Solution 1

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1 Second quantization of an elastic solid

1.1

For the classical Lagrangian

$$L = \int d^3x \left\{ \sum_{i=1}^{3} \frac{\rho}{2} \dot{u}_i^2(x,t) - \frac{K}{2} \sum_{i,j=1}^{3} \partial_i u_j(x,t) \partial_i u_j(x,t) - \frac{\Gamma}{2} (\vec{\nabla} \cdot \vec{u}(x,t))^2 \right\} \tag{1}$$

The canonical momentum is

$$\Pi_i(x,t) = \frac{\delta L}{\delta \dot{u}_i} = \rho \dot{u}_i(x,t) \tag{2}$$

Hence the classical Hamiltonian is

$$H = \int d^3x \sum_{i=1}^{3} \Pi_i \dot{u}_i - L = \int d^3x \sum_{i=1}^{3} \frac{\Pi_i^2}{2\rho} + \frac{K}{2} \sum_{i,j=1}^{3} \partial_i u_j(x,t) \partial_i u_j(x,t) + \frac{\Gamma}{2} (\vec{\nabla} \cdot \vec{u}(x,t))^2 \tag{3}$$

1.2

For the quantum theory, it obeys the equal-time commutation relation

$$[u_i(x), \Pi_j(x')] = i\delta(\vec{x} - \vec{x}')\delta_{ij} \tag{4}$$

Since $u_i(x)$ and $\Pi_i(x)$ are real, they satisfy

$$u_i(x) = u_i^\dagger(x), \quad \Pi_i(x) = \Pi_i^\dagger(x) \tag{5}$$

After performing Fourier transformation, we can quantize the theory in momentum space. Define

$$\begin{aligned} u_i(p) &= \sqrt{\rho} \int d^3x e^{-i\vec{x}\cdot\vec{p}} u_i(x) \\ \Pi_i(p) &= \frac{1}{\sqrt{\rho}} \int d^3x e^{-i\vec{x}\cdot\vec{p}} \Pi_i(x) \end{aligned} \tag{6}$$

We have

$$[u_i(p), \Pi_j(q)] = i(2\pi)^3 \delta^3(\vec{p}+\vec{q})\delta_{ij} \tag{7}$$

where we used

$$(2\pi)^3 \delta^3(\vec{p}+\vec{q}) = \int d^3x e^{-i\vec{x}\cdot(\vec{p}+\vec{q})} \tag{8}$$

The Hamiltonian in the momentum space is

$$H = \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2}\Pi_i(-p)\Pi_i(p) + \frac{1}{2}\omega_{ij}^2(p) u_i(-p) u_j(p) \right] \tag{9}$$

where

$$\omega_{ij}^2(p) = \frac{K}{\rho} p^2 \delta_{ij} + \frac{\Gamma}{\rho} p_i p_j \tag{10}$$

with $p^2 = p_1^2 + p_2^2 + p_3^2$.

This matrix has two eigenvalues

$$\omega_L^2(p) = \left(\frac{K+\Gamma}{\rho} \right) p^2, \quad \omega_T^2(p) = \left(\frac{K}{\rho} \right) p^2 \tag{11}$$

For $\omega_L^2(p)$, it corresponds to the longitudinal mode, with eigenvector parallel to the $\vec{p}$. For $\omega_T^2(p)$, it corresponds to the transverse mode. In three dimensional space, there are two transverse modes perpendicular to the longitudinal mode.

The Hamiltonian can be split into two terms $H = H_L + H_T$,

$$\begin{aligned} H_L &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \Pi_L(-p)\Pi_L(p) + \omega_L^2(p) u_L(-p) u_L(p) \\ H_T &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \sum_{a=1,2} \Pi_T^a(-p)\Pi_T^a(p) + \omega_T^2(p) u_T^a(-p) u_T^a(p) \end{aligned} \tag{12}$$

1.3

The creation and annihilation operators for the normal modes are

$$
\begin{aligned}
a_L(p) &= \frac{1}{\sqrt{2}}\left(\sqrt{\omega_L(p)}u_L(p) + \frac{i}{\sqrt{\omega_L(p)}}\Pi_L(p)\right) \\
a_L(p)^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{\omega_L(p)}u_L(-p) - \frac{i}{\sqrt{\omega_L(p)}}\Pi_L(-p)\right) \\
a_T^a(p) &= \frac{1}{\sqrt{2}}\left(\sqrt{\omega_T(p)}u_T^a(p) + \frac{i}{\sqrt{\omega_T(p)}}\Pi_T^a(p)\right) \\
a_T^a(p)^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{\omega_T(p)}u_T^a(-p) - \frac{i}{\sqrt{\omega_T(p)}}\Pi_T^a(-p)\right)
\end{aligned}
\tag{13}
$$

The Hamiltonian in terms of $a(p)$ and $a^\dagger(p)$ is

$$
\begin{aligned}
H_L &= \int \frac{d^3p}{(2\pi)^3}\omega_L(p)a_L(p)^\dagger a_L(p) \\
H_T &= \int \frac{d^3p}{(2\pi)^3}\sum_{a=1,2}\omega_T(p)a_T^a(p)^\dagger a_L^a(p)
\end{aligned}
\tag{14}
$$

1.4

(a) The ground state is defined in this way

$$
a_L(p)|0\rangle = 0, \quad a_T^a(p)|0\rangle = 0 \tag{15}
$$

(b) A state with one longitudinal phonon with momentum $\vec{k}$ is

$$
a_L(k)^\dagger|0\rangle \tag{16}
$$

(c) A state with one longitudinal phonon of momentum $\vec{k}$ and one transverse phonon of momentum $\vec{q}$ is

$$
a_L(k)^\dagger a_T^a(q)^\dagger|0\rangle. \tag{17}
$$

2 Creation and Annihilation Operators

2.1

$$\begin{aligned}\langle \chi_1, \cdots, \chi_{N-1}|a(\varphi)|\psi, \cdots, \psi_N\rangle &= \langle \varphi, \chi_1, \cdots, \chi_{N-1}|\psi_1, \cdots, \psi_N\rangle \\ &= \begin{vmatrix} \langle\varphi|\psi_1\rangle & \langle\chi_1|\psi_1\rangle & \cdots & \langle\chi_{N-1}|\psi_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varphi|\psi_N\rangle & \langle\chi_1|\psi_N\rangle & \cdots & \langle\chi_{N-1}|\psi_N\rangle \end{vmatrix}_\zeta \\ &= \sum_{k=1}^{N} \zeta^{k-1}\langle\varphi|\psi_k\rangle \begin{vmatrix} \langle\chi_1|\psi_1\rangle & \cdots & \langle\chi_{N-1}|\psi_1\rangle \\ \vdots & \ddots & \vdots \\ \langle\chi_1|\psi_N\rangle & \cdots & \langle\chi_{N-1}|\psi_N\rangle \end{vmatrix}_\zeta \\ &= \sum_{k=1}^{N} \zeta^{k-1}\langle\varphi|\psi_k\rangle\langle \varphi, \chi_1, \cdots, \chi_{N-1}|\psi_1, \cdots, \text{ (no } \psi_k), \psi_N\rangle \end{aligned} \tag{18}$$

Hence we have

$$a(\varphi)|\psi_1, \cdots, \psi_N\rangle = \sum_{k=1}^{N} \zeta^{k-1}\langle\varphi|\psi_k\rangle|\psi_1, \cdots, \text{ (no } \psi_k), \psi_N\rangle \tag{19}$$

2.2

By using the above equation, we can get

$$a(\varphi_1)a^\dagger(\varphi_2) - \zeta a^\dagger(\varphi_2)a(\varphi_1) = \langle\varphi_1|\varphi_2\rangle \tag{20}$$

Hence

$$[a(\varphi_1), a^\dagger(\varphi_2)]_{-\zeta} = \langle\varphi_1|\varphi_2\rangle \tag{21}$$

3 The Free Electron Gas

3.1

The Hamiltonian in the momentum space is

$$H = \sum_{i,\sigma} E_{i,\sigma} a^\dagger_{i,\sigma} a_{i,\sigma} \tag{22}$$

where $E_i = p_i^2/2M$.

Since the electron has spin, each energy state can be filled up with two electrons with spin up and spin down. If N is even number, the ground state has the lowest $N/2$ single-particle energies filled with

$$E_1 \leq E_2 \leq \cdots \leq E_{N/2}, \quad p_1 \leq p_2 \leq \cdots \leq p_{N/2} \tag{23}$$

The Fermi energy is $E_{N/2}$ and Fermi momentum is $p_F = p_{N/2}$. The ground state energy is

$$E_{GS} = 2(E_1 + E_2 + \cdots + E_{N/2}) \tag{24}$$

If N is odd number, the ground state has the lowest $(N-1)/2$ single-particle energies filled with two electrons. The energy level $E_{(N+1)/2}$ has only electron. The Fermi energy is $E_{(N+1)/2}$ and Fermi momentum is $p_F = p_{(N+1)/2}$. The ground state energy is

$$E_{GS} = 2(E_1 + E_2 + \cdots + E_{(N-1)/2}) + E_{(N+1)/2} \tag{25}$$

3.2

The excited state with one electron with spin $\uparrow$ and momentum $\vec{p}$ and one hole with spin $\downarrow$ and momentum $\vec{q}$ is

$$|\Psi\rangle = a^\dagger_{\uparrow,p} a_{\downarrow,q} |GS\rangle \tag{26}$$

The energy is

$$E = E_{GS} + p^2/2M - q^2/2M \tag{27}$$

3.3

The current operator is

$$J = -i\frac{e\hbar}{2m} \int d^3x \sum_{\sigma=\uparrow\downarrow} \left[\psi^\dagger_\sigma(x) \nabla \psi_\sigma(x) - \nabla \psi^\dagger_\sigma \psi_\sigma(x)\right] \tag{28}$$

The expectation value for this operator is

$$\begin{aligned} \langle\varphi_\sigma|J(x)|\varphi_\sigma\rangle &= \int d^3y \int d^3z \langle\varphi_\sigma|y\rangle\langle 0|\psi_\sigma(y) J(x) \psi^\dagger_\sigma(z)|0\rangle\langle z|\varphi_\sigma\rangle \\ &= -i\frac{e\hbar}{2m} \int d^3x \sum_{\sigma=\uparrow\downarrow} \left[\varphi^\dagger_\sigma(x) \nabla \varphi_\sigma(x) - \nabla \varphi^\dagger_\sigma \varphi_\sigma(x)\right] \\ &= j \end{aligned} \tag{29}$$

3.4

The equation of motion for $\psi_\sigma(y)$ satisfies

$$\begin{aligned}\frac{\partial \psi_\sigma(y)}{\partial t} &= i[H, \psi_\sigma(y,t)] = ie^{iHt}[H, \psi_\sigma(y)]e^{-iHt} \\ &= ie^{iHt} \int d^3x \sum_{\sigma=\uparrow\downarrow} \frac{1}{2M} [\nabla \psi_\sigma^\dagger(x) \cdot \nabla \psi_\sigma(x), \psi_\sigma(y)]e^{-iHt} \\ &= \frac{i}{2M} \nabla^2 \psi(y,t) \end{aligned} \tag{30}$$

3.5

Using the equation of motion Eq.(30) and its Hermitian conjugate, we have

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \sum_\sigma \frac{i}{2M} \left[-(\nabla^2 \psi^\dagger)\psi + \psi^\dagger \nabla^2 \psi \right] \\ &= \sum_\sigma \frac{i}{2M} \left[-\nabla \cdot (\nabla \psi^\dagger \psi) + \nabla \psi^\dagger \cdot \nabla \psi + \nabla \cdot (\psi^\dagger \nabla \psi) - \nabla \psi^\dagger \cdot \nabla \psi \right] \\ &= \nabla \cdot J \end{aligned} \tag{31}$$

Hence we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{32}$$

4 Free fermions in one dimension

4.1

The one particle state with periodic boundary condition is

$$\psi_\sigma(x) = e^{i\frac{2\pi n}{L}x} \tag{33}$$

where $n \in \mathbb{Z}$.

4.2

The Hamiltonian in position space for a general free fermion model is

$$H = \int dx a^\dagger(x) [-\frac{\nabla^2}{2M} + V(x)] a(x) \tag{34}$$

The above Hamiltonian in momentum space can be written as

$$H = \int \frac{dp}{2\pi} \frac{p^2}{2M} a^\dagger(p)a(p) + \int \frac{dp}{2\pi} \int \frac{dq}{2\pi} V(q) a^\dagger(p+q)a(p) \tag{35}$$

where

$$V(q) = \int dx V(x) e^{-iqx} \tag{36}$$

4.3

The anticommutators

$$\{a(p), a(p')\} = \{a^\dagger(p), a^\dagger(p')\} = 0 \tag{37}$$

The anticommutator

$$\{a(p), a^\dagger(p')\} = \delta_{p,p'} \tag{38}$$

4.4

Assume that the above Hamiltonian can be diagonalized and written in this form

$$H = \sum_{p,\sigma} E(p) b_\sigma^\dagger(p) b_\sigma(p) \tag{39}$$

As shown in Fig.1, the ground state has the lowest $N/2$ single-particle energyis filled with

$$E_{p_1} \leq E_{p_2} \leq \cdots \leq E_{p_{(N/2-1)/2}} \tag{40}$$

The Fermi energy $E_F = E_{p_{(N/2-1)/2}}$. Since particle has 1/2 spin, each level is double degenerate. There are in total four single particle state with E_F.

4.5

The Hamiltonian is

$$H = \sum_{p,\sigma} E(p) b_\sigma^\dagger(p) b_\sigma(p) \tag{41}$$

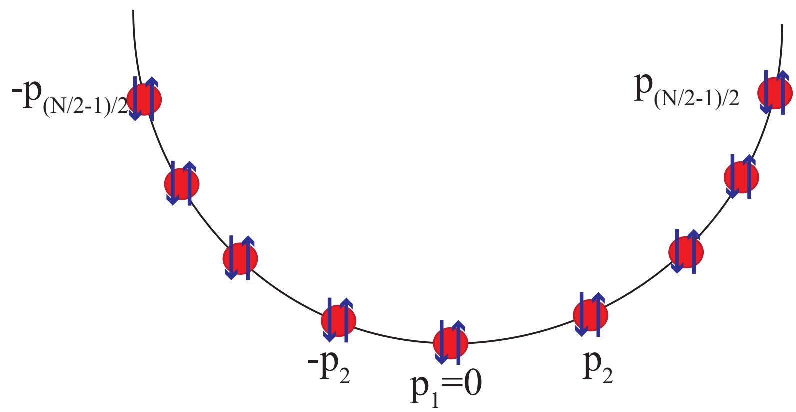

Figure 1: (Color online) The ground state for N particles

The excited state (with a given spin) is

$$|\psi\rangle = b^\dagger_\sigma(q_1)b^\dagger_\sigma(q_2)\cdots b^\dagger_\sigma(q_m)b_\sigma(q'_1)b_\sigma(q'_2)\cdots b_\sigma(q'_m)|GS\rangle \quad (42)$$

It has m pairs of particle-hole excitations, where $q_i \geq p_{N/2}$ and $q'_i \leq p_{N/2}$. The excitation energy is $\sum_{i=1}^m E(q_i) - \sum_{i=1}^m E(q'_i)$.

The degeneracy of the single particle and single hole states with arbitrary spin is 8.

4.6

The excited state with two particles and two holes is

$$|\psi\rangle = b^\dagger_{\sigma_1}(q_1)b^\dagger_{\sigma_2}(q_2)b_{\sigma_3}(q_3)b_{\sigma_4}(q_4)|GS\rangle \quad (43)$$

This state has N particle and the excitation energy is $E(q_1)+E(q_2)-E(q_3)-E(q_4)$. Since the excitation have momenta p, $q_1+q_2-q_3-q_4=p$.

The excited state with one particles and one holes is

$$|\psi\rangle = b^\dagger_{\sigma_1}(q_1)b_{\sigma_2}(q_2)|GS\rangle \quad (44)$$

This state has N particle and the excitation energy is $E(q_1)-E(q_2)$. Since the excitation have momenta p', $q_1-q_2=p'$.

5 Thermodynamics of the Ideal Fermi Gas

5.1

The thermodynamic potential for ideal fermi gas is

$$\begin{aligned}\Omega &= -k_B T V \int \frac{d^3p}{(2\pi)^3} \log\left[1 + e^{-\beta(\frac{p^2}{2m}-\mu)}\right] \\ &= -k_B T V \int d\epsilon N_0(\epsilon) \log\left[1 + z e^{-\beta\epsilon}\right] \end{aligned} \tag{45}$$

where $z = e^{\beta\mu}$ and the one-particle density of state is

$$N_0(\epsilon) = \frac{2\pi(2m)^{3/2}}{(2\pi)^3}\sqrt{\epsilon} \tag{46}$$

Hence we have

$$\begin{aligned}\frac{P}{k_B T} &= -\frac{1}{k_B T}\frac{\partial\Omega}{\partial V} = \int d\epsilon N_0(\epsilon) \log\left[1 + z e^{-\beta\epsilon}\right] \\ &= \frac{1}{\lambda_T^3} f_{5/2}(z) \end{aligned} \tag{47}$$

where

$$f_{5/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dx \sqrt{x} \log(1 + z e^{-x}) \tag{48}$$

and

$$\lambda_T = \left(\frac{2\pi}{mkT}\right)^{1/2} \tag{49}$$

The specific volume satisfies

$$\begin{aligned}\frac{1}{v} = \rho = \frac{\langle N\rangle}{V} &= -\frac{1}{V}\frac{\partial\Omega}{\partial\mu} = \int d\epsilon N_0(\epsilon) \frac{z e^{-\beta\epsilon}}{1 + z e^{-\beta\epsilon}} \\ &= \frac{1}{\lambda_T^3} f_{3/2}(z) \end{aligned} \tag{50}$$

where

$$f_{5/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dx \sqrt{x} \frac{z e^{-x}}{1 + z e^{-x}} \tag{51}$$

5.2

Since

$$f_{5/2}(z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{5/2}}$$
$$f_{3/2}(z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{3/2}} \tag{52}$$

In the high temperature limit, $z \to 1$, the equation of state of free Fermi gas also has the form of a virial expansion. The second coefficent is $-1/2^5/2$ for P and $-1/2^{3/2}$ for $1/v$.

5.3

From Eq.(50), we have

$$\rho = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\infty} d\epsilon \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}+1} \tag{53}$$

The energy density satisfies

$$u = \frac{U}{V} = \int d\epsilon N_0(\epsilon)\frac{\epsilon}{1+z^{-1}e^{\beta\epsilon}}$$
$$= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\infty} d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1} \tag{54}$$

5.4

Using the Sommerfeld expansion, we have

$$U = \int d\epsilon V N_0(\epsilon)\frac{\epsilon}{1+z^{-1}e^{\beta\epsilon}}$$
$$\approx \int_0^{\infty} g(\epsilon)d\epsilon + \frac{\pi^2}{6\beta^2}g'(\mu)$$
$$= \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\frac{(k_BT)^2}{\epsilon_F} \tag{55}$$

where $g(\epsilon) = VN_0(\epsilon)\epsilon$.

Hence the specific heat is

$$C_v = \frac{\pi^2}{2} \frac{k_B^2 T}{\epsilon_F} \tag{56}$$

5.5

Using the same approximation, we can show that for Eq.(47),

$$P = \frac{2}{5} \frac{N\epsilon_F}{V} + \frac{\pi^2}{6} \frac{N(k_B T)^2}{\epsilon_F V} \tag{57}$$

At $T = 0$, $P = \frac{2}{5}\frac{N\epsilon_F}{V}$, which is known as Fermi pressure. It is nonzero due to the effects of the Pauli principle which keeps fermions from occupying the same single-particle state.