Solution 4

December 6, 2015

1 One-dimensional conductors

1.1

In the $g = 0$ limit, the Hamiltonian is

$$H = \int dx \left[R^\dagger(x)(-iv_F\partial_x)R(x) + L^\dagger(x)(iv_F\partial_x)L(x)\right] + \int dx \left[\frac{\Pi^2(x)}{2(M/4K)} + \frac{1}{2\Delta^2(x)}\right] \quad (1)$$

(a) Suppose

$$R(x,t) = \int \frac{dq}{2\pi} R(q,t) e^{i(qx - E(q)t)} \quad (2)$$

where $E(q)$ can be obtained from diagonalizing the Hamiltonian in momentum space and $E(q) = qv_F$.

The propagator for the right-moving electron is

$$\begin{aligned}
&\langle TR(x,t)R^\dagger(x',t')\rangle = \theta(t-t')\langle R(x,t)R^{dag}(x',t')\rangle - \theta(t'-t)\langle R^\dagger(x',t)R(x,t)\rangle \\
&= \int \frac{dqdq'}{(2\pi)^2} e^{iq(x-v_Ft)-iq'(x'-v_Ft')} \left[\theta(t-t')\langle R(q,t)R^\dagger(q',t')\rangle - \theta(t'-t)\langle R^\dagger(q',t')R(q,t)\rangle\right] \\
&= \int \frac{dq}{2\pi} e^{iq(x-v_Ft-x'+v_Ft')} \left[\theta(t-t')\theta(q) - \theta(t'-t)\theta(-q)\right] \quad (3)
\end{aligned}$$

Performing Fourier transformation for the above result, we have

$$S_{RR}(p,\omega) = \int dx \int dt e^{-i[p(x-x')-\omega(t-t')]} \langle TR(x,t)R^\dagger(x',t')\rangle$$
$$= \int dx \int dt e^{-i[p(x-x')-\omega(t-t')]} \int \frac{dq}{2\pi} e^{iq(x-x')-iqv_F(t-t')} \left[\theta(t-t')\theta(q) - \theta(t'-t)\theta(-q)\right]$$
$$= \int_0^\infty dt e^{i(\omega - pv_F + i\epsilon)} t\theta(p) - \int_{-\infty}^0 dt e^{i(\omega - pv_F - i\epsilon)t}\theta(-p)$$
$$= \frac{i\theta(p)}{\omega - pv_F + i\epsilon} + \frac{i\theta(-p)}{\omega - pv_F - i\epsilon} \tag{4}$$

(b) Similarly, we have

$$L(x,t) = \int \frac{dq}{2\pi} R(q,t) e^{i(qx - E(q)t)} \tag{5}$$

where $E(q) = -qv_F$. Therefore we can just take $v_F \to -v_F$ and calculate the propagator for the left-moving electrons similarly

$$\langle TL(x,t)L^\dagger(x',t')\rangle = \int \frac{dq}{2\pi} e^{iq(x+v_Ft-x'-v_Ft')} \left[\theta(t-t')\theta(-q) - \theta(t'-t)\theta(q)\right] \tag{6}$$

Then we have

$$S_{LL}(p,\omega) = \frac{i\theta(-p)}{\omega + pv_F + i\epsilon} + \frac{i\theta(p)}{\omega + pv_F - i\epsilon} \tag{7}$$

(c) The Hamiltonian for phonon is

$$H_\Delta(x) = \frac{1}{2}\frac{4K}{M} \int dx (\Pi^2(x) + \omega_0^2 \Delta^2(x)) \tag{8}$$

where $\omega_0 = \sqrt{M/4K}$.

Define

$$a(x,t) = \int \frac{dq}{2\pi} a(q,t) e^{iqx - t/\omega_0}$$
$$a^\dagger(x',t') = \int \frac{dq'}{2\pi} a^\dagger(q',t') e^{-i(q'x' - t'/\omega_0)} \tag{9}$$

The above Hamiltonian can be written as

$$H_\Delta(x) = \int dx \frac{1}{\omega_0}(a^\dagger(x)a(x) + \frac{1}{2}) \tag{10}$$

The propagator for the phonon field is

$$\begin{aligned}
&\langle T\Delta(x,t)\Delta(x',t')\rangle = \frac{1}{\omega_0}\langle T(a(x,t) + a^\dagger(x,t))(a(x',t') + a^\dagger(x',t'))\rangle \\
&= \frac{1}{2\omega_0}\left[\theta(t-t')\int \frac{dq}{2\pi} e^{iq(x-x')-i(t-t')/\omega_0} + \theta(t'-t)\int \frac{dq}{2\pi} e^{-iq(x-x')+i(t-t')/\omega_0}\right]
\end{aligned} \tag{11}$$

Hence we have

$$\begin{aligned}
G_\Delta(p,\omega) &= \int dx \int dt e^{-i[p(x-x')-\omega(t-t')]}\langle T\Delta(x,t)\Delta(x',t')\rangle \\
&= \frac{1}{2\omega_0}\left[\int_0^\infty dt e^{i\omega t} e^{-it/\omega_0} e^{-\epsilon t} - \int_{-\infty}^0 dt e^{i\omega t} e^{it/\omega_0} e^{\epsilon t}\right] \\
&= \frac{i}{2\omega_0}\left[\frac{1}{\omega - 1/\omega_0 + i\epsilon} - \frac{1}{\omega + 1/\omega_0 - i\epsilon}\right] \\
&= \frac{i}{\frac{M}{4K}\omega^2 - 1 + i\epsilon}
\end{aligned} \tag{12}$$

1.2

For the electron-phonon vertex shown in Fig.1, it represents the interaction between electron and phonon, the corresponding interaction terms are

$$\begin{aligned}
&g\Delta(x)L^\dagger R \\
&g\Delta(x)R^\dagger L
\end{aligned} \tag{13}$$

1.3

In the limit $M \to \infty$, the Hamiltonian is

$$H = \int dx \psi^\dagger(x) \begin{pmatrix} -iv_F\partial_x & g\Delta(x) \\ g\Delta(x) & iv_F\partial_x \end{pmatrix} \psi(x) + \int dx \frac{1}{2}\Delta^2 \tag{14}$$

Use the Fourier expansion

$$\psi(x) = \int \frac{dp}{2\pi} \psi(p)^{ipx} \tag{15}$$

The Hamitonian can then be written as

$$H = \sum_{\alpha,\alpha'=R,L} \int \frac{dp}{2\pi} \psi^\dagger_{p,\alpha} \begin{pmatrix} pv_F & g\Delta \\ g\Delta & -pv_F \end{pmatrix}_{\alpha,\alpha'} \psi_{p,\alpha'} + \frac{1}{2} N a_0 \Delta^2 \tag{16}$$

This Hamiltonian can be diagonalized by defining

$$\begin{aligned} a_p &= \sqrt{\frac{E+pv_F}{2E}} R_p + \sqrt{\frac{E-pv_F}{2E}} L_p \\ b_p &= -\frac{E-pv_F}{2E} R_p + \sqrt{\frac{E+pv_F}{2E}} L_p \end{aligned} \tag{17}$$

Therefore, we have

$$H = \int \frac{dp}{2\pi} \begin{pmatrix} a^\dagger_p & b^\dagger_p \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} a_p \\ b_p \end{pmatrix} + \frac{1}{2} N a_0 \Delta^2 \tag{18}$$

where

$$E = \sqrt{p^2 v_F^2 + g^2 \Delta^2} \tag{19}$$

After diagonalizing the Hamiltonian, we see that there are both positive and negative energy excitations. The negative energy excitation looks like excitation from holes and we can perform a particle-hole transformation by taking $b_p = d^\dagger_p$. Then we have

$$H = \int \frac{dp}{2\pi} E(p)(a^\dagger_p a_p + d^\dagger_p d_p - 1) + \frac{1}{2} N a_0 \Delta^2 \tag{20}$$

The ground state is defined by

$$a_p|GS\rangle = 0 \quad d_p|GS\rangle = 0 \tag{21}$$

The ground state energy is

$$E_{gnd} = -\int_{-E_c/p_F}^{E_c/p_F} \frac{dp}{2\pi} \sqrt{p^2 v_F^2 + g^2 \Delta^2} + \frac{1}{2} N a_0 \Delta^2 \tag{22}$$

To minimize the energy E_0, we need to find Δ, which satisfies

$$\frac{dE_{gnd}}{d\Delta} = 0 \tag{23}$$

This leads to

$$Na_0\Delta - \frac{g^2\Delta}{\pi v_F}\text{arcsinh}\left(\frac{E_c v_F}{g|\Delta|p_F}\right) = 0 \tag{24}$$

Hence we have

$$\begin{aligned}\Delta_0 &= \pm\frac{E_c v_F}{g p_F}\frac{1}{\sinh\left(\frac{Na_0\pi v_F}{g^2}\right)} \\ &\approx \pm\frac{2E_c v_F}{g p_F}\exp\left(-\frac{Na_0\pi v_F}{g^2}\right)\end{aligned} \tag{25}$$

1.4

The fermion propagator

$$\begin{aligned}&\langle TR(x,t)L^\dagger(x',t')\rangle = \int\frac{dqdq'}{(2\pi)^2}e^{iqx-iq'x'}\langle TR(q,t)L^\dagger(q',t')\rangle \\ &= \int\frac{dqdq'}{(2\pi)^2}e^{iqx-iq'x'}\left[\theta(t-t')\langle R(q,t)L^\dagger(q',t')\rangle - \theta(t'-t)\langle L^\dagger(q',t')R(q,t)\rangle\right]\end{aligned} \tag{26}$$

According to Eq.(17), we have

$$\begin{aligned}R_p &= \sqrt{\frac{E+pv_F}{2E}}a_p - \sqrt{\frac{E-pv_F}{2E}}b_p \\ L_p &= \sqrt{\frac{E-pv_F}{2E}}a_p + \sqrt{\frac{E+pv_F}{2E}}b_p\end{aligned} \tag{27}$$

Using the above equations, we can show that

$$\langle R(q,t)L^\dagger(q',t')\rangle = \frac{g\Delta}{2E(q)}\delta(q-q')2\pi e^{-iE(q)(t-t')} \tag{28}$$

Simiarly, we have

$$\langle L^\dagger(q',t')R(q,t)\rangle = -\frac{g\Delta}{2E(q)}\delta(q-q')2\pi e^{iE(q)(t-t')} \tag{29}$$

Therefore we have

$$\langle TR(x,t)L^\dagger(x',t')\rangle = \int \frac{dqdq'}{(2\pi)^2} e^{iqx-iq'x'} \left[\theta(t-t')\langle R(q,t)L^\dagger(q',t')\rangle - \theta(t'-t)\langle L^\dagger(q',t')R(q,t)\rangle\right]$$

$$= \int \frac{dq}{2\pi} e^{iq(x-x')} \frac{g\Delta}{2E} \left[\theta(t-t')e^{-iE(q)(t-t')} + \theta(t'-t)e^{iE(q)(t-t')}\right] \tag{30}$$

In momentum space, it becomes

$$S_{RL}(p,\omega) = \int dx \int dt e^{-i[p(x-x')-\omega(t-t')]} \int \frac{dq}{2\pi} e^{iq(x-x')} \frac{g\Delta}{2E} \left[\theta(t-t')e^{-iE(q)(t-t')} + \theta(t'-t)e^{iE(q)(t-t')}\right]$$

$$= \frac{g\Delta}{2E(p)} \left[\int_0^\infty dt e^{i\omega t} e^{-\epsilon t} e^{-iE(p)t} + \int_{-\infty}^0 dt e^{i\omega t} e^{\epsilon t} e^{iE(p)t}\right]$$

$$= \frac{ig\Delta}{2E(p)} \left(\frac{1}{\omega - E(p) + i\epsilon} - \frac{1}{\omega + E(p) - i\epsilon}\right) \tag{31}$$

Thus when $g \neq 0$, the left and right movers are coupled together and there is non-zero component for $S_{RL}(p,\omega)$.

Similarly, we can calculate the fermion propagator for the right mover

$$\langle TL(x,t)R^\dagger(x',t')\rangle = \langle TR(x,t)L^\dagger(x',t')\rangle \tag{32}$$

This leads to

$$S_{LR}(p,\omega) = \frac{ig\Delta}{2E(p)} \left(\frac{1}{\omega - E(p) + i\epsilon} - \frac{1}{\omega + E(p) - i\epsilon}\right) \tag{33}$$

Using the same method, we can show that

$$\langle TR(x,t)R^\dagger(x',t')$$

$$= \int \frac{dq}{2\pi} e^{iq(x-x')} \left[\theta(t-t')\frac{E+qv_F}{2E} e^{-iE(q)(t-t')} - \theta(t'-t)\frac{E-qv_F}{2E} e^{iE(q)(t-t')}\right] \tag{34}$$

In momentum space, it equals to

$$S_{RR}(p,\omega) = \frac{E(p)+pv_F}{2E(p)} \frac{i}{\omega - E(p) + i\epsilon} + \frac{E(p)-pv_F}{2E(p)} \frac{i}{\omega + E(p) - i\epsilon} \tag{35}$$

The fermion propagator for the left mover is

$$\langle TL(x,t)L^\dagger(x',t')\rangle$$
$$= \int \frac{dq}{2\pi} e^{iq(x-x')} \left[\theta(t-t')\frac{E-qv_F}{2E}e^{-iE(q)(t-t')} - \theta(t'-t)\frac{E+qv_F}{2E}e^{iE(q)(t-t')}\right] \tag{36}$$

In momentum space, it equals to

$$S_{LL}(p,\omega) = \frac{E(p)-pv_F}{2E(p)}\frac{i}{\omega - E(p)+i\epsilon} + \frac{E(p)+pv_F}{2E(p)}\frac{i}{\omega + E(p) - i\epsilon} \tag{37}$$

1.5

The energy spectrum is

$$E(p)_\pm = \pm\sqrt{(v_F p)^2 + (g\Delta)^2} \tag{38}$$

There is an energy gap because

$$E(p=0)_+ - E(p=0)_- = 2g\Delta \approx \frac{4E_c v_F}{p_F}\exp\left(-\frac{Na_0\pi v_F}{g^2}\right) \tag{39}$$

The single particle state can be represented as

$$a_p^\dagger|GS\rangle, \quad d_p^\dagger|GS\rangle \tag{40}$$

1.6

Under the global discrete transformation,

$$R(x)\to R(x), \quad L(x)\to -L(x), \quad \Delta(x)\to -\Delta(x), \tag{41}$$

The Hamiltonian becomes

$$H = \sum_{\alpha,\alpha'=R,L}\int\frac{dp}{2\pi}\begin{pmatrix}R^\dagger(x) & -L^\dagger(x)\end{pmatrix}\begin{pmatrix}-iv_F\partial_x & -g\Delta \\ -g\Delta & iv_F\partial_x\end{pmatrix}\begin{pmatrix}R(x) \\ -L(x)\end{pmatrix} + \cdots \tag{42}$$

It is identical to the original Hamiltonian and thus the Hamiltonian is invariant under the global discrete symmetry transformation.

In terms of the lattice model,

$$c[x(n)] = \frac{1}{\sqrt{2a_0}} \left[e^{ip_F x(n)} R + e^{-ip_F x(n)} L \right] \tag{43}$$

under the global $\mathbb{Z}_2$ transformation,

$$\begin{aligned} c'[x(n)] &= \frac{1}{\sqrt{2a_0}} \left[e^{ip_F x(n)} R - e^{-ip_F x(n)} L \right] \\ &= (-i) c[x(n+1)] \end{aligned} \tag{44}$$

where we use $p_F = \pi/2$.

For $u[x(n)]$,

$$u[x(n)] = u_0 + \cos(\pi n)\Delta \tag{45}$$

under the $\mathbb{Z}_2$ transformation,

$$u'[x(n)] = u_0 - \cos(\pi n)\Delta = u[x(n+1)] \tag{46}$$

Therefore this transformation is equivalent to a translation of all the fields by one lattice spacing.

For the CDW order parameter, under the $\mathbb{Z}_2$ transformation, we have

$$O'_{CDW}(x) = -R^\dagger(x)L(x) - L^\dagger(x)R(x) = -O_{CDW}(x) \tag{47}$$

It is odd under the global discrete symmetry.

1.7

The ground state expectation value for O_{CDW} in the $M \to \infty$ limit is

$$\begin{aligned} \langle GS|O_{CDW}|GS\rangle &= \langle GS|R^\dagger(x)L(x) + L^\dagger(x)R(x)|GS\rangle \\ &= -\int_{E_c/p_F}^{E_c/p_F} \frac{dp}{2\pi} \frac{g\Delta_0}{2E(p)} - \int_{E_c/p_F}^{E_c/p_F} \frac{dp}{2\pi} \frac{g\Delta_0}{2E(p)} \\ &= -\int_{-E_c/p_F}^{E_c/p_F} \frac{dp}{2\pi} \frac{g\Delta}{g^2\Delta_0^2 + p^2 v_F^2} = -\frac{N a_0 \Delta_0}{g} \end{aligned} \tag{48}$$

This suggests that the order parameter has non-vanishing expectation value only if $\Delta \neq 0$.

1.8

Using the bosonization method, we have

$$R(x) \sim \frac{1}{\sqrt{2\pi a_0}} : e^{i2\sqrt{\pi}\phi_R(x)} :$$
$$L(x) \sim \frac{1}{\sqrt{2\pi a_0}} : e^{-i2\sqrt{\pi}\phi_L(x)} : \tag{49}$$

where

$$\phi_R(x) = \frac{\beta}{4\sqrt{\pi}}\phi(x) - \frac{\sqrt{\pi}}{\beta}\int_{-\infty}^{x_1} dx_1' \Pi(x_0, x_1')$$
$$\phi_L(x) = \frac{\beta}{4\sqrt{\pi}}\phi(x) + \frac{\sqrt{\pi}}{\beta}\int_{-\infty}^{x_1} dx_1' \Pi(x_0, x_1') \tag{50}$$

We can derive that

$$\begin{aligned} R^\dagger(x)L(x) + L^\dagger(x)R(x) &= \frac{1}{2\pi a_0}e^{-i2\sqrt{\pi}\phi(x)} + \frac{1}{2\pi a_0}e^{i2\sqrt{\pi}\phi(x)} \\ &= \frac{1}{\pi a_0}\cos(2\sqrt{\pi}\phi(x)) \end{aligned} \tag{51}$$

1.9

According to the rule in Eq.(49), the bosonized Hamiltonian is

$$\begin{aligned} H = &\sum_{\alpha,\alpha'=R,L} \int \frac{dp}{2\pi} \begin{pmatrix} R^\dagger(x) & -L^\dagger(x) \end{pmatrix} \begin{pmatrix} -iv_F\partial_x & -g\Delta \\ -g\Delta & iv_F\partial_x \end{pmatrix} \begin{pmatrix} R(x) \\ -L(x) \end{pmatrix} + \int dx \left[\frac{\Pi^2}{2M/4K} + \frac{1}{2}\Delta^2\right] \\ = &\int dx \frac{v_F}{2}\left(\Pi_\phi^2 + (\partial_x\phi)^2\right) + \frac{g\Delta(x)}{\pi a_0}\cos(2\sqrt{\pi}\phi) + \int dx \left[\frac{\Pi^2}{2M/4K} + \frac{1}{2}\Delta^2\right] \end{aligned} \tag{52}$$

This Hamiltonian is invariant under the symmetry $\Delta \to -\Delta$ and $\phi \to \phi + \sqrt{\pi}/2$. This symmetry is equivalent to the chiral symmetry defined in the fermion model and corresponds to two dimerization pattern.

1.10

(a) For the Dirac Hamiltonian

$$H_{Dirac} = \begin{pmatrix} -iv_F\partial_x & g\Delta(x) \\ g\Delta(x) & iv_F\partial_x \end{pmatrix} \tag{53}$$

When $\Delta(x)$ takes a soliton configuration, there is a fermionic zero mode of the form

$$\psi(x) = \begin{pmatrix} R \\ L \end{pmatrix} e^{-f(x)} \tag{54}$$

To find the explicit form of $f(x)$ function, we need to solve the following equation

$$\begin{pmatrix} -iv_F\partial_x & g\Delta(x) \\ g\Delta(x) & iv_F\partial_x \end{pmatrix} \begin{pmatrix} R \\ L \end{pmatrix} e^{-f(x)} = 0 \tag{55}$$

This leads to

$$f'(x) = \frac{ig\Delta(x)L}{v_F R} = -\frac{ig\Delta(x)R}{v_F L} \tag{56}$$

Solving this equation gives $R/L = \pm i$. When $R/L = i$, we have

$$f'(x) = \frac{g\Delta(x)}{v_F} \tag{57}$$

In the thin-soliton approximation $\Delta(x) = \Delta_0 \text{sign}(x)$, we have

$$f(x) = \frac{g}{v_F}\Delta_0|x|, \quad L = \sqrt{\frac{g\Delta_0}{2v_F}} \tag{58}$$

Therefore the zero mode wavefunction is normalizable and equals to

$$\psi(x) = \begin{pmatrix} i \\ 1 \end{pmatrix} \sqrt{\frac{g\Delta_0}{2v_F}} e^{-\frac{g}{v_F}\Delta_0|x|} \tag{59}$$

When $R/L = -i$, $f(x) = -\frac{g}{v_F}\Delta_0|x|$, the solution is not normalizable and is non-physical.
(b) The charge in terms of boson is

$$Q = -e\int dx j_0(x) = -\frac{e}{\sqrt{\pi}}\int dx\partial_x\phi(x) = -\frac{e}{\sqrt{\pi}}(\phi(\infty) - \phi(-\infty)) \tag{60}$$

We know the Hamiltonian is invariant under $\Delta \to -\Delta$ and $\phi \to \phi + \sqrt{\pi}/2$. For the soliton configuration, $\Delta(x = \infty) = -\Delta(x = -\infty)$, thus we have $\phi(-\infty) = \phi(\infty) + \sqrt{\pi}/2$. The charge Q is equal to

$$Q = \frac{e}{2} \tag{61}$$

This demonstrates that the solition carries fractional charge.