Physics 561, Fall Semester 2015
Professor Eduardo Fradkin

Problem Set No. 3:
Measurements and Correlation Functions
Due Date: November 8, 2015

1 The Density and Current Correlation Function

In this problem you will work out some details of the relation between equal time commutators of densities and currents and conservation laws in an interacting electron gas that was discussed in class.

Let $|G\rangle$ be the exact ground state of a system of *interacting fermions* at finite density $\bar{\rho}(\vec{x})$. (Notice that this state is uniform only if $\bar{\rho}$ is constant.) Let $J_0(\vec{r})$ be the *charge density* operator and $\vec{J}(\vec{x})$ be the *gauge-invariant current density* operator discussed in class,

$$\begin{aligned} J_0(\vec{x}) &= e\rho(\vec{x}) = e\sum_{\sigma=\uparrow,\downarrow}\psi^\dagger_\sigma(\vec{x})\psi_\sigma(\vec{x}) \\ \vec{J}(\vec{x}) &= \frac{ie\hbar}{2m}\sum_{\sigma=\uparrow,\downarrow}\left[\psi^\dagger_\sigma(\vec{x})\vec{\bigtriangledown}\psi_\sigma(\vec{x}) - \vec{\bigtriangledown}\psi^\dagger(\vec{x})\psi(\vec{x})\right] + \frac{e^2}{mc}\vec{A}(\vec{x})\sum_{\sigma=\uparrow,\downarrow}\psi^\dagger_\sigma(\vec{x})\psi_\sigma(\vec{x}) \end{aligned} \tag{1}$$

where $\vec{A}(\vec{x},t)$ is the electromagnetic vector potential. In what follows we will denote the 4-current retarded correlation function by

$$D^R_{\mu\nu}(x,x') = -i\theta(t-t')\langle G|\left[J_\mu(x), J_\nu(x')\right]|G\rangle$$

1. Use the equal time commutation relations of the fermion operators $\psi_\sigma(\vec{x})$ and $\psi^\dagger_\sigma(\vec{x})$ to derive the identity:

   $$\langle G|\left[J_k(\vec{x},t), J_0(\vec{x}\,',t)\right]|G\rangle = -i\frac{e\hbar}{m}\partial'_k\left(\bar{\rho}(\vec{x}\,')\delta^{(3)}(\vec{x}-\vec{x}\,')\right) \tag{2}$$

   where $\bar{\rho}(\vec{x})$ the ground state expectation value of the local charge density.

2. Show that the retarded 4-current correlation function $D^R_{\mu\nu}(x,x')$ obeys the conservation law

   $$\partial^\nu D^R_{\mu\nu}(x,x') = \begin{cases} 0 & \mu = 0 \\ -\frac{e\hbar}{m}\partial'_k\left(\bar{\rho}(x')\delta^{(3)}(\vec{x}-\vec{x}\,')\right) & \mu = k \end{cases} \tag{3}$$

3. Use the conservation law that you just derived to derive a relationship between the Fourier transform of the longitudinal spatial components of the current-current retarded correlation function, $D^R_{k\ell}(\vec{p},\omega)$, and the (Fourier transform) of the retarded density-density correlation function, $D^R_{00}(\vec{p},\omega)$.

4. Find an expression relating the longitudinal conductivity at wave vector $\vec{k}$ and frequency ω to the longitudinal components of the current correlation function.

2 STM

In this problem we will discuss a model representing the measurement of the local density of states of a Fermi system by Scanning Tunneling Microscopy (STM). The general setup that we will use is easily generalized for any other tunneling probe. We will assume that the Fermi system to be probed has an unspecified Hamiltonian H. We will denote by $\psi(\vec{x},t)$ the fermion (electron) operator (for this system) which destroys an electron from location $\vec{x}$ at time t. For simplicity we will ignore spin. We will investigate the properties of this unknown system by tunneling electrons from an external reservoir at a coordinate $\vec{x}_0$ on the surface of the system.This is the setup of an STM.

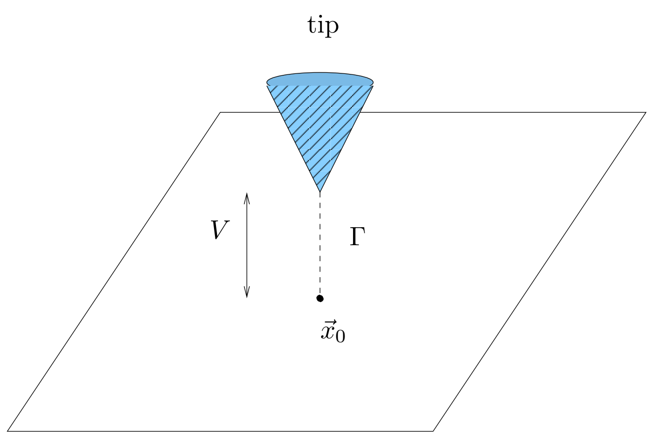

Figure 1: Tunneling from an STM tip at a point $\vec{x}_0$ with amplitude Γ. The voltage V is the difference of the Fermi energies in the tip and the system.

A scanning tunneling microscope works as follows. The observer scans the surface of the system with a scanning tip, which is a reservoir of electrons.We will represent the tip by a system of free fermions with Hamiltonian H_{tip}. The actual form of H_{tip} is not needed for what we will do below. the only quantity of interest is the spectral function of the fermions in the tip. We will represent the fermion operator for electrons in the tip by $\varphi(\vec{x},t)$.

In the figure we placed the tip above a point $\vec{x}_0$ on the plane. The tip is sufficiently far away from the system so that the tip and the system can be considered to be weakly coupled at the *point contact* $\vec{x}_0$. We will represent this interaction by a *tunneling Hamiltonian*

$$\begin{aligned} H_{\text{tunnel}} =& \Gamma \left(e^{-ieVt/\hbar} \psi^\dagger(\vec{x}_0) \varphi(\vec{x}_0) + \text{h.c.} \right) \\ \equiv& \Gamma e^{-ieVt/\hbar} A(\vec{x}_0) + \text{h.c.} \end{aligned}$$

which defines the tunneling operator $A(\vec{x}) \equiv \psi^\dagger(\vec{x})\varphi(\vec{x})$. Here, the voltage $V = E_F^{\text{tip}} - E_F^{\text{2DEG}} > 0$ is the difference of the Fermi energy of the tip and the Fermi energy of the 2DEG; $\varphi(\vec{x})$ is an operator which removes an electron from the tip and $\varphi^\dagger(\vec{x})$ is its adjoint. The tunneling amplitude Γ is weak enough so it can be treated perturbatively.

The (Heisenberg) operator that measures the tunneling current between the tip and the 2DEG at $\vec{x}_0$ is

$$\mathcal{I}_{\text{tunnel}}(\vec{x}_0, t) = -ie\Gamma \left(A(\vec{x}_0, t) e^{-ieVt/\hbar} - A^\dagger(\vec{x}_0, t) e^{-ieVt/\hbar} \right)$$

1. Use Linear Response Theory in the tunneling amplitude Γ to show that the expectation value of the tunneling current is

$$I(t) = \langle \mathcal{I}_{\text{tunnel}}(t) \rangle = \frac{2e\Gamma^2}{\hbar} \int dt' \, \text{Im} \Big\{ e^{-ieV(t-t')/\hbar} \, D_A^R(t-t'; \vec{x}_0) \Big\}$$

   where

$$D_A^R(t-t'; \vec{x}_0) = -i\theta(t-t') \, \langle \left[A(\vec{x}_0, t), A^\dagger(\vec{x}_0, t') \right] \rangle$$

   is the retarded Green function of the tunneling operator $A(\vec{x}_0, t)$.

2. Use the time Fourier transform $\widetilde{D}_A^R(\vec{x}_0, \omega)$ of the retarded Green function of the tunneling operator to show that the expectation value of the tunneling current is constant (in time) and given by

$$I(t) = \frac{2e\Gamma^2}{\hbar} \, \text{Im} \, \widetilde{D}_A^R(\vec{x}_0, -\frac{eV}{\hbar})$$

3. Use the spectral relations discussed in class (Fluctuation-Dissipation Theorem) to show that

$$\text{Im} \, \widetilde{D}_A^R(\vec{x}_0, \omega) = -\frac{1}{2} \left(1 - e^{-\beta\hbar\omega} \right) J_1^A(\vec{x}_0, \omega)$$

   where $J_1^A(\vec{x}_0, \omega)$ is the spectral function of the tunneling operator

$$J_1^A(\vec{x}_0, \omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; G_A^>(\vec{x}_0; t) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \; \langle A(\vec{x}_0, t) \; A^\dagger(\vec{x}_0, 0) \rangle$$

4. Show that, at $\Gamma = 0$, the correlation function of the tunneling operator $G_A^>(\vec{x}_0, t-t' = \langle A(\vec{x}_0, t) A^\dagger(\vec{x}_0, t') \rangle$ factorizes

$$G_A^>(\vec{x}_0, t-t') = G_{\text{system}}^<(\vec{x}_0, t-t')^* \;\; G_{\text{tip}}^>(\vec{x}_0, t-t')$$

   where $G_{\text{system}}^<(\vec{x}_0, t-t')$ and $G_{\text{tip}}^>(\vec{x}_0, t-t')$ are *electron* (fermion) correlation functions for the system and the tip respectively.

5. Use this relation you just derived to show that the electron spectral functions in the system and the tip and the tunneling operator spectral function are related by

$$J_1^A(\vec{x}_0,\omega) = \int \frac{d\Omega}{2\pi} e^{-\beta\hbar\Omega}\; J_1^{\text{system}}(\vec{x}_0,\Omega)^*\;\; J_1^{\text{tip}}(\vec{x}_0,\omega+\Omega)$$

6. Let $\rho(\vec{x};\omega)$ be the exact one-particle local density of states (LDOS)

$$\rho(\vec{x};\omega) \equiv -\frac{1}{\pi}\ \text{Im}\ \widetilde{G}^R(\vec{x},\vec{x};\omega)$$

We will denote by $\rho_{\text{tip}}(\vec{x}_0;\omega)$ and $\rho_{\text{system}}(\vec{x}_0;\omega)$ the LDOS at $\vec{x}_0$ of the electrons in the tip and the system respectively. Use the relation between the electron spectral function and the imaginary part of the corresponding retarded Green function to show that the expectation value of the *tunneling current* I_t is given by

$$\begin{aligned} I_t = \langle \mathcal{I}_{\text{tunnel}} \rangle =& e\frac{2\pi}{\hbar}\Gamma^2\ (e^{\beta eV}-1) \\ & \times \int d\Omega\ \rho_{\text{tip}}(\vec{x}_0;\Omega-\frac{eV}{\hbar})\ \rho_{\text{system}}(\vec{x}_0;\Omega)\ f(\beta\hbar\Omega)\left[1-f\left(\beta\hbar(\Omega-\frac{eV}{\hbar})\right)\right] \end{aligned}$$

where $f(z)$ is the Fermi function

$$f(z) = \frac{1}{e^z+1}$$

7. Show that at $T=0$ the tunneling current is equal to

$$I_t = e\frac{2\pi}{\hbar}\Gamma^2 \int_0^{eV/\hbar} d\Omega\ \rho_{\text{tip}}(\vec{x}_0;\Omega-\frac{eV}{\hbar})\ \rho_{\text{system}}(\vec{x}_0;\Omega)$$

8. We will now assume that the DOS of the tip is constant and independent of the energy (within the allowed spectrum) and given by $\rho_{\text{tip}}(\vec{x}_0;\omega) \equiv \rho_{\text{tip}}(0)$, its value at the Fermi energy. Show that the *differential tunneling conductance* $G_t(V) = \frac{dT_t}{dV}$ is given by

$$G_t(V) = \frac{dI_t}{dV} = \frac{2\pi e^2}{\hbar^2}\Gamma^2\rho_{\text{tip}}(0)\ \rho_{\text{system}}(\vec{x}_0;\frac{eV}{\hbar})$$

This shows that the tunneling conductance measures the local DOS of the probed system.

9. Use these results to determine the voltage dependence of the tunneling conductance at $T=0$ for a system with the following possible LDOS: a) $\rho_{\text{system}}(\vec{x}_0,\omega) = \rho_{\text{system}}(0)$ constant, b) $\rho_{\text{system}}(\vec{x}_0,\omega) \propto |\omega|^\alpha$ where α is an exponent (a *power law*), and c) $\rho(\vec{x}_0,\omega) \propto \theta(|\omega|-\Delta/\hbar)(|\omega|-\Delta/\hbar)^\nu$, a *gap* Δ, where ν is another exponent. What are the effects of a non-zero temperature in each case?