Solution 2

October 14, 2015

1 Antiferromagnetic Spin Waves

1.1

The Heisenberg Hamiltonian is

$$\begin{aligned} H &= J \sum_{n=-N/2+1}^{N/2} \hat{S}_k(n) \cdot \hat{S}_k(n+1) \\ &= \frac{J}{2} \sum_j \Big(\hat{S}^+(n) \cdot \hat{S}^- + \hat{S}^-(n) \cdot \hat{S}^+(n+1) \Big) + J\hat{S}_3(n) \cdot \hat{S}_3(n+1) \end{aligned} \tag{1}$$

where $\hat{S}^- = \hat{S}_1 - i\hat{S}_2$ and $\hat{S}^+ = \hat{S}_1 + i\hat{S}_2$.

To calculate the quantum mechanical equations of motion obeyed by $\hat{S}^\pm(j)$ and $\hat{S}_3(j)$, we first need to calculate the commutators

$$[\hat{S}^+(j), H] = \frac{J}{2}\hat{S}_3(j)(\hat{S}^+(j-1) + \hat{S}^+(j+1)) - \frac{J}{2}\hat{S}^+(j)(\hat{S}_3(j-1) + \hat{S}_3(j+1)) \tag{2}$$

where we used

$$\begin{aligned} [\hat{S}_3, \hat{S}_\pm] &= \pm\hat{S}_\pm \\ [\hat{S}_+, \hat{S}_-] &= 2\hat{S}_3 \end{aligned} \tag{3}$$

Similarly, we can calculate $[\hat{S}^-(j), H]$ and $[\hat{S}_3(j), H]$.

The Heisenberg equations of motion are

$$\partial_0 \hat{S}^+(j) = -i\frac{J}{2}\left[\hat{S}_3(j)(\hat{S}^+(j-1)+\hat{S}^+(j+1)) - \hat{S}^+(j)(\hat{S}_3(j-1)+\hat{S}_3(j+1))\right]$$
$$\partial_0 \hat{S}^-(j) = -i\frac{J}{2}\left[-\hat{S}_3(j)(\hat{S}^-(j-1)+\hat{S}^-(j+1)) - \hat{S}^-(j)(\hat{S}_3(j-1)+\hat{S}_3(j+1))\right]$$
$$\partial_0 \hat{S}_3(j) = -i\frac{J}{4}\left[\hat{S}^+(j)(\hat{S}^-(j-1)+\hat{S}^-(j+1) + \hat{S}^-(j)(\hat{S}^+(j-1)+\hat{S}^+(j+1))\right] \tag{4}$$

They are non-linear equations. This suggests that the Heisenberg model in general is not a free bosonic system.

1.2

For the even sites, by using Eq.(8), we have

$$\hat{S}^+|n\rangle = \sqrt{2S}\left[1-\frac{\hat{n}(j)}{2S}\right]^{\frac{1}{2}}\hat{a}(j)|n\rangle = \left[2S\left(1-\frac{n-1}{2S}\right)n\right]^{\frac{1}{2}}|n-1\rangle$$
$$\hat{S}^-|n\rangle = \left[2S(n+1)(1-\frac{n}{2S})\right]^{\frac{1}{2}}|n+1\rangle \tag{5}$$

From the above equation, we have (for the even sites)

$$\hat{n}(j)|S,S\rangle = (S-\hat{S}_3(j))|S,S\rangle = 0$$
$$\hat{n}(j)|S,-S\rangle = 2S|S,-S\rangle \tag{6}$$

Hence Eq.(7) and Eq.(8) are consistent with Eq.(4).

1.3

The Heisenberg Hamiltonian is

$$H = H_{\text{even}} + H_{\text{odd}}$$
$$= \sum_{j\in\text{even}} JS\left[\hat{a}^\dagger(j)\hat{a}(j) + \hat{b}^\dagger(j+1)\hat{b}(j+1)\right]$$
$$+ JS\left[(1-\frac{\hat{n}(j)}{2S})^{1/2}(1-\frac{\hat{n}(j+1)}{2S})^{1/2}\hat{a}(j)\hat{b}(j+1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j+1)(1-\frac{\hat{n}(j)}{2S})^{1/2}(1-\frac{\hat{n}(j+1)}{2S})^{1/2}\right]$$
$$+ J\hat{a}^\dagger(j)\hat{a}(j)\hat{b}^\dagger(j+1)\hat{b}(j+1) - S^2 + \sum_{j\in\text{odd}}(\hat{a}\leftrightarrow\hat{b}) \tag{7}$$

1.4

In the semiclassical limit $S \to \infty$, for the above Hamiltonian, if we only include terms which are of order $1/S$, the Hamiltonian is

$$H_{\text{even}} = JS \sum_{j \in \text{even}} \left[\hat{a}(j)\hat{b}(j+1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j+1) + \hat{a}^\dagger(j)\hat{a}(j) + \hat{b}^\dagger(j+1)\hat{b}(j+1) - S \right]$$
$$H_{\text{odd}} = JS \sum_{j \in \text{odd}} \left[\hat{b}(j)\hat{a}(j+1) + \hat{b}^\dagger(j)\hat{a}^\dagger(j+1) + \hat{b}^\dagger(j)\hat{b}(j) + \hat{a}^\dagger(j+1)\hat{a}(j+1) - S \right] \tag{8}$$

The above Hamiltonian takes a quadratic form.

1.5

In the semiclassical limit, the equations of motion becomes

$$\partial_0 \hat{a}(j) = -i\frac{JS}{2}\left[\hat{b}^\dagger(j-1) + \hat{b}^\dagger(j+1) - 2\hat{a}(j)\right]$$
$$\partial_0 \hat{a}^\dagger(j) = -i\frac{JS}{2}\left[\hat{b}(j-1) + \hat{b}(j+1) - 2\hat{a}^\dagger(j)\right] \tag{9}$$

The above result is for the even j. For the odd j, we only need to switch $a \leftrightarrow b$ to get the similar result. The equations of motion are now linear. The term $\hat{n}(j)\hat{n}(j+1)$ is neglected.

1.6

By using the Fourier transformation

$$\hat{a}(q) = \sqrt{\frac{2}{N}} \sum_{j \in \text{even}} e^{iqj}\hat{a}(j)$$
$$\hat{b}(q) = \sqrt{\frac{2}{N}} \sum_{j \in \text{odd}} e^{-iqj}\hat{b}(j) \tag{10}$$

The Hamiltonian can be written as

$$H = 2SJ\sum_q \left[\hat{a}(q)\hat{b}(q)\cos(q) + \hat{a}^\dagger(q)\hat{b}^\dagger(q)\cos(q) + \hat{a}^\dagger\hat{a}(q) + \hat{b}^\dagger(q)\hat{b}(q) - \frac{N}{2}S \right] \tag{11}$$

By performing the canonical transformation

$$\begin{aligned}\hat{c}(q) &= \cosh(\theta)\hat{a}(q) + \sinh(\theta)\hat{b}^{\dagger}(q)\\ \hat{d}(q) &= \cosh(\theta)\hat{b}(q) + \sinh(\theta)\hat{a}^{\dagger}(q)\end{aligned} \tag{12}$$

The Hamiltonian has off-diagonal term $\cos(q)(\cosh^2(\theta)+\sinh^2(\theta))\hat{c}(q)\hat{d}(q) - 2\cosh(\theta)\sinh(\theta)\hat{d}(q)\hat{c}(q)$. To diagonalize the Hamiltonian, this term is equal to zero and this requires that

$$\cos(q) = \tanh(2\theta) \tag{13}$$

The Hamiltonian after diagonalization takes the following form

$$H_{SW} = E_0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dq}{2\pi}\omega(q)\left(\hat{n}_c(q) + \hat{n}_d(q)\right) \tag{14}$$

where $\omega(q) = |\sin(q)|$.

1.7

The ground state satisfies

$$\hat{c}(q)|GS\rangle = \hat{d}(q)|GS\rangle = 0 \tag{15}$$

1.8

The single particle eigenstate can be generated by $\hat{c}^{\dagger}(q)$ or $\hat{d}^{\dagger}(q)$.

$$\begin{aligned}\hat{c}^{\dagger}(q)|GS\rangle &= |q,1\rangle,\\ \hat{d}^{\dagger}(q)|GS\rangle &= |q,2\rangle\end{aligned} \tag{16}$$

The dispersion relation is

$$E(q) = 2NJS|\sin(q)| \tag{17}$$

It equals to zero when $q = n\pi$. Since $q \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, when q is around zero, the energy of excited state goes to zero. Around $q = 0$,

$$E(q) \approx 2NJS|q| \tag{18}$$

The energy vanishes linearly as the momentum q approaches to zero.

1.9

In the spin wave approximation,

$$\hat{S}^+(j) = \sum_q e^{-iqj}\sqrt{2S}(\cosh\theta\hat{c}^\dagger(q) - \sinh\theta\hat{d}(q))$$
$$\hat{S}^-(j) = \sum_q e^{iqj}\sqrt{2S}(\cosh\theta\hat{c}(q) - \sinh\theta\hat{d}^\dagger(q)) \tag{19}$$

When both n and n' are even or odd numbers,

$$\begin{aligned} D_{+-}(jt, j't') &= -i\langle GS|T\hat{S}^+(j,t)\hat{S}^-(j',t')|GS\rangle \\ &= -4Si\sum_q e^{-iE_q(t-t')}\theta(t-t')\langle GS|\cosh^2\theta\hat{c}^\dagger(q)\hat{c}(q) + \sinh^2\theta\hat{d}(q)\hat{d}^\dagger(q)|GS\rangle \\ &\quad - 4Si\sum_q e^{iE_q(t-t')}\theta(t'-t)\langle GS|\cosh^2\theta\hat{c}^\dagger(q)\hat{c}(q) + \sinh^2\theta\hat{d}(q)\hat{d}^\dagger(q)|GS\rangle \end{aligned} \tag{20}$$

The propagator in the momentum space is

$$\begin{aligned} D_{+-}(q) &= 4Si\int dt\left[\theta(\Delta t)e^{-i(E_q+\omega)\Delta t}\cosh^2(\theta) + \theta(-\Delta t)e^{i(E_q-\omega)\Delta t}\sinh^2(\theta)\right] \\ &= \frac{4S}{|\sin(q)|}\left[\frac{1-|\sin(q)|}{\omega - E_q + i\epsilon} + \frac{1+|\sin(q)|}{\omega + E_q - i\epsilon}\right] \end{aligned} \tag{21}$$

When one is odd and one is even, the calculation is similar,

$$\begin{aligned} &D_{+-}(jt, j't') = -i\langle GS|T\hat{S}^+(j,t)\hat{S}^-(j',t')|GS\rangle \\ &= -4Si\sum_q e^{-iE_q(t-t')}\theta(t-t')\langle GS|\cosh\theta\sinh\theta\hat{c}^\dagger(q)\hat{c}(q) + \cosh\theta\sinh\theta\theta\hat{d}(q)\hat{d}^\dagger(q)|GS\rangle \\ &\quad - 4Si\sum_q e^{iE_q(t-t')}\theta(t'-t)\langle GS|\cosh\theta\sinh\theta\theta\hat{c}^\dagger(q)\hat{c}(q) + \cosh\theta\sinh\theta\theta\hat{d}(q)\hat{d}^\dagger(q)|GS\rangle \end{aligned} \tag{22}$$

The propagator in the momentum space is

$$D_{+-}(q) = \frac{4S\cos(q)}{|\sin(q)|}\left[\frac{1}{\omega - E_q - i\epsilon} + \frac{1}{\omega + E_q + i\epsilon}\right] \tag{23}$$

For D_{33},

$$D_{33}(jt, j't') = -i\langle GS|T(S-\hat{n}(j,t))(S-\hat{n}(j',t'))|GS\rangle \tag{24}$$

It includes term $\langle GS|\hat{a}^\dagger(j)\hat{a}(j)|GS\rangle$ and term $\langle GS|\hat{a}^\dagger(j)\hat{a}(j)\hat{a}^\dagger(j')\hat{a}(j')|GS\rangle$.

The propagator in the momentum space is

$$\begin{aligned}
&D_{33} = NS^2\delta_{q,\pi}\delta_{q',\pi}\left[-\frac{1}{\omega-i\delta}+\frac{1}{\omega+i\delta}\right]+\frac{\delta_{q+q',0}}{N}\sum_k \\
&= \left(\cosh\theta_{k-q/2}\sinh\theta_{k+q/2}-\cosh\theta_{k+q/2}\sinh\theta_{k-q/2}\right) \\
&\times\left(\frac{1}{\omega-E_{k-q/2}-E_{k+q/2}+i\epsilon}-\frac{1}{\omega-E_{k-q/2}-E_{k+q/2}-i\epsilon}\right)
\end{aligned} \tag{25}$$

1.10

The propogator is already calculated in the above problem.

1.11

From the result on $D_{+-}(p,\omega)$, there is a pole when $\cos^2(q/2)\to 0$. This requires that $q=\pi$.

2 The electron gas

2.1

The Feynman propagator for the non-interacting system

$$\begin{aligned}
&G_0^{\sigma,\sigma'} = -i_0\langle G|T\psi_\sigma(x)\psi^\dagger_{\sigma'}(x')|G\rangle_0 \\
&= -i\sigma_{\sigma,\sigma'}\sum_\alpha \varphi^*_\alpha(r)\varphi_\alpha(r')\times \\
&\left[\theta(\alpha-G)e^{i(E_\alpha-E_G)(t-t')}\theta(t-t')-\theta(G-\alpha)e^{i(E_\alpha-E_G)(t-t')}\theta(t'-t)\right] \\
&= sigma_{\sigma,\sigma'}\sum_\alpha \varphi^*_\alpha(r)\varphi_\alpha(r')\left[\frac{\theta(\alpha-G)}{\omega-(E_\alpha-E_G)+i\delta}+\frac{\theta(G-\alpha)}{\omega-(E_\alpha-E_G-i\delta)}\right]
\end{aligned} \tag{26}$$

For free fermion, we have

$$G_0^{\sigma,\sigma'} = \delta_{\sigma,\sigma'} \int d^3p e^{ip(r-r')} \left[\frac{\theta(|p| - p_F)}{\omega - \frac{p^2}{2m} + i\delta} + \frac{\theta(p_F - |p|)}{\omega - \frac{p^2}{2m} - i\delta} \right] \tag{27}$$

Hence after Fourier transformation, we have

$$\begin{aligned} G_F(p,\omega) &= \frac{\theta(|p| - p_F)}{\omega - \frac{p^2}{2m} + i\delta} + \frac{\theta(p_F - |p|)}{\omega - \frac{p^2}{2m} - i\delta} \\ &= \frac{1}{\omega - E(p) + i\epsilon(|p| - p_F)} = \frac{1}{\omega - E(p) + i\epsilon \text{sign } \omega} \end{aligned} \tag{28}$$

2.2

(a) The density operator is

$$\langle \rho(x,y) \rangle = -i \lim_{t' \to t} G_F(x,t,\sigma,y,t',\sigma') \tag{29}$$

(b) Using the result in Eq.(28), we have

$$\begin{aligned} \rho(x,y) &\sim \int_0^\pi d\theta \sin\theta \int \frac{dp\ p^2}{(2\pi)^3} e^{i\cos\theta p|x-y|} \theta(p_F - |p|) \\ &\sim \frac{1}{(2\pi)^3} \int_0^{p_F} dp\ p \frac{\sin(p|x-y|)}{|x-y|} \\ &\sim \frac{1}{r^2}(-p_F \cos(p_F r) + \frac{1}{r}\sin(p_F r)) \end{aligned} \tag{30}$$

where $r = |x - y|$.

In the short distance limit when $r << 1/p_F$, we have

$$\rho(r) \approx \frac{p_F^3}{3\pi^2} \tag{31}$$

In the long distance limit when $r >> 1/p_F$, the second term is negligible, we have

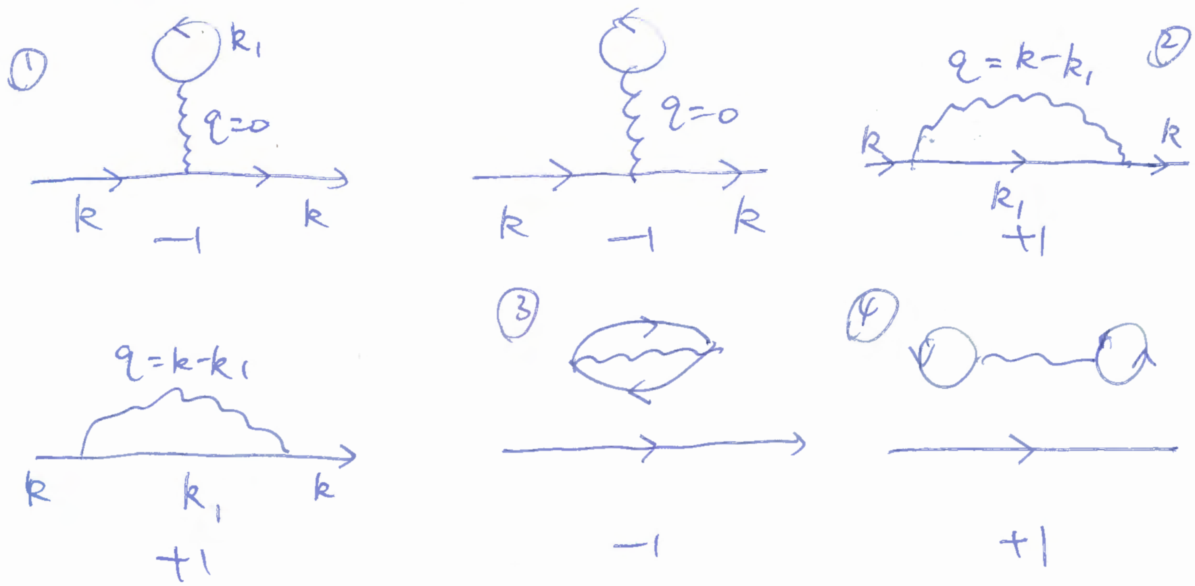
$$\rho(r) \approx -\frac{p_F \cos(rp_F)}{\pi^2 r^2} \tag{32}$$

It decays as $1/r^2$ and is also oscillatory.

3(a).

First order:

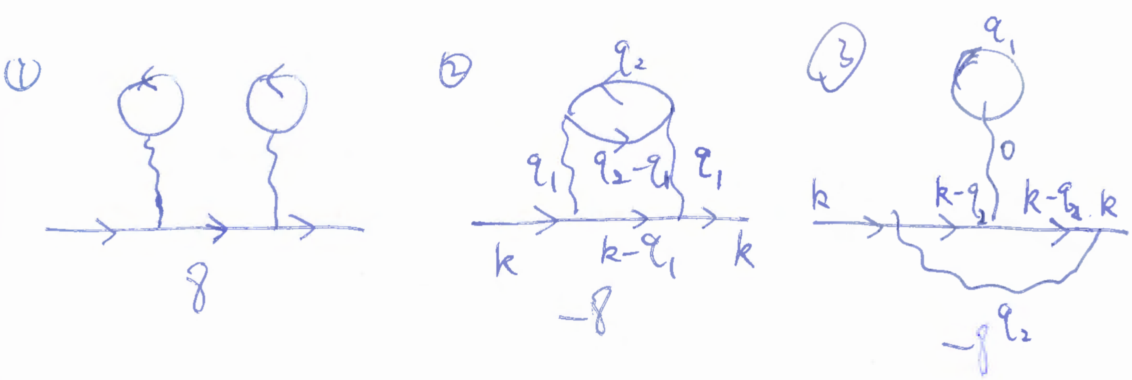
There are six first order diagrams,

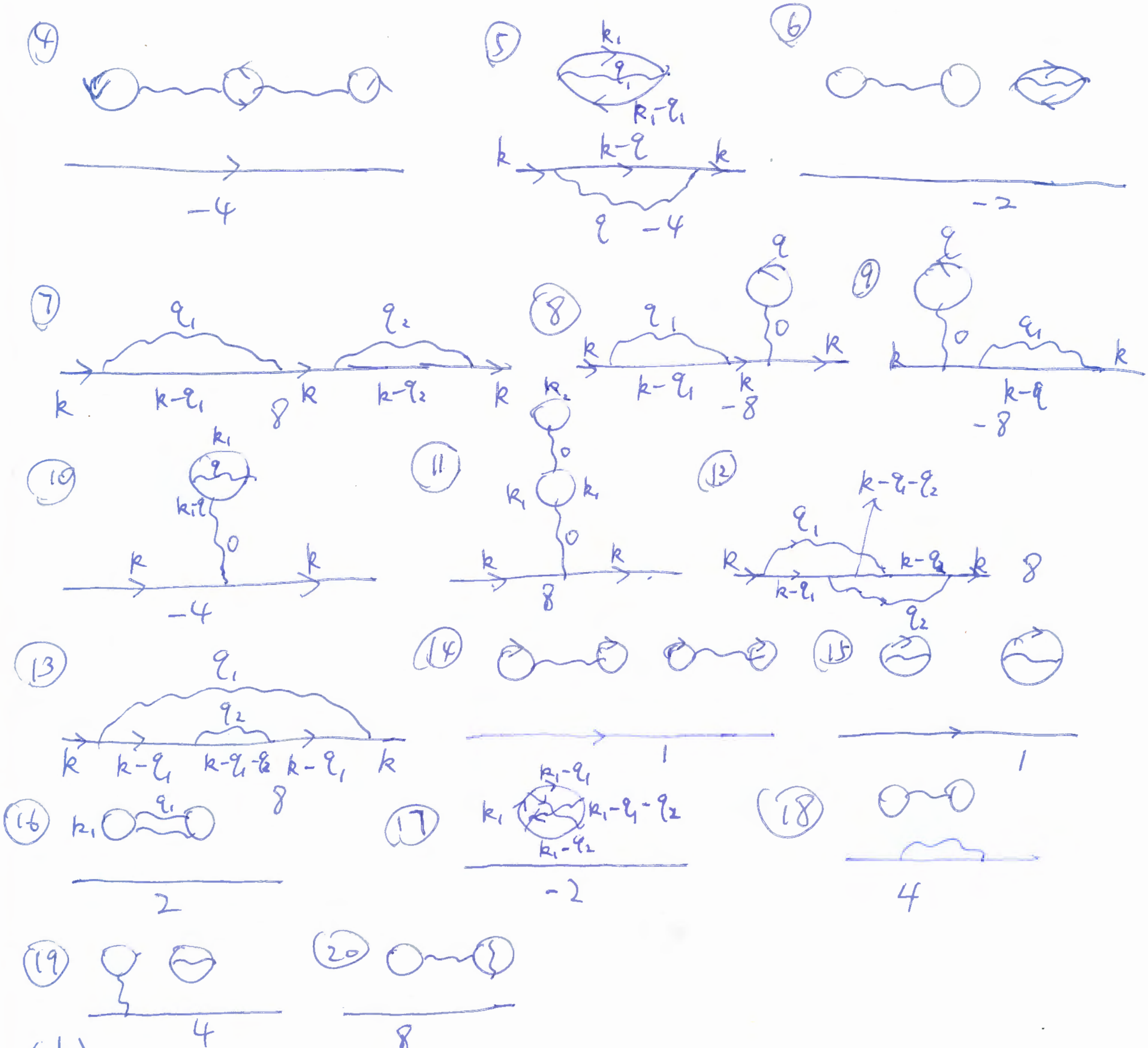


The correct factor is $(-1)^F$, where F is the number of closed fermionic loops.

Second order:

There are $120 = 5!$ second order diagrams.





(b).

For the first order:

$$G_1 \sim (-2) \sum_{k_1} \int G_0(k)\, G_0(k_1)\, G_0(k) \qquad V(q=0)$$

$$G_2 \sim \int \sum_{k_1} G_0(k)\, G_0(k_1)\, G_0(k)\, V(q)$$

$$G_3 \sim -1 \int \sum_{k_1 q} G_0(k)\, G_0(k_1)\, G_0(k_1-q)\, V(q).$$

$$G_4 \sim \int G_0(k) \sum_{k_1} G_0(k_1)\, V(0)\, G_0(k_1)$$

For the second order:

① $G \sim \frac{8}{2} G_0(k) G_0(k) G_0(k) \int \sum_{q_1 q_2} G_0(q_1) G_0(q_2) V(0) V(0)$

②. $G \sim -\frac{8}{2} G_0(k) G_0(k) \int \sum_{q_1 q_2} G_0(q_2) G_0(q_2 - q_1) V(q_1) V(q_1)$

③. $G \sim -\frac{8}{2} \int \sum_{q_1 q_2} G_0(k) G_0(k) G_0(k-q_2) G_0(k-q_2) G_0(q_1) V(q_2)$

④ $G \sim -\frac{4}{2} \int \sum_{q} G_0(k) G_0(q) G_0(q) G_0(q) V(0) V(0)$

⑤ $G \sim -\frac{4}{2} \int \sum_{q_1 k_1 q} G_0(k) G_0(k) G_0(k-q) V(q) G_0(k_1) G_0(k_1-q_1) V(q_1)$

⑥ $G \sim -\frac{2}{2} \int \sum_{k_1 q} G_0(k) G_0(k) G_0(k) V(0) G_0(k_1) V(q) G_0(k_1-q)$

⑦ $G \sim \frac{8}{2} \int \sum_{q_1 q_2} G_0(k) V(q_1) G_0(k-q_1) G_0(k) V(q_2) G_0(k-q_2) G_0(k)$

⑧. $G \sim -\frac{8}{2} \int \sum_{q_1 q} G_0(k) V(q_1) G_0(k-q_1) G_0(k) V(0) G_0(q) G_0(k)$

⑨ $G \sim -\frac{8}{2} \int \sum_{q_1 q} G_0(k) V(0) G_0(q) G_0(k) V(q_1) G_0(k-q_1) G_0(k)$

⑩ $G \sim \frac{4}{2} \int \sum G_0(k) V(0) G_0(k) G_0(k_1) V(q) G_0(k_1-q)$

⑪. $G \sim \frac{8}{2} \int \sum_{k_1 k_2} G_0(k) V(0) G_0(k) G_0(k_1) G_0(k_1) V(0) G_0(k_2)$

⑫. $G \sim \frac{8}{2} \int \sum_{q_1 q_2} G_0(k) V(q_1) G_0(k-q_1) G_0(k-q_1-q_2) V(q_2) G_0(k-q_2) G_0(k)$

(13) $$G \sim \frac{8}{2}\int \sum_{q_1 q_2} G_0(k)\, V(q_1)\, G_0(k)\, G_0(k-q_1)\, V(q_2)\, G_0(k-q_1-q_2)\, G_0(k-q_1)$$

(14) $$G \sim \frac{1}{2}\int \sum_{k_1,k_2} G_0(k)\, G_0(k_1)\, V(0)\, G_0(k_1)\, G_0(k_2)\, V(0)\, G_0(k_2)$$

(15) $$G \sim \frac{1}{2}\int \sum_{k_1 q_1 k_2 q_2} G_0(k)\, G_0(k_1)\, G_0(k_1-q_1)\, V(q_1)\, G_0(k_2)\, G_0(k_2-q_2)\, V(q_2)$$

(16) $$G \sim \frac{2}{2}\int \sum_{k_1 q_1} G_0(k)\, G_0(k_1)\, V(q_1)\, G_0(k_1-q_1)\, V(q_1)\, G_0(k_1-q_1)\, G_0(k_1)$$

(17) $$G \sim -\frac{2}{2}\int \sum_{k_1, q_1 q_2} G_0(k)\, G_0(k_1)\, G_0(k_1-q_1)\, G_0(k_1-q_1-q_2)\, G_0(k-q_2)\, V(q_1)\, V(q_2)$$

(18) $$G \sim \frac{4}{2}\int \sum_{k_1,q_1} G_0(k)\, V(q_1)\, G_0(k-q_1)\, G_0(k)\, G_0(k_1)\, V(0)\, G_0(k_1)$$

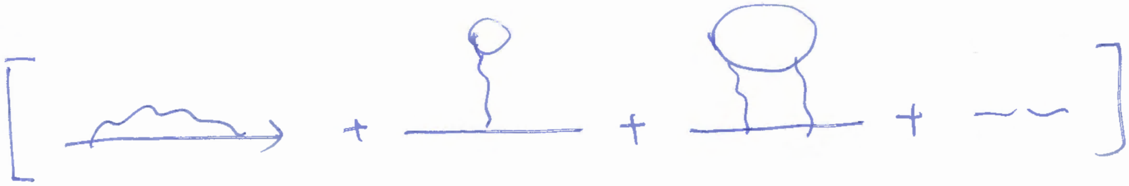
(19) $$G \sim \frac{4}{2}\int \sum_{k_1, q_1, k_2} G_0(k)\, G_0(k)\, V(0)\, G_0(k_1)\, G_0(k_2)\, V(q_1)\, G_0(k_2-q_1)$$

(20) $$G \sim \frac{8}{2}\int \sum_{k_1,k_2,q} G_0(k)\, G_0(k_1)\, V(0)\, G_0(k_2)\, G_0(k_2)\, V(q)\, G_0(k_2-q)$$

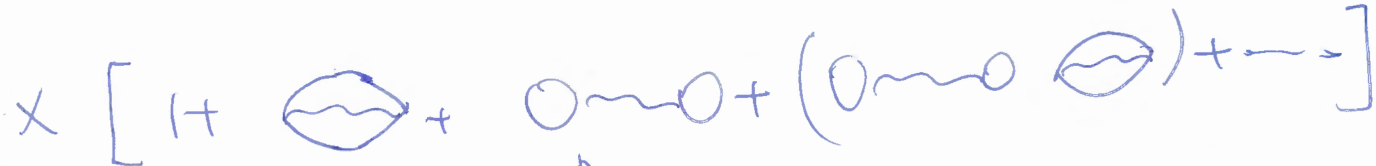
(c)

We can show that all the diagrams can be decomposed in this way.

All the diagrams =



↑ connected diagrams.



↑ vacuum diagrams.

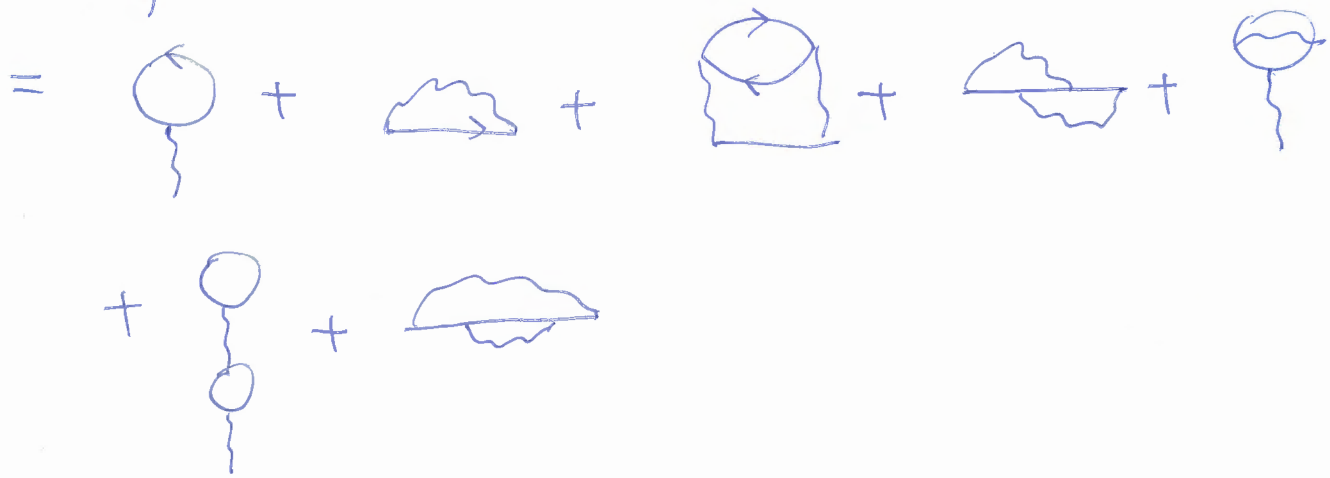
$$G^{(2)} = \frac{\text{All the diagrams}}{\text{vacuum diagrams}} = \text{connected diagrams}.$$

All the vacuum diagrams cancel out to this order.

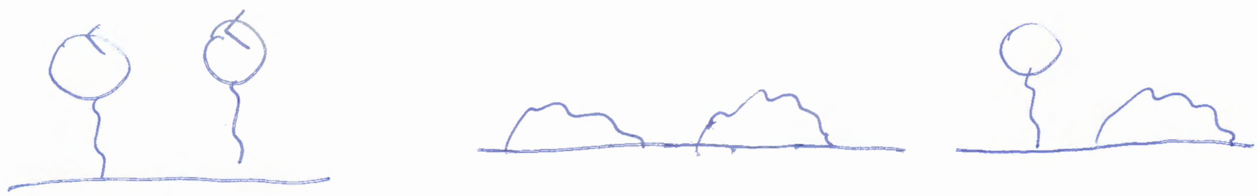
(d). Using Dyson's equation.

$$G = \frac{1}{\omega - \epsilon(k) - \Sigma(k)}$$

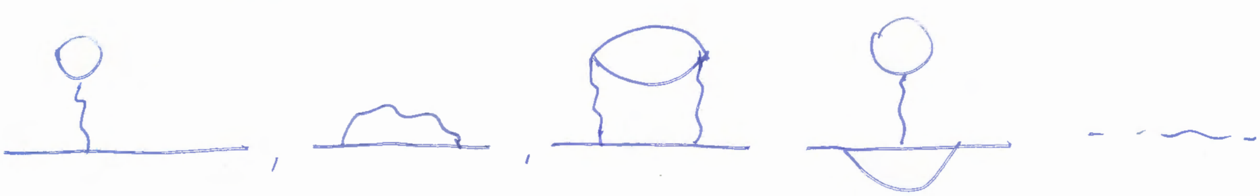
$\Sigma(k)$ at the second order.



The diagrams like



are reducible diagrams.



are irreducible diagrams.

2.3

(a) For any operator O, the expectation value is

$$\langle O(t)\rangle = \langle \Psi(t)|O|\Psi(t)\rangle \tag{33}$$

Applying some perturbation term to the Hamiltonian, we have

$$\begin{aligned}\delta O(t) &= \langle G|O_{H'}|G\rangle - \langle G|O_H|G\rangle \\ &= i\int dt' \langle G\left[H^{ext}(t'), O(t)\right]||G\rangle\end{aligned} \tag{34}$$

For an electron system interacts with an external potential, the change of the local density $\delta\rho(x)$ caused by the external potential is

$$\delta\rho(p,\omega) = \Pi(p,\omega)V(p,\omega) = \Pi^0(p,\omega)V_{eff}(p,\omega) \tag{35}$$

$\delta\rho(x)$ can be obtained by performing Fourier transformation on $\delta\rho(p,\omega)$.

(b) $\delta\rho(x)$ is

$$\delta\rho(x) = \int d^3p\Pi(p,\omega=0)V(p)e^{-i\vec{p}\cdot\vec{x}} \tag{36}$$

In our model, $V(x)$ is a point charge of strength Q at the origin, hence

$$V(p) = \frac{4\pi Q}{p^2} \tag{37}$$

Inserting the potential in $\delta\rho(x)$, we have

$$\begin{aligned}\delta\rho(x) &= \int d^3p\Pi(p,\omega=0)V(p)e^{-i\vec{p}\cdot\vec{x}} \\ &= \int d^3p\Pi^0(p,\omega=0)V_{eff}(p,\omega=0)e^{-i\vec{p}\cdot\vec{x}} \\ &= \int d^3p\frac{V(p,\omega=0)\Pi^0(p,\omega=0)}{1-V(p,\omega=0)\Pi^0(p,\omega=0)}e^{-i\vec{p}\cdot\vec{x}}\end{aligned} \tag{38}$$

where the density propagator Π^0 equals to

$$\Pi^0(p,\omega) = 2\int \frac{d^3k}{(2\pi)^3}\theta(|k+q|-p_F)\theta(p_F-|k|)\times \left[\frac{1}{\omega+(E(k)-E(k+q))+i\epsilon} - \frac{1}{\omega-(E(k)-E(k+q)-i\delta)}\right] \tag{39}$$

V_{eff} equals to

$$V_{eff}(p,\omega) = \frac{V(p)}{1 - V(p)\Pi^0(p,\omega)} = \frac{V(p)}{\epsilon(p,\omega)} \tag{40}$$

In the static limit,

$$\Pi^0(q,\omega=0) = -8\int \frac{d^3k}{(2\pi)^3}\frac{\theta(|k+q|-p_F)\theta(p_F-|k|)}{q^2+2k\cdot q} \tag{41}$$

According to Eq.(4.88) in the lecture notes, the static dielectric function $\epsilon(q,\omega=0)$ is

$$\epsilon(q,0) = 1 + \left(\frac{4}{9\pi}\right)^{1/3} r_s \frac{u(x)}{x^2} \tag{42}$$

where $x = \frac{|q|}{2p_F}$ and r_s satisfies

$$p_F a_0 r_s = \left(\frac{9\pi}{4}\right)^{1/3} \tag{43}$$

The dimensionless function $u(x)$ is

$$u(x) = \frac{1}{2}\left(1 + \frac{1}{2x}(1-x^2)\log\left|\frac{1+x}{1-x}\right|\right) \tag{44}$$

when $x \to 0$,

$$u(x) = \frac{1}{2} + \frac{1}{4x}(1-x^2)\log\left|\frac{1+x}{1-x}\right| = \frac{1}{2} + \frac{1}{2} = 1 \tag{45}$$

Therefore we have

$$\left(\frac{4}{9\pi}\right)^{1/3} r_s \frac{u(x)}{x^2} \approx \left(\frac{4}{9\pi}\right)^{1/3} r_s \frac{1}{x^2} = \frac{\lambda^2}{p^2} \tag{46}$$

where λ is proportional to the Thomas-Fermi screening length ξ_{TF} defined in the lecture notes.

Combing Eq.(39), Eq.(40) and Eq.(46) together, we have

$$\delta\rho(x) = -\int \frac{d^3p}{(2\pi)^3}\frac{\lambda^2}{p^2+\lambda^2}e^{i\vec{p}\cdot\vec{x}} = -\frac{\lambda^2}{4\pi}\frac{1}{|x|}e^{-|x|/\epsilon_{TF}} \tag{47}$$

where we used Eq.(4.95) in the lecture notes. Actually there is some issue with the above calculation. There is a weak singularity as $x \to 1$. If the Fermi surface is sharp, there will be some extra contribution to the dielectric function in the denominator of V_{eff}.

Below we do the calculation more carefully,

$$\delta\rho(x) = \int \frac{d^3p}{(2\pi)^3}\delta\rho(p)e^{-i\vec{p}\cdot\vec{x}} = \int_0^\infty \frac{dp}{(2\pi)^2}p^2\int_{-1}^1 d\cos(\theta)\delta\rho(p)e^{-ip|x|\cos\theta}$$
$$= \int_0^\infty \frac{dp}{2\pi^2|x|}\frac{1-\epsilon(p)}{\epsilon(p)}\sin(p|x|) \tag{48}$$

where $\epsilon(p) = \epsilon(p, 0)$.

Define

$$g(p) = \frac{1}{2\pi^2}p\frac{1-\epsilon(p)}{\epsilon(p)} \tag{49}$$

We have

$$\delta\rho(x) = \frac{1}{|x|}\int_0^\infty dpg(p)\sin(p|x|) = -\frac{1}{|x|^3}\int_0^\infty dpg''(p)\sin(p|x|) \tag{50}$$

where the second term is obtained by performing integration by parts twice.

Since there is a singularity in $\epsilon(p)$ when $x \to 1$, when we expand around $x = 1$, the most singular part in g'' is

$$g''(p) \sim \frac{A}{p-2p_F} \tag{51}$$

By using $\sin(px) = \sin[(p-2p_F)x]\cos(2p_Fx) + \cos[(p-2p_F)x]\sin(2p_Fx)$, we can write Eq.(50) as

$$\delta\rho(x) = \frac{A}{|x|^3}\int_{2p_F-\Lambda}^{2p+\Lambda}\frac{\sin[(p-2p_F)|x|]\cos(2p_F|x|) + \cos[(p-2p_F)|x|]\sin(2p_F|x|)}{p-2p_F}dp \tag{52}$$

When $p_F|x| >> 1$, we have

$$\delta\rho(x) \sim \frac{A\cos(2p_F|x|)}{|x|^3} \tag{53}$$

It decays as $1/|x|^3$. The oscillation behavior of the charge density is due the sharp Fermi surface and this behavior is called Friedel oscillation.