17}$$

(In a d-dimensional Heisenberg (σ) model T scales like L^{d-2}.) It is precisely this scaling behavior, first observed by Thouless [4], who led Abrahams et al. [6] to the conclusion that there is no dc conductivity in two dimensions. Using the replica method Wegner conjectured that localization near two dimensions is in the same universality class as a generalized matrix non-linear sigma model. At the time the nature of the symmetry was unclear. Parisi [13] suggested that the symmetry group was

$$\frac{O(n_+, n_-)}{O(n_+) \times O(n_-)}$$

in the limit $(n_-, n_-) \to 0$. This result has been later rederived by a number of authors [9, 14, 15]. For our purposes I will follow the work of Hikami [15].

Consider the interaction of 4 diffusive modes shown in fig. 11. The result is the ("one-particle irreducible") vertex

$$\Gamma = -4\pi\rho_x D\tau^+ N[2(q_1 \cdot q_4 + q_2 \cdot q_3) - (q_3 + q_2) \cdot (q_1 + q_4)]. \tag{18}$$

where I have set all frequency differences to zero. Note the absence of momentum independent vertices. There is also a six-point vertex shown in the fig. 12. Note also that diagrams of the type shown in fig. 13 are zero.

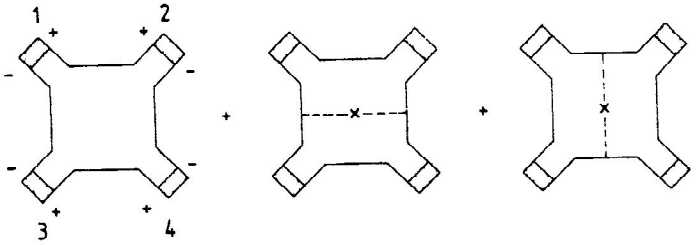

Fig. 11. The interaction vertex of four diffusive modes.

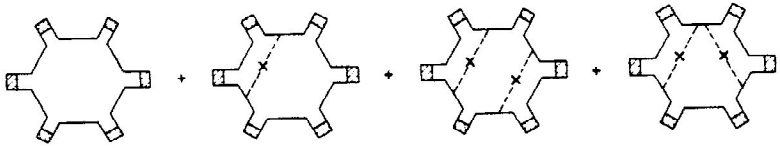

Fig. 12. The six-point vertex function.

Fig. 13. A diagram which is absent.

2.2. *The non-linear sigma-model*

One can reproduce all these diagrammatic rules by introducing a field $\phi(x)$ which takes values on a symmetric space (the coset space G/H

between a Lie group and its maximum compact subgroup H). The Hamiltonian is

$$H = \tfrac{1}{4}\int \mathrm{d}^d x \,\mathrm{tr}\,\hat{c}_\mu\phi\,\hat{c}_\mu\phi. \tag{19}$$

For our case the space is

$$\frac{\mathrm{O}(n_-, n_-)}{\mathrm{O}(n_-)\times\mathrm{O}(n_-)}.$$

The field ϕ has the form

$$\phi = \begin{bmatrix} \phi_{++} & \phi_{--} \\ \phi_{--} & \phi_{--} \end{bmatrix}. \tag{20}$$

clearly ϕ_{+-} (ϕ_{-+}) corresponds to the diffusive mode K_{++} and ϕ_{++} (ϕ_{--}) goes into K_{+-} (K_{--}). The matrix elements ϕ_{+-} (ϕ_{--}) are proportional to $\mathrm{i}\rho(E_\mathrm{F})$ ($-\mathrm{i}\rho(E_\mathrm{F})$). Thus ϕ has the property

$$\phi \sim \rho(E_\mathrm{F})O^\mathrm{T}(x)\begin{bmatrix} \mathrm{i}\mathbf{1}_{n_-} & \\ & -\mathrm{i}\mathbf{1}_{n_-} \end{bmatrix} O(x) \tag{21}$$

and $\phi^2(x) = -\text{constant}\times(\rho(E_\mathrm{F}))^2$.

If the density of states is constant one has a matrix model with a nonlinear hyperbolic constraint. This hyperbolic constraint indicates a noncompact symmetry

$$\frac{\mathrm{O}(n_+, n_-)}{\mathrm{O}(n_+)\times\mathrm{O}(n_-)}$$

rather than its compact version

$$\frac{\mathrm{O}(n_+ + n_-)}{\mathrm{O}(n_+)\times\mathrm{O}(n_-)}.$$

If the ϕ fields are parametrized as

$$\phi(x) = \begin{bmatrix} \mathrm{i}(\frac{1}{4}x + QQ^\mathrm{T})^{1/2} & Q \\ Q^\mathrm{T} & -\mathrm{i}(\frac{1}{4}x + Q^\mathrm{T}Q)^{1/2} \end{bmatrix}. \tag{22}$$

In terms of the Q's and expanding up to second order it is found that [9]

$$H = \int \mathrm{d}^d x \,[\tfrac{1}{2}\hat{c}_\mu Q\hat{c}_\mu Q^\mathrm{T} - \tfrac{1}{4}xQ_{\alpha a}Q_{\beta b}\,\hat{c}_\mu Q_{\alpha b}\,\hat{c}_\mu Q_{\beta a}$$
$$-\tfrac{1}{8}xQ_{\alpha a}Q_{\alpha b}\,\hat{c}_\mu Q_{\beta a}\,\hat{c}_\mu Q_{\beta b} - \tfrac{1}{8}xQ_{\alpha a}Q_{\beta a}\,\hat{c}_\mu Q_{\alpha b}\,\hat{c}_\mu Q_{\beta b} + \mathrm{O}(x^2)]. \tag{23}$$

In the limit $(n_+, n_-) \to 0$ the perturbative rules are satisfied. A renormalization group analysis to the order of two loops ($1/N^2$ in our terms) shows that, in the limit $n \to 0$ the wave function renormalization of ϕ is trivial ($Z = 1$) and that only a renormalization of the coupling constant is necessary. The β-function of x is, in $2+\varepsilon$ dimensions,

$$\beta_x = -\frac{\Lambda \hat{c} x}{\hat{c} \Lambda} = -\varepsilon x + \frac{x^2}{\pi N} + 0 + O\left(\frac{1}{N^3}\right), \tag{24}$$

where $x = x_R Z_D \Lambda^\varepsilon$, $\varepsilon = d-2$,

$$Z_D = 1 + \frac{1}{2\pi N} \frac{1}{2\pi\rho D} \ln \frac{|\omega|}{D\Lambda} .$$

(I hold x_R fixed, Λ is the momentum transfer cut off; x dimensionless.) Eq. (24) reveals that at $d = 2$ ($\varepsilon = 0$) there is only an infrared unstable fixed point at $x = 0$ ($D = \infty$). No matter how low the level of the disorder is, the system always iterates to the strongly disordered limit where all states are localized. This result was first obtained by Abrahams et al. [6]. All states are localized provided that no other fixed points exist. This seems to be the case though there is still some controversy concerning numerical results.

2.3. *Scaling*

In $2+\varepsilon$ there is an infrared stable fixed point at $x = 0$. The infrared unstable fixed point is sitting at $x^* = \pi N \varepsilon$. This is the critical fixed point. The flows are shown in fig. 14.

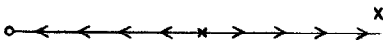

Fig. 14. The infrared flows of the non-interacting theory.

For $x < x^*$ the infrared flows iterate towards $x = 0$. The system is a conductor. For $x > x^*$ the flows iterate to large x where the system is an insulator. All states are localized.

At x^* there is a transition and scale invariance. The localization length diverges. As the fixed point is approached from the localized side, the localization length diverges like $\xi \sim |x - x^*|^{-\nu}$. The exponent ν is found to be equal to $1/\varepsilon$. These results were first proposed by Wegner on the basis of scaling arguments [5]. On the other hand we can integrate

the RG equation

$$-\frac{\partial x}{\partial \ln \Lambda} \equiv \frac{\partial x}{\partial \ln L} = \beta(x) \tag{25}$$

in the metallic phase $x < x^*$. The result is

$$L = L_0 \left\{ \int_{x_0}^{x} \frac{\mathrm{d}x'}{\beta(x')} \right\} \tag{26}$$

with $x = x(L)$ and $x_0 = x(L_0)$.

Hence

$$L = L_0 \left(\frac{x}{x_0} \right)^{-1/\varepsilon} \left| \frac{\pi N \varepsilon - x_0}{\pi N \varepsilon} \right|^{-1/\varepsilon} \tag{27}$$

for $x \ll \pi N \varepsilon = x^*$ and $x_0 \simeq x^*$. Therefore

$$x(L) \simeq \pi N \varepsilon \left(\frac{L_0}{L} \right)^{\varepsilon} \left| \frac{\pi N \varepsilon - x_0}{\pi N \varepsilon} \right|^{-1}. \tag{28}$$

Recall the definition of $x = 1/2\pi\rho D$

$$\rightarrow \rho D \simeq \text{constant} \times L^{\varepsilon} \left| \frac{\pi N \varepsilon - x_0}{\pi N \varepsilon} \right| = \sigma L^{\varepsilon}, \tag{29}$$

$$\sigma = A \left| \frac{\pi N \varepsilon - x_0}{\pi N \varepsilon} \right| \simeq A' \left| \frac{E_F - E_c}{E_c} \right|, \tag{30}$$

if we assume that the bare density of states is smooth.

Thus the conductivity vanishes linearly at the edge ($O(\varepsilon)$). One further aspect of the non-interacting theory is the triviality of the wave-function renormalization. From the point of view of the σ-model analogy we know that the wave function renormalization originates from terms that contain internal loops of the ϕ field. These terms are suppressed in the $(n_+, n_-) \rightarrow 0$ limit. Thus Z_ϕ must be one. We are going to see that this result does not apply to the interacting theory. By the way, the triviality of Z_ϕ implies that the exponent $\eta = 2-d$. This result is consistent with the fact that the "order parameter" of this theory $\rho(E_F)$ is smooth as E_F crosses E_c. The exponent β equals zero and the result $\eta = 2-d$ follows through.

3. Interacting theory

Many of the results explained before are modified once interactions enter into the picture. I will follow my general strategy and study, first, the $N = \infty$ limit of the full theory.

I argued in the previous lecture that the $N = \infty$ limit, even though it is in a sense a trivial theory, is a good starting point for a perturbative expansion. This feature is also present in the interacting case.

Let me first make a few comments on the nature of the interaction. I assume the electrons to be spinless, or rather that spin play no role in the problem. This is clearly an oversimplification. There are many cases in which the role of the spin is crucial (e.g. the Hubbard model, etc.). I also assume the interparticle potential to be short-ranged. More precisely

$$U(\boldsymbol{x}-\boldsymbol{y}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\exp\left[\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})\right]}{\boldsymbol{p}^2+\lambda_0^2} > 0. \tag{31}$$

where λ_0^{-1} is the range of the interaction. This is also not true in real systems in which the interactions are the repulsive Coulomb forces. There are a number of technical difficulties that appear in the presence of long-range forces. I will show that even this model system has a structure which is already complicated. The $N = \infty$ limit that I am about to describe reproduces the standard Hartree approximation for the one-particle Green's function.

3.1. The $N = \infty$ limit

How does the Dyson equation, discussed in the previous lecture, get modified if $g \neq 0$? In principle many more diagrams appear. Some of them are shown in fig. 15.

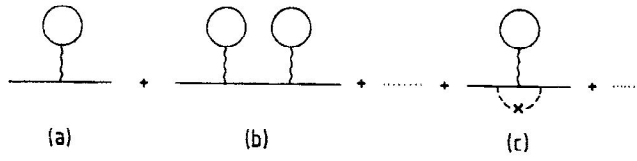

Fig. 15. Interaction corrections to the electron self-energy.

In these graphs the lines represent $N = \infty$ propagators with $g = 0$. The graphs (15.a) and (15.b) are typical tadpole insertions. Their sum produces the standard Hartree approximation. In this system they produce a correction to the self-energy which is real, momentum independent and diagonal in the orbital indices. Thus their effect can be compensated by a further shift in the chemical potential. In other models in which a metal–insulator transition is produced by spin dependent forces (like the Hubbard model) these terms play a very important role. Finally we have to analyze diagram (15c). This diagram has a purely real momentum independent contribution which is set to zero by our procedure of computing integrals. Its contribution to Σ is

$$\delta\Sigma = \mathrm{i}MgU(0)n \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, G_\gamma^+(p-q,\omega) G_\gamma^-(p-q,\omega) \tag{32}$$

$$\approx \mathrm{i}MgU(0)n\rho(E_\mathrm{F}) \int_{-\infty}^{+\infty} \mathrm{d}\varepsilon \left[G^-(\varepsilon,\omega)\right]^2 \equiv 0. \tag{33}$$

where n is the particle density. $\delta\Sigma$ vanishes since both poles are in the upper half complex energy plane. Therefore for energies near E_F the average one-particle Green's function is the same as in the non-interacting case (at $N = \infty$!).

We now have to compute the diffusive modes. It is not hard to see that any interaction insertion is down by one power in N^{-1} (fig. 16). Thus the K^{+-} and K^{++} modes remain the same as for $g = 0$. The Goldstone behavior of K^{+-} is not modified.

Let us now consider the vertex shown in fig. 17. It obeys the "integral equation"

$$\Gamma^{zz}(q,\omega) = 1 + M\left[\int_k G_\gamma(\boldsymbol{k}+\boldsymbol{q},\omega+E) G_\gamma(\boldsymbol{k},E)\right]\Gamma^{zz}. \tag{34}$$

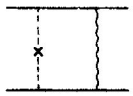

Fig. 16. An interaction correction.

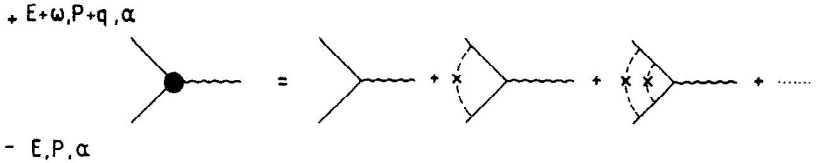

Fig. 17. The $(+-)$ vertex at $N = \infty$.

The solution is

$$\Gamma^{zz}_{\pm\pm} = 1. \tag{35}$$

$$\Gamma^{zz}_{\pm\mp} = \frac{1}{\tau(Dq^2 - i|\omega|)}.$$

We can see that the Goldstone behavior of K^{--} produces diffusive behavior in Γ^{--} itself. This result is going to play a very important role.

Finally let us consider the effective interaction at $N = \infty$. In this limit the only diagrams that survive are those with an arbitrary number of electron loops (bubbles). Note by the way that one of the vertices inside the bubble must be the full $N = \infty$ vertex. See fig. 18.

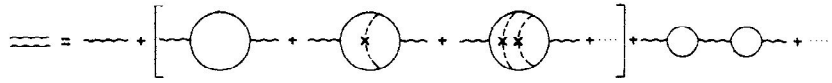

Fig. 18. The effective interaction at $N = \infty$.

The effective potential $\mathscr{U}_\infty(\boldsymbol{p}, \omega)$ at $N = \infty$ is then seen to satisfy the Dyson equation

$$\mathscr{U}^{\infty}(\boldsymbol{p}, \omega) = U(\boldsymbol{p}, \omega) + gU(\boldsymbol{p}, \omega)\Pi_\infty(\boldsymbol{p}, \omega)\mathscr{U}_\infty(\boldsymbol{p}, \omega) \tag{36}$$

with the polarization bubble $\Pi_\infty(\boldsymbol{p}, \omega)$ equal to

$$\Pi_\infty(\boldsymbol{p}, \omega) = -i \int \frac{d\Omega}{2\pi} \int \frac{d^d q}{(2\pi)^d} G_\infty(\boldsymbol{p}+\boldsymbol{q}, \omega+\Omega) G_\infty(\boldsymbol{q}, \Omega) \times \Gamma_\infty(\boldsymbol{q}, \omega+\Omega, \Omega). \tag{37}$$

The result for Π_∞ is

$$\Pi_\infty(\boldsymbol{p}, \omega) = -\rho_\infty(E_F) \frac{D\boldsymbol{p}^2}{D\boldsymbol{p}^2 - i|\omega|}. \tag{38}$$

If we combine Π_∞ with $\mathscr{U}_\infty(\boldsymbol{p}, \omega)$

$$\mathscr{U}_\infty^{-1}(\boldsymbol{p}, \omega) = U^{-1}(\boldsymbol{p}, \omega) - \Pi_\infty(\boldsymbol{p}, \omega) \tag{39}$$

we find

$$\mathscr{U}_\infty(\boldsymbol{p}, \omega) = \frac{1}{\lambda^2} \frac{D\boldsymbol{p}^2 - i|\omega|}{D\boldsymbol{p}^2 - i|\omega|\lambda_0^2/\lambda^2}. \tag{40}$$

in the limit $(\boldsymbol{p}, \omega)$ small. Here $\lambda^2 = \lambda_0^2 + g\rho_\infty(E_F)$.

Eq. (40) shows a remarkable change in the form of the potential. In the static limit ($\omega \to 0$, $\boldsymbol{p}$ finite) we get $\mathscr{W}_\infty(\boldsymbol{p}, 0) \sim 1/\lambda^2$, that is static screening. In the opposite limit ($\boldsymbol{p} \to 0$, ω finite) (dynamic) $\mathscr{W}_\infty \sim 1/\lambda_0^2$ and the bare interaction is seen. The corrections to these limits will play a role in the theory.

This completes the discussion of the properties of the model at $N = \infty$. We see that we get a system with the expected properties of a weakly disordered metal (finite lifetime τ, screening, diffusion etc.). Thus the interactions of the type we have consider do not alter the metallic properties of the non-interacting system so far as $N = \infty$. We are going to find rather dramatic effects in the N^{-1} corrections.

The rules that define the N^{-1} expansion are now very simple. All lines represent G_∞ propagators. Wavy lines represent the full $N = \infty$ effective interaction $\mathscr{W}_\infty$. The vertices must be dressed according to the rules already explained: the $(+-)$ vertex (which couples a diffusive mode to an interaction line) must be dressed with a ladder of impurity lines with a cross in each while the $++$ vertex (and the $--$) is not dressed at $N = \infty$. We then combine all these ingredients taking care of the N dependence of the graph.

3.2. N^{-1} *corrections*

3.2.1. N^{-1} *corrections to the electron propagator*

To make a long story short I will only quote the contributions which give rise to infrared divergencies in two dimensions (fig. 19).

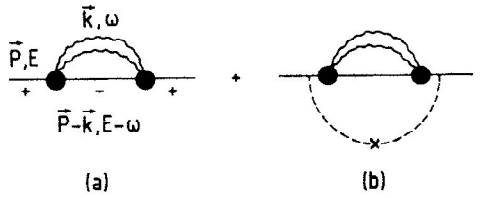

Fig. 19. The N^{-1} corrections to the electron self-energy. The double wavy lines represent the effective interaction at $N = \infty$.

There are other contributions to the self-energy but they are infrared finite. Note that the internal integration range has been restricted to $\varepsilon < \omega < E_c$. The other range ($\omega < \varepsilon$) gives only a finite contribution.

The contribution of graph (18.a) is equal to

$$\delta\Sigma_a = \frac{ig}{\lambda^2 N}\int \frac{d^2k}{(2\pi)^2}\int_{\epsilon}^{E_c}\frac{d\omega}{2\pi}\left[\frac{Dk^2-i|\omega|}{Dk^2-i|\omega|\lambda_0^2/\lambda^2}\right] \times\left(\frac{1}{\tau}\frac{1}{Dk^2-i|\omega|}\right)^2 G_-(\boldsymbol{p}-\boldsymbol{k}, E-\omega). \tag{41}$$

This graph was first considered by Altshuler et al. [16]. In eq. (41) I have introduced an ultraviolet cutoff in the energy integral (E_c). By dimensional analysis we can set $E_c = D\lambda^2$.

An evaluation of the integrals, in the limit $|\boldsymbol{p}| \approx p_F$ and $E \to 0$, yields the result

$$\delta\Sigma_a \approx \frac{g}{N\lambda^2\tau^2}\, G_-(\boldsymbol{p}, E)\frac{1}{8\pi^2 D}\frac{\ln\lambda^2/\lambda_0^2}{(\lambda_0^2/\lambda^2-1)}\left(-\ln\frac{E}{E_c}\right) + \frac{ig}{N\lambda^2\tau^2}\frac{E_c}{8\pi^2 D}\frac{G^2(\boldsymbol{p}, E)}{2mD}\left\{1+\frac{p^2}{m}G_-(\boldsymbol{p}, E)\left[1+\frac{E}{E_c}\left(\ln\frac{E}{E_c}-1\right)\right]\right\} + \text{finite terms}. \tag{42}$$

The contribution of graph (19.b) is equal to

$$\delta\Sigma_b = M\int\frac{d^2q}{(2\pi)^2}\, G_-^2(\boldsymbol{p}-\boldsymbol{q}, E)\delta\Sigma_a(\boldsymbol{p}-\boldsymbol{q}, E). \tag{43}$$

Evaluating the integrals explicitly one finds

$$\delta\Sigma_b = -\frac{i}{8\pi^2 N\rho D\tau}\ln\frac{\lambda^2}{\lambda_0^2}\ln\frac{E}{E_c}. \tag{44}$$

Collecting the results from $\delta\Sigma_a$ and $\delta\Sigma_b$ we find that, to order N^{-1}, $G^{-1}(\boldsymbol{p}, E)$ is equal to

$$G^{-1}(\boldsymbol{p}, E) = E\left[1-\frac{1}{\pi N}\left(\frac{1}{2\pi\rho D}\right)\left(\frac{\rho g_{--}}{\lambda_0^2}\frac{\ln\lambda^2/\lambda_0^2}{\lambda^2/\lambda_0^2-1}+\frac{5}{4}\frac{\rho g_{+-}}{\lambda^2}\right)\ln\frac{E}{E_c}\right] - (E_0(\boldsymbol{p})-\mu)\left[1-\frac{1}{\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln\lambda^2/\lambda_0^2}{(\lambda^2/\lambda_0^2-1)}\ln\frac{E}{E_c}\right] + \frac{i}{2\tau}\left[1-\frac{1}{2\pi N}\frac{1}{2\rho\pi D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln\lambda^2/\lambda_0^2}{\lambda^2/\lambda_0^2-1}\ln\frac{E}{E_c}\right]. \tag{45}$$

In eq. (45) I have set the coupling constant g to be equal to g_{+-} (equal to $g_{++} = g_{--}$ at $N = \infty$) anticipating the result that g_{+-} and g_{--} renormalize differently. I will come back to this point later.

It is apparent from eq. (45) that G^{-1} needs a number of renormalizations. Obviously a wave-function renormalization of the Fermi field ψ is necessary since

$$\partial G^{-1}/\partial E|_{E_F} = 1$$

(in fact it diverges logarithmically in this model with short-ranged interactions). It is also necessary to do a renormalization of both the Fermi velocity and of the life-time τ. This is true because we have kept diagram (19.b). In an expansion in powers of $1/k_F l$ this diagram would not contribute, even though it has the same degree of infrared divergence as (19.a). It is an important feature of the N^{-1} expansion that *all* diagrams with the *same order of divergence* are also of the *same order in* N^{-1}.

3.2.2. N^{-1} *corrections for vertices and the diffusive mode*

Before we can renormalize this theory I will present the divergent contributions to the other vertices and propagators.

It is outside the purpose of these lectures to give a description too detailed of the calculation. Instead I will give the highlights of it. In addition to the contributions already discussed, for the two-point functions, there are non-trivial contributions to essentially all quantities.

In this way the contributions that one finds for the diffusive modes K_{+-} imply that not only it is necessary to renormalize the diffusion constant (as we did in the non-interacting case) but, in addition, the diffusive modes acquire an extra wave function renormalization. Typical infrared divergent graphs are shown in fig. 20.

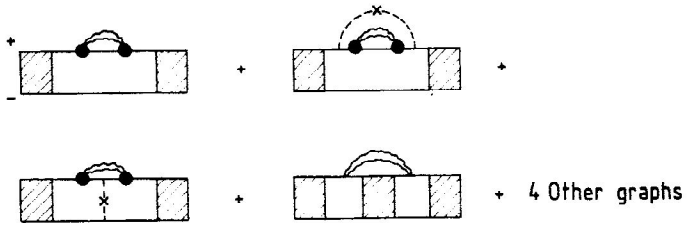

Fig. 20. Interaction dependent N^{-1} correction to the diffusive mode.

The vertex functions, shown in fig. 17, also acquire non-trivial contributions. Firstly let me note that Γ^{++} and Γ^{+-} renormalize differently. Hence the need for two separate coupling constants g_{++} and g_{+-}. The vertex function Γ^{+-} has logarithmically divergent

contributions which can be absorbed in a renormalization of g_{--}. The vertex Γ^{--} (and Γ^{++}) also has a logarithmically divergent contribution, which can be cancelled by a suitable renormalization of g_{--}. However, Γ^{+-} has a new term, to order N^{-1}, which is cut-off independent and diverges like Ω^{-1} as $\Omega \to 0$. Thus Γ^{+-} acquires a "diffusive component" to order N^{-1}. The origin of this term is consistent with the Ward identity

$$\Gamma \sim \delta G^{-1}\, \delta\Omega. \tag{46}$$

Since G^{-1} has terms that diverge like $\ln(|\Omega|\, D\Lambda^2)$ (see eq. (45)) eq. (46) implies the presence of $1/\Omega$ terms.

Finally the effective interaction $\mathscr{U}$ has divergent contributions. This contribution can be cancelled, in part, by the renormalization of the diffusion constant. However, this is not the end of the story. In the dynamic regime ($Dq^2 \ll \omega$) $\mathscr{U}$ has the form

$$\mathscr{U}_x \approx \frac{1}{\lambda_0^2}\left[1 - \frac{Dq^2}{i\omega}\, v + \ldots\right], \tag{47}$$

where

$$v = \frac{\lambda^2}{\lambda_0^2} - 1 \quad \text{at } N = \infty.$$

The additional divergences can be now cancelled provided one renormalizes v.

There is one point that needs some discussion at this point. The polarization operator Π, at $N = \infty$, has the form

$$\Pi_x = -\rho_x(E_F)\frac{Dq^2}{-i\Omega + Dq^2}. \tag{38}$$

One might wonder why, instead of renormalizing v one does not renormalize Π_x multiplicatively by rescaling $\rho_x(E_F)$. In fact such procedure has been proposed by McMillan [17]. This procedure, however intuitive, is not correct. One can show, using particle conservation, that the ρ_x in Π is equal to $\partial n/\partial\mu$, where n is the physical particle density and μ the chemical potential, rather that the physical one particle density of states. Therefore the coefficient in front of Π remains equal to ρ_x and does not acquire renormalization. The same observation applies for the ρ_x which enters multiplying D.

3.3. Renormalization

We begin our discussion of renormalization by doing some dimensional analysis. In my first lecture I argued that the parameter x which equals $1/2\pi\rho_x D$, scales naïvely like L^{d-2}, where L is a length. Of course this is still true in the interacting theory. I will argue not that there are two parameters $y = \rho g_{--} \lambda_0^2$ and $z = \rho g_{-+} \lambda_0^2$ which are dimensionless in *all* dimensions. In this system the field ϕ has dimension of $L^{-d/2}$, the energy is T^{-1}, the mass m is TL^{-2} and both g's have dimension $L^{d-2}T^{-1}$, where T has dimensions of time. Since the density of states ρ has dimension $L^{-d}T$ it follows immediately that $[x] = L^{d-2}$, $[y] = [z] = 1$.

We want now to renormalize this expansion while keeping the physics of $N = \infty$ invariant. This procedure is equivalent to require the metallic phase to be stable.

I will adopt the following set of renormalization conditions

$$\operatorname{Im} G_{\mathrm{R}}^{-1}|_{E=0,|\boldsymbol{P}|=P_{\mathrm{F}}} = \frac{1}{2\tau_{\mathrm{R}}}. \tag{48a}$$

$$\frac{\hat{c}}{\hat{c}E} \operatorname{Re} G_{\mathrm{R}}^{-1}|_{E=0,|\boldsymbol{P}|=P_{\mathrm{F}}} = 1, \tag{48b}$$

$$-\frac{\hat{c}}{\hat{c}|\boldsymbol{P}|} \operatorname{Re} G_{\mathrm{R}}^{-1}|_{E=0,|\boldsymbol{P}|=P_{\mathrm{F}}} = v_{\mathrm{F}}^{\mathrm{R}}, \tag{48c}$$

where τ_{R} and $v_{\mathrm{F}}^{\mathrm{R}}$ are the (renormalized) mean-free time and Fermi velocity respectively.

I introduce now the renormalization constants Z_ψ, Z_τ and $Z_{v_{\mathrm{F}}}$.

$$G_{\mathrm{R}}^{-1} = Z_\psi G_{\mathrm{B}}^{-1} \tag{49a}$$

$$\tau_{\mathrm{R}} = \tau Z_\tau Z_\psi^{-1}. \tag{49b}$$

$$v_{\mathrm{F}}^{\mathrm{R}} = v_{\mathrm{F}} Z_{v_{\mathrm{F}}} Z_\psi. \tag{49c}$$

In the same manner the diffusive mode K^{--} can be renormalized.

$$\frac{M}{\tau_{\mathrm{R}}} \frac{\hat{c}}{\hat{c}\Omega} (K_{\mathrm{R}}^{--})^{-1} = -\mathrm{i}. \tag{50a}$$

$$\frac{M}{\tau_{\mathrm{R}}} \frac{\hat{c}}{\hat{c}q^2} (K_{\mathrm{R}}^{--})^{-1} = D_{\mathrm{R}}. \tag{50b}$$

where D_{R} is the renormalized diffusion constant. Two renormalization constants are necessary to make K^{--} finite, Z_Q and Z_D.

$$K_R^{--} = Z_c Z_\psi Z_Q^{-1} K_B^{--}, \tag{51a}$$

$$x_R = x Z_\psi^2 Z_Q^{-1} Z_D^{-1} \Lambda^{-\varepsilon}, \tag{51b}$$

where we have made $x = 1/2\pi\rho D$ dimensionless in all dimensions by pulling out its naïve dimension Λ^ε. Alternatively we could have scaled it by $E_c^{-\varepsilon/2}$.

The coupling constants $y = \rho g_{--} \Lambda_0^{\varepsilon/2}$ and $z = \rho g_{--} \Lambda_0^\varepsilon$ can also be multiplicatively renormalized

$$y_R = y Z_\psi^2 Z_y^{-1}, \tag{52a}$$

$$z_R = z(Z_\psi Z_c Z_Q^{-1})^2 Z_z^{-1}. \tag{52b}$$

Eq. (52b) shows explicitly that the renormalization of the diffusive mode affects that of the Γ^{--} vertex. This is entirely natural since g_{--} measures the strength of the coupling between the diffusive mode K^{--} and the interaction line $\mathscr{U}_\sigma$.

Finally the "screening parameter" v is renormalized

$$v_R = v Z_D^{-1} Z_v^{-1}. \tag{53}$$

All these renormalization constants are sufficient to render the theory finite to order $1/N$. To the next order in $1/N$, at least one more constant is necessary, to cancel the divergences of the coefficient of the $1/\Omega$ term in the Γ^{--} vertex. It remains an open question to prove that the theory is renormalizable to all orders in N^{-1}.

3.3.1. The renormalization group equations

I will not present the complete expressions of the renormalization constants and the beta functions since they are rather complicated. Instead I will present the beta-functions directly in the limit v small. In this limit the equations simplify considerably. This will turn out to be sufficient since v is found to be irrelevant at the critical fixed point. In this limit one finds the following beta-functions.

$$\beta_x = -E_c \frac{\partial x}{\partial E_c} = -\frac{\varepsilon x}{2} + \frac{x^2}{2\pi N} + \frac{x^2 y}{2\pi N} + \frac{x^2 z}{4\pi N}, \tag{54a}$$

$$\beta_y = -E_c \frac{\partial y}{\partial \ln E_c} = -\frac{xyz}{2\pi N} + \frac{3}{2\pi N} xy\sqrt{xz}, \tag{54b}$$

$$\beta_z = -E_c \frac{\partial z}{\partial E_c} = \frac{3}{\pi N} xyz + \frac{4xz^2}{\pi N} + \frac{2xz}{\pi N}\sqrt{yz}. \tag{54c}$$

$$\beta_v = -E_c \frac{\partial v}{\partial E_c} = \frac{vx}{2\pi N}\,[3y - z - 2\sqrt{yz}]. \tag{54d}$$

It is more convenient to make a change of variables. A look into the perturbative expressions shown in the last lecture suggests that $w = xy$ and $u = xz$ are the natural variables.

Define $s = -\ln(E_c/\Omega)$, where Ω is a frequency scale. The β-functions for x, u and w are

$$\frac{dx}{ds} = -\frac{\varepsilon x}{2} + \frac{x^2}{2\pi N} + \frac{xw}{2\pi N} + \frac{xu}{4\pi N}, \tag{55a}$$

$$\frac{du}{ds} = -\frac{\varepsilon u}{2} + \frac{xy}{2\pi N} + \frac{7uw}{2\pi N} + \frac{17u^2}{4\pi N} + \frac{2u}{\pi N}\sqrt{uw}, \tag{55b}$$

$$\frac{dw}{ds} = -\frac{\varepsilon w}{2} - \frac{uw}{4\pi N} + \frac{xw}{2\pi N} + \frac{w^2}{2\pi N} + \frac{3}{2\pi N}\,w\sqrt{uw}. \tag{55c}$$

These RG equations have to be integrated with the initial condition $u = w$ since at the unperturbed level $g_{+-} = g_{++} = g$.

As usual we first discuss the fixed point structure. We consider first $\varepsilon > 0$.

(i) Trivial FP: If $\varepsilon > 0$ ($d > 2$) then the origin is a trivial, infrared stable fixed point.

(ii) Anderson fixed point: This is the critical FP of the non-interacting theory. It turns out that there is a whole line of unstable fixed points in the plane $u = 0$ ranging from the Anderson fixed point ($x^* = \pi N\varepsilon$, $u = w = 0$) to the point $w^* = \pi N\varepsilon$, $x = u = 0$.

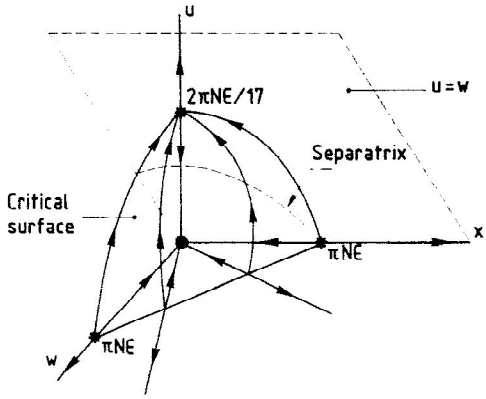

Fig. 21. The RG flows.

(iii) Non-trivial fixed point: $w = x = 0$, $u^* = 2\pi\mathcal{N}\varepsilon$ 17. This fixed point is critical. Has a relevant eigenvalue of $+\varepsilon$ 2 along the direction of the u axis and two irrelevant directions with eigenvalues -8ε 17 and -9ε 17.

The qualitative flows are shown in fig. 21.

From this analysis we can already conclude two things. Firstly that the Anderson fixed point is unstable. On the other hand there is a nontrivial fixed point at $u^* = 2\pi\mathcal{N}\varepsilon$ 17, $x = w = 0$ with relevant eigenvalue equal to ε 2. Note that, in terms of the coupling constant g_{--} this is an infinite fixed point, since it requires $x \to 0$ and $g_{--} \to \infty$ simultaneously. This is a consequence of the fact that the interaction is too simple. In more complex cases this fixed point is expected to be finite. The exponent ν of the localization length, as well as that of the conductivity, does not change since the eigenvalue at the non-trivial FP is the same one we found for the non interacting case (please note that I am scaling with energies now!)

3.3.2. *Density of states*

Let us discuss now the behavior of the density of states. The renormalized density of states $\mathcal{N}_R$ can be extracted from the Green's function

$$\mathcal{N}_R = \frac{1}{\pi}\,\mathrm{Im}\,G_R(\boldsymbol{x},\boldsymbol{x},E) = \frac{1}{\pi}\,\mathrm{Im}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,G_R(\boldsymbol{p},E)$$

$$= \frac{1}{\pi}\,\mathrm{Im}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{E - E_R(\boldsymbol{p}) + \mu_R + \mathrm{i}\,2\tau_R}\,. \tag{56}$$

Thus the renormalized density of states and the bare density of states are related by

$$\mathcal{N}_R = \frac{v_F}{v_F^R}\,Z_\psi^{-1}\,\mathcal{N}_B. \tag{57}$$

Using the expression

$$v_F^R = v_F Z_\psi Z_{v_F}. \tag{49c}$$

we find

$$\mathcal{N}_R = \mathcal{N}_B (Z_\psi^2 Z_{v_F})^{-1}. \tag{58}$$

If we define $Z_\mathcal{N}$ to be

$$Z_\mathcal{N} = (Z_\psi^2 Z_{v_F})^{-1} \tag{59}$$

and

$$\gamma_N = -E_c \frac{\hat{c}}{\hat{c}E_c} \ln Z_N,$$

then I can write the renormalization group equation

$$\left[E_c \frac{\hat{c}}{\hat{c}E_c} - \beta_u \frac{\hat{c}}{\hat{c}u} - \beta_w \frac{\hat{c}}{\hat{c}w} - \beta_x \frac{\hat{c}}{\hat{c}x} - \gamma_N \right] N_B = 0. \tag{60}$$

It is easy to solve (eq. (60)) at the Fixed Point

$$u^* = \frac{2\pi N\varepsilon}{17}, \quad w = x = 0.$$

Alternatively we may be interested in the frequency dependence of the density of states at the FP

$$\left[E_c \frac{\hat{c}}{\hat{c}E_c} - \gamma_N^* \right] N_B = 0 \tag{61}$$

$$\rightarrow N_B(E_c)E_c^{-\gamma_N^*} = N_B(\lambda E_c)(\lambda E_c)^{-\gamma_N^*}. \tag{62}$$

where

$$\gamma_N^* = -E_c \frac{\hat{c}}{\hat{c}E_c} \ln Z_n|_{FP} = -\frac{7\varepsilon}{17}. \tag{63}$$

We then want to find the frequency dependence. By dimensional analysis we know that

$$N_B(\Omega, \lambda, E_c) = N_B(\Omega, \lambda E_c). \tag{64}$$

Using eq. (62) we get

$$N_B(\Omega, \lambda, E_c) = \lambda^{\gamma_N^*} N_B(\Omega, E_c). \tag{65}$$

Choosing $\lambda = \Omega/\Omega^*$ we find

$$N_B(\Omega, E_c) = (\Omega/\Omega^*)^{-\gamma_N^*} N_B(\Omega^*, E_c). \tag{66}$$

We conclude that at the Fixed Point, and along the whole separatrix, the density of states vanishes as the Fermi surface is approached with an exponent $\theta = -\gamma_N^* = 7\varepsilon/17$

$$N_B(\Omega, E_c) \sim (\Omega/\Omega^*)^{7\varepsilon/17}. \tag{67}$$

A similar analysis yields the critical exponent for $N(E_F)$, the density of states at the Fermi surface, as the critical point is approached.

The renormalization group equation

$$\left[E_\epsilon \frac{\hat{c}}{\hat{c} E_\epsilon} - \beta_u \frac{\hat{c}}{\hat{c} u} - \gamma_N \right] N_B = 0 \tag{68}$$

can be solved along the relevant trajectory. The result is

$$N^B(\Delta) = \text{constant} \cdot \Delta^{1/\nu}, \tag{69}$$

where

$$\Delta = \left| \frac{u_0 - u^*}{u^*} \right|,$$

is the distance to the separatrix.

3.3.3. *The exponent* η

Finally I want to describe very briefly the issue of the anomalous dimension η of the diffusive mode K^{--}. Recall the renormalization of K^{--}

$$K_R^{--} = Z_\epsilon Z_v Z_Q^{-1} K_B^{--}. \tag{70}$$

We can define an anomalous dimension γ_K^*

$$\gamma_K^* = E_\epsilon \frac{\hat{c}}{\hat{c} E_\epsilon} \ln Z_K|_{FP}. \tag{71}$$

By a standard argument we find that the exponent η is

$$\eta = \gamma_K^* = \epsilon/17. \tag{72}$$

I want to end these lectures with a few comments. What I have described to you is a renormalization group treatment of the localization transition including interactions effects. This is not yet a finished theory. It is only a model system in which we can start building some intuition. There are many aspects of the theory which I have not even mentioned, like the Hall Effect. There are many open problems. A calculation to order N^{-2} must be carried out to check the consistency of the renormalization procedure that I have described. Also more realistic interactions have to be considered. I hope to have motivated some of you to take a closer look at some of these problems.

Note added in proof: The complete version of this work has been published recently [M. Ma and E. Fradkin, Phys. Rev. B28 (1983) 2990]. The

calculation of the renormalized density of states presented in these lectures is incorrect. The correct calculation is presented in the paper quoted above.

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