Solution 3

November 10, 2015

1 The Density and Current Correlation Function

1.1

Using the definition of J_0 and $\vec{J}$ operators, we can calculate

$$\langle G|[J_k(\vec{x},t), J_0(\vec{x}',t)]|G\rangle = \langle G|[J_k(\vec{x}), J_0(\vec{x}')]|G\rangle \tag{1}$$

$$= \langle G|\left[\frac{ie\hbar}{2m}\sum_\sigma(\psi_\sigma^\dagger(x)\partial_k\psi_\sigma(x) - \partial_k\psi_\sigma^\dagger(x)\psi_\sigma(x)) + \frac{e^2}{mc}A(x)\sum_\sigma\psi_\sigma^\dagger(x)\psi_\sigma(x), e\sum_{\sigma'}\psi_{\sigma'}^\dagger(x')\psi_\sigma(x')\right]|G\rangle \tag{2}$$

For the first two terms,

$$\begin{aligned}&\langle G|\left[\frac{ie\hbar}{2m}\sum_\sigma(\psi_\sigma^\dagger(x)\partial_k\psi_\sigma(x) - \partial_k\psi_\sigma^\dagger(x)\psi_\sigma(x)), e\sum_{\sigma'}\psi_{\sigma'}^\dagger(x')\psi_\sigma(x')\right]|G\rangle \\ &= -\frac{ie^2\hbar}{2m}\big(2\rho(x')\partial_k\delta^3(x-x') - \partial_k'\rho(x')\delta^3(x-x')\big) \\ &= \frac{ie^2\hbar}{2m}(-\partial_k'(\rho(x')\delta^3(x-x')))\end{aligned} \tag{3}$$

For the third term

$$
\begin{aligned}
&\langle G| \left[\frac{e^2}{mc} A(x) \sum_{\sigma} \psi^\dagger_\sigma(x)\psi_\sigma(x), e\sum_{\sigma'} \psi^\dagger_{\sigma'}(x')\psi_{\sigma'}(x') \right] |G\rangle \\
&= \sum_{\sigma,\sigma'} \frac{e^3}{mc} \langle G| \left[\psi^\dagger_\sigma(x)\psi_\sigma(x), \psi^\dagger_{\sigma'}(x')\psi_{\sigma'}(x') \right] |G\rangle \\
&= 0
\end{aligned} \tag{4}
$$

Hence we have

$$
\langle G|[J_k(\vec{x},t), J_0(\vec{x}',t)]|G\rangle = -\frac{ie^2\hbar}{2m}\partial'_k \left(\rho(x')\delta^3(x-x')\right) \tag{5}
$$

1.2

Notice that

$$
\begin{aligned}
\partial^{\prime,\nu} D^R_{\mu,\nu}(x,x') &= \partial^{\prime,\nu} \left(-i\theta(t-t')\langle G|[J_\mu(x), J_\nu(x')]|G\rangle\right) \\
&= \partial^{\prime,\nu} \left(-i\theta(t-t')\right) \langle G|[J_\mu(x), J_\nu(x')]|G\rangle \\
&= \partial^{\prime,0} \left(-i\theta(t-t')\right) \langle G|[J_\mu(x), J_0(x')]|G\rangle \\
&= -i\delta(t-t')\langle G|[J_\mu(x), J_0(x')]|G\rangle
\end{aligned} \tag{6}
$$

If $\mu = 0$, we have

$$
\partial^{\prime,\nu} D^R_{0,\nu}(x,x') = -i\delta(t-t')\langle G|[J_0(x), J_0(x')]|G\rangle = 0 \tag{7}
$$

If $\mu = k$, according to the result in Eq.(5), we have

$$
\partial^{\prime,\nu} D^R_{k,\nu}(x,x') = -\frac{e^2\hbar}{2m}\partial'_k \left(\rho(x')\delta^3(x-x')\right) \tag{8}
$$

Hence,

$$
\partial^{\prime,\nu} D^R_{\mu,\nu}(x,x') = \begin{cases} 0, & \mu = 0 \\ -\frac{e^2\hbar}{2m}\partial'_k \left(\rho(x')\delta^3(x-x')\right), & \mu = k \end{cases} \tag{9}
$$

1.3

From the above equation, if $\mu = 0$, we have

$$\partial'^{,0} D^R_{00}(x,x') - \partial'^{,k} D^R_{0k}(x,x') = 0 \tag{10}$$

Using Fourier transformation, we have

$$-i\omega D^R_{00}(p,\omega) - ip_k D^R_{0k}(p,\omega) = 0 \tag{11}$$

If $\mu = k$,

$$\partial'^{,\nu} D^R_{k,\nu}(x,x') = -\frac{e^2\hbar}{2m}\partial'_k \left(\rho(x')\delta^3(x-x')\right) \tag{12}$$

Assume that ρ is a constant, we have

$$\partial^0 D^R_{k0}(x,x') - \partial^l D^R_{kl}(x,x') = -\frac{e^2\hbar\rho}{m}\partial'_k(\delta^{(3)}(x-x')) \tag{13}$$

In the momentum space, it becomes

$$-i\omega D^R_{k0}(p,\omega) - ip_l D^R_{kl}(p,\omega) = -\frac{e^2\hbar\rho}{m} ip_k \tag{14}$$

Combined with result in Eq.(11), we have

$$\omega^2 D^R_{00}(p,\omega) - p_l p_k D^R_{kl}(p,\omega) = -\frac{e^2\hbar\rho}{m}\vec{p}^2 \tag{15}$$

Notice that

$$D^R_{lk}(p,\omega) = D^R_{\perp}(p,\omega)\frac{p_l p_k}{p^2} + D^R_{\parallel}(p,\omega)\left(\frac{p_l p_k}{p^2} - \delta_{lk}\right) \tag{16}$$

Substituting the above equation into Eq.(15), we have

$$\omega^2 D^R_{00}(p,\omega) - \vec{p}^2 D^R_{\parallel}(p,\omega) = -\frac{e^2\hbar\rho}{m}\vec{p}^2 \tag{17}$$

1.4

From the lecture note, we have

$$\vec{E}_{ext} = \left(\frac{e^2\rho}{mc^2}\mathbb{I} - \frac{1}{\hbar}D^R + p\otimes p - \vec{p}^2\mathbb{I} + \omega^2\mathbb{I} \right)^{-1} (p\otimes p - \vec{p}^2\mathbb{I} + \omega^2\mathbb{I})\vec{E} \quad (18)$$

$$i\omega\vec{J}_{ind} = \left(\frac{1}{\hbar}D^R - \frac{e^2\rho}{mc^2}\mathbb{I} \right) \left(\frac{e^2\rho}{mc^2}\mathbb{I} - \frac{1}{\hbar}D^R + p\otimes p - \vec{p}^2\mathbb{I} + \omega^2\mathbb{I} \right)^{-1}$$
$$\times (p\otimes p - \vec{p}^2\mathbb{I} + \omega^2\mathbb{I})\vec{E} \quad (19)$$

Using $\vec{J} = \sigma\vec{E}$, we have

$$i\omega\sigma(p,\omega) = \left(\frac{1}{\hbar}D^R - \frac{e^2\rho}{mc^2}\mathbb{I} \right) \left(\frac{e^2\rho}{mc^2}\mathbb{I} - \frac{1}{\hbar}D^R + p\otimes p - \vec{p}^2\mathbb{I} + \omega^2\mathbb{I} \right)^{-1}$$
$$\times (p\otimes p - \vec{p}^2\mathbb{I} + \omega^2\mathbb{I}) \quad (20)$$

Breaking σ into longitudinal and transverse components,

$$\sigma_{ij} = \sigma_\parallel \frac{p_i p_j}{p^2} + \sigma_\perp \left(\frac{p_i p_j}{p^2} - \delta_{ij} \right) \quad (21)$$

this leads to

$$p_i \sigma_{ij} p_j = \sigma_\parallel p^2 \quad (22)$$

Thus we have

$$\begin{aligned} i\omega\sigma_\parallel &= \frac{p_i p_j}{\vec{p}^2} i\omega\sigma_{ij} \\ &= \left(\frac{1}{\hbar}D^R_\parallel - \frac{e^2\rho}{mc^2} \right) \left(1 - \frac{mc^2}{e^2\rho}\left(-\frac{1}{\hbar}D^R_\parallel + \omega^2 \right) + \cdots \right) \frac{mc^2}{e^2\rho}\omega^2 \\ &\approx \left(\frac{1}{\hbar}D^R_\parallel - \frac{e^2\rho}{mc^2} \right) \frac{1}{1 + \frac{mc^2}{e^2\rho}\left(-\frac{1}{\hbar}D^R_\parallel + \omega^2 \right)} \frac{mc^2}{e^2\rho}\omega^2 \end{aligned} \quad (23)$$

Hence,

$$\sigma_\parallel(\vec{k},\omega) = -i\omega \left(\frac{\frac{1}{\hbar}D^R_\parallel(k,\omega) - \frac{e^2\rho}{mc^2}}{\frac{e^2\rho}{mc^2} - \frac{1}{\hbar}D^R_\parallel(\vec{k},\omega) + \omega^2} \right) \quad (24)$$

2 STM

2.1

According to the linear response theory,

$$I(t) = \frac{i}{\hbar}\int_{-\infty}^{t} dt'\theta(t-t')\langle[H(t'), \mathcal{I}_{tunnel}(t)]\rangle \tag{25}$$

where

$$H(t') = \Gamma e^{-ieVt'/\hbar} A(x_0, t') + h.c. \tag{26}$$

and

$$\mathcal{I}_{tunnel}(t) = -ie\Gamma\left(A(x_0, t)e^{-ieVt/\hbar} - A^\dagger(x_0, t)e^{ieVt/\hbar}\right) \tag{27}$$

Substituting the above two equations into $I(t)$, we have

$$\begin{aligned}
I(t) = \frac{e\Gamma^2}{\hbar}\int dt'\theta(t-t')\Big(&e^{-ieV(t+t')/\hbar}\langle[A(x_0,t'), A(x_0,t)]\rangle \\
- e^{ieV(t-t')/\hbar}\langle[A(x_0,t'), A^\dagger(x_0,t)]\rangle &+ e^{-ieV(t-t')/\hbar}\langle[A^\dagger(x_0,t'), A(x_0,t)]\rangle \\
e^{-ieV(t+t')/\hbar}\langle[A^\dagger(x_0,t'), A^\dagger(x_0,t)]\rangle\Big)& \\
= \frac{ie\Gamma^2}{\hbar}\int dt'(e^{-ieV(t-t')/\hbar}D^R_A(t-t',x_0))^* &- e^{-ieV(t-t')/\hbar}D^R_A(t-t',x_0)
\end{aligned} \tag{28}$$

where $D^R_A(t-t', x_0)$ is the retarded Green function

$$D^R_A(t-t', x_0) = -i\theta(t-t')\langle\left[A(x_0,t), A^\dagger(x_0,t')\right]\rangle \tag{29}$$

Thus we show that

$$I(t) = \frac{2e\Gamma^2}{\hbar}\int dt'\mathrm{Im}\left\{e^{-ieV(t-t')/\hbar}D^R_A(t-t',x_0)\right\} \tag{30}$$

2.2

Using Fourier transformation of $D_A^R(t-t', x_0)$, we have

$$\begin{aligned} I(t) &= \frac{2e\Gamma^2}{\hbar}\int dt' \text{Im}\left\{e^{-ieV(t-t')/\hbar}D_A^R(t-t',x_0)\right\} \\ &= \frac{2e\Gamma^2}{\hbar}\text{Im}\left(\int dt'\int \frac{d\omega}{2\pi}e^{-i\omega(t-t')}e^{-ieV(t-t')/\hbar}\widetilde{D}_A^R(x_0,\omega)\right) \\ &= \frac{2e\Gamma^2}{\hbar}\text{Im}\left(\int dt'\int \frac{d\omega}{2\pi}2\pi\delta(eV/\hbar+\omega)e^{-i\omega t}e^{-ieVt/\hbar}\widetilde{D}_A^R(x_0,\omega)\right) \\ &= \frac{2e\Gamma^2}{\hbar}\text{Im}\widetilde{D}_A^R(x_0,-\frac{eV}{\hbar}) \end{aligned} \tag{31}$$

2.3

Using the Fluctuation-Dissipation Theorem, we have

$$J_1(k,\omega) = -\frac{2}{1-e^{-\beta\hbar\omega}}\text{Im}D^R(k,\omega) \tag{32}$$

where

$$J_1(k,\omega) = \int dt\int d^3(x-x')e^{i(-k\cdot(x-x')+\omega t)}\langle O(x,t)O^\dagger(x,0)\rangle \tag{33}$$

Undo the Fourier transformation, we have

$$J_1(x,x',\omega) = -\frac{2}{1-e^{-\beta\hbar\omega}}\text{Im}D^R(x,x',\omega) \tag{34}$$

Notice that

$$\langle O(x,t)O^\dagger(x',0)\rangle = G^>(x_0,t) \tag{35}$$

where $O(x,t) = A(x,t)$. Hence we have

$$J_1^A(x_0,\omega) = -\frac{2}{1-e^{-\beta\hbar\omega}}\text{Im}\widetilde{D}_A^R(x_0,\omega) \tag{36}$$

2.4

The correlation function

$$
\begin{aligned}
&G_A^>(x_0, t-t') = \langle A(x_0,t)A^\dagger(x_0,t')\rangle \\
&= \langle e^{i(H_{tip}+H)t/\hbar}A(x_0)e^{-i(H_{tip}+H)t/\hbar}e^{i(H_{tip}+H)t'/\hbar}A^\dagger(x_0)e^{-i(H_{tip}+H)t'/\hbar}\rangle \\
&= \langle e^{i(H_{tip}+H)t/\hbar}\psi^\dagger(x_0)\varphi(x_0)e^{-i(H_{tip}+H)t/\hbar}e^{i(H_{tip}+H)t'/\hbar}\varphi^\dagger(x_0)\psi(x_0)e^{-i(H_{tip}+H)t'/\hbar}\rangle \\
&= \langle e^{iHt/\hbar}\psi^\dagger(x_0)e^{iH(t'-t)/\hbar}\psi(x_0)e^{-iHt'/\hbar}e^{iH_{tip}t/\hbar}\varphi(x_0)e^{iH_{tip}(t'-t)/\hbar}\varphi^\dagger(x_0)e^{-iH_{tip}t'/\hbar}\rangle
\end{aligned} \tag{37}
$$

When $\Gamma = 0$, there is no tunneling and so we can separate the ground state into the tip and the system ground state

$$
|G\rangle = |G_{sys}\rangle|G_{tip}\rangle \tag{38}
$$

Hence

$$
\begin{aligned}
&G_A^>(x_0, t-t') = \langle G_{sys}|\psi^\dagger(x_0,t')\psi(x_0,t)|G_{sys}\rangle^*\langle G_{tip}|\varphi(x_0,t)\varphi^\dagger(x_0,t')|G_{tip}\rangle \\
&= G_{sys}^<(x_0,t-t')^*G_{tip}^>(x_0,t-t')
\end{aligned} \tag{39}
$$

2.5

Notice that

$$
\begin{aligned}
J_1^A(x_0,\omega) &= \int_{-\infty}^{\infty} dte^{i\omega t}G_A^>(x_0,t) \\
&= \int_{-\infty}^{\infty} dte^{i\omega t}G_{sys}^<(x_0,t)^*G_{tip}^>(x_0,t) \\
&= \int_{-\infty}^{\infty} dte^{i\omega t}\int\frac{d\omega'}{2\pi}e^{i\omega' t}J_2^{sys}(x_0,\omega')\int\frac{d\omega'' t}{2\pi}e^{-i\omega'' t}J_1^{tip}(x_0,\omega'') \\
&= \int_{-\infty}^{\infty} dte^{i\omega t}\int\frac{d\omega'}{2\pi}e^{i\omega' t}e^{-\beta\hbar\omega'}J_1^{sys}(x_0,\omega')\int\frac{d\omega'' t}{2\pi}e^{-i\omega'' t}J_1^{tip}(x_0,\omega'') \\
&= \int\frac{d\omega'}{2\pi}J_1^{sys}(x_0,\omega')J_1^{tip}(x_0,\omega+\omega')e^{-\beta\hbar\omega'}
\end{aligned} \tag{40}
$$

Hence we have

$$
J_1^A(x_0,\omega) = \int\frac{d\Omega}{2\pi}J_1^{sys}(x_0,\Omega)J_1^{tip}(x_0,\omega+\Omega)e^{-\beta\hbar\Omega} \tag{41}
$$

where $\Omega = \omega'$.

2.6

According to question 2.2, we have

$$
\begin{aligned}
I_t &= \frac{2e\Gamma^2}{\hbar}\mathrm{Im}\widetilde{D}_A^R(x_0, -\frac{eV}{\hbar}) \\
&= -\frac{e\Gamma^2}{\hbar}\left(1 - e^{-\beta\hbar(-\frac{eV}{\hbar})}\right) J_1^A(x_0, -\frac{eV}{\hbar}) \\
&= \frac{e\Gamma^2}{\hbar}(e^{\beta eV} - 1)\int \frac{d\Omega}{2\pi}e^{-\beta\hbar\Omega}\frac{2}{1+e^{-\beta\hbar\Omega}}\mathrm{Im}\widetilde{G}_{sys}^R(x_0, x_0, \Omega)\frac{2}{1+e^{-\beta\hbar(\Omega-\frac{eV}{\hbar})}}\mathrm{Im}\widetilde{G}_{tip}^R(x_0, x_0, \Omega - \frac{eV}{\hbar})
\end{aligned} \tag{42}
$$

Notice that

$$
\rho_{tip}(x, \omega) = -\frac{1}{\pi}\widetilde{G}_{tip}^R(x, x, \omega) \tag{43}
$$

$$
\rho_{sys}(x, \omega) = -\frac{1}{\pi}\widetilde{G}_{sys}^R(x, x, \omega) \tag{44}
$$

We have

$$
\begin{aligned}
I_t &= \frac{e\Gamma^2}{\hbar}(e^{\beta eV} - 1)\int \frac{d\Omega}{2\pi}e^{-\beta\hbar\Omega}\frac{2}{1+e^{-\beta\hbar\Omega}}\pi^2\rho_{sys}(x_0, \Omega)\frac{2}{1+e^{-\beta\hbar(\Omega-\frac{eV}{\hbar})}}\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar}) \\
&= \frac{2\pi e\Gamma^2}{\hbar}(e^{\beta eV} - 1)\int d\Omega\rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar})\frac{1}{e^{\beta\hbar\Omega}+1}\left[1 - \frac{1}{1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}}\right] \\
&= \frac{2\pi e\Gamma^2}{\hbar}(e^{\beta eV} - 1)\int d\Omega\rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar})f(\beta\hbar\Omega)\left[1 - f\left(\beta\hbar(\Omega - \frac{eV}{\hbar})\right)\right]
\end{aligned} \tag{45}
$$

where

$$
f(z) = \frac{1}{e^z + 1} \tag{46}
$$

2.7

Notice that

$$
\begin{aligned}
I_t &= \frac{2\pi e\Gamma^2}{\hbar}(e^{\beta eV} - 1)\int d\Omega\rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar})f(\beta\hbar\Omega)\left[1 - f\left(\beta\hbar(\Omega - \frac{eV}{\hbar})\right)\right] \\
&= \frac{2\pi e\Gamma^2}{\hbar}\int d\Omega\rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar})\left[\frac{1}{1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}} - \frac{1}{e^{\beta\hbar\Omega}+1}\right]
\end{aligned} \tag{47}
$$

At $T = 0$, we have

$$\begin{aligned} I_t &= \frac{2\pi e\Gamma^2}{\hbar} \int d\Omega \rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar}) \left[\theta(\frac{eV}{\hbar} - \Omega) - \theta(-\Omega)\right] \\ &= \frac{2\pi e\Gamma^2}{\hbar} \int_0^{\frac{eV}{\hbar}} d\Omega \rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar}) \end{aligned} \tag{48}$$

2.8

Assuming that $\rho_{tip}(x_0, \omega) = \rho_{tip}(0)$, we have

$$\begin{aligned} G_t(V) &= \frac{dI_t}{dV} \\ &= \frac{d}{dV}\left(\frac{2\pi e\Gamma^2}{\hbar} \int_0^{\frac{eV}{\hbar}} d\Omega \rho_{sys}(x_0, \Omega)\rho_{tip}(x_0, \Omega - \frac{eV}{\hbar})\right) \\ &= \frac{2\pi e\Gamma^2}{\hbar}\rho_{tip}(0)\rho_{sys}(x_0, \frac{eV}{\hbar})\frac{d}{dV}\left(\frac{eV}{\hbar}\right) \\ &= \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\rho_{sys}(x_0, \frac{eV}{\hbar}) \end{aligned} \tag{49}$$

2.9

2.9.1 $T = 0$

(a) If $\rho_{sys}(x_0, \omega) = \rho_{sys}(0)$, according to the result in Eq.(49), we have

$$G_t(V) = \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\rho_{sys}(0) \tag{50}$$

(b) If $\rho_{sys}(x_0, \omega) \propto |\omega|^\alpha$, according to the result in Eq.(49), we have

$$\begin{aligned} G_t(V) &= \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\rho_{sys}(x_0, \frac{eV}{\hbar}) \\ &\propto \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\left|\frac{eV}{\hbar}\right|^\alpha \\ &= \frac{2\pi\Gamma^2 e^{2+\alpha}}{\hbar^{2+\alpha}}\rho_{tip}(0)V^\alpha \end{aligned} \tag{51}$$

(c) If $\rho_{sys}(x_0,\omega) \propto \theta(|\omega| - \Delta/\hbar)(|\omega| - \Delta/\hbar)^\nu$, we have

$$
\begin{aligned}
G_t(V) &= \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\rho_{sys}(x_0,\frac{eV}{\hbar}) \\
&\propto \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\theta(\left|\frac{eV}{\hbar}\right| - \Delta/\hbar)(\left|\frac{eV}{\hbar}\right| - \Delta/\hbar)^\nu \\
&= \frac{2\pi e^{2+\nu}\Gamma^2}{\hbar^{2+\nu}}\rho_{tip}(0)\theta(V - \Delta/e)(V - \Delta/e)^\nu
\end{aligned}
\tag{52}
$$

2.9.2 $T \neq 0$

Using the result in Eq.(45),

$$
I_t = \frac{2\pi e\Gamma^2}{\hbar}(e^{\beta eV} - 1)\int d\Omega \rho_{sys}(x_0,\Omega)\rho_{tip}(x_0,\Omega - \frac{eV}{\hbar})f(\beta\hbar\Omega)\left[1 - f\left(\beta\hbar(\Omega - \frac{eV}{\hbar})\right)\right]
\tag{53}
$$

We can derive that

$$
\begin{aligned}
G_t(V) = \frac{dI_t}{dV} &= \frac{2\pi e\Gamma^2}{\hbar}\rho_{tip}(0)\int d\Omega \rho_{sys}(x_0,\Omega)\frac{\partial}{\partial V}\left(\frac{1}{1 + e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}}\right) \\
&= \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\int d\Omega \rho_{sys}(x_0,\Omega)\frac{e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}}{\left(1 + e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}\right)^2}
\end{aligned}
\tag{54}
$$

(a) If $\rho_{sys}(x_0,\omega) = \rho_{sys}(0)$, we have

$$
\begin{aligned}
G_t(V) &= \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\int d\Omega \rho_{sys}(x_0,\Omega)\frac{e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}}{\left(1 + e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}\right)^2} \\
&= \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\rho_{sys}(0)\int d\Omega \frac{e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}}{\left(1 + e^{\beta\hbar(\Omega - \frac{eV}{\hbar})}\right)^2} \\
&= \frac{2\pi e^2\Gamma^2}{\hbar^2}\rho_{tip}(0)\rho_{sys}(0)
\end{aligned}
\tag{55}
$$

In this case, the nonzero temperature has no effect on the conductance.
(b) If $\rho_{sys}(x_0,\omega) \propto |\omega|^\alpha$, we have

$$\begin{aligned}
G_t(V) &= \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\int d\Omega\rho_{sys}(x_0,\Omega)\frac{e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}}{\left(1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}\right)^2} \\
&\propto \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\int d\Omega|\Omega|^\alpha\frac{e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}}{\left(1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}\right)^2} \\
&= \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\left(\int_0^\infty d\Omega\Omega^\alpha\frac{e^{\beta\hbar(-\Omega-\frac{eV}{\hbar})}}{1+e^{\beta\hbar(-\Omega-\frac{eV}{\hbar})}}+\int_0^\infty d\Omega\Omega^\alpha\frac{e^{\beta\hbar(-\Omega+\frac{eV}{\hbar})}}{(1+e^{\beta\hbar(-\Omega+\frac{eV}{\hbar})})^2}\right)
\end{aligned} \tag{56}$$

At finite temperature, the simple power law disappears.
(c) If $\rho_{sys}(x_0,\omega) \propto \theta(|\omega|-\Delta/\hbar)(|\omega|-\Delta/\hbar)^\nu$, we have

$$\begin{aligned}
G_t(V) &= \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\int d\Omega\rho_{sys}(x_0,\Omega)\frac{e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}}{\left(1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}\right)^2} \\
&\propto \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\int d\Omega\theta(|\Omega|-\Delta/\hbar)(|\Omega|-\Delta/\hbar)^\nu\frac{e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}}{\left(1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar})}\right)^2}
\end{aligned} \tag{57}$$

We can split the above integral into two parts (1) $\Omega \in [-\infty, -\Delta/\hbar]$ and (2) $\Omega \in [\Delta/\hbar, \infty]$. The final result is

$$G_t(V) \propto \frac{2\pi e^2\Gamma^2\beta}{\hbar}\rho_{tip}(0)\left(\int_0^\infty d\Omega\Omega^\nu\frac{e^{\beta\hbar(-\Omega-\frac{eV}{\hbar}-\Delta/\hbar)}}{(1+e^{\beta\hbar(-\Omega-\frac{eV}{\hbar}-\Delta/\hbar)})^2}+\int_0^\infty d\Omega\Omega^\nu\frac{e^{\beta\hbar(\Omega-\frac{eV}{\hbar}+\Delta/\hbar)}}{(1+e^{\beta\hbar(\Omega-\frac{eV}{\hbar}+\Delta/\hbar)})^2}\right) \tag{58}$$

From the above result, we can see that at finite temperature, we do not have a simple power law anymore. We have an infinite sum over positive frequencies. Furthermore, it is also determined by the temperature and the gap Δ.