Solution to Problem Set No.2  
Principles of Quantum Mechanics

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1 Wave Packets

1.1

\[ 1 = \int_{-\infty}^{\infty} dp \langle \Psi | p \rangle \langle p | \Psi \rangle \]  
\[ = |A|^2 \int_{-\infty}^{\infty} dp \left| e^{-\frac{(p-p_0)^2}{4\sigma}} e^{iap} \right|^2 \]  
\[ = |A|^2 \int_{-\infty}^{\infty} dp e^{-\frac{(p-p_0)^2}{2\sigma}} \]  
\[ = |A|^2 \sqrt{2\pi\sigma} \]  
\[ \Rightarrow A = \sqrt{\frac{1}{2\pi\sigma}} \]  

1.2

Insert a completeness relation \( 1 = \int dp |p\rangle \langle p| \) into \( \langle x|\Psi \rangle \) we have

\[ \langle x|\Psi \rangle = \int_{-\infty}^{\infty} dp \langle x|p\rangle \langle p|\Psi \rangle \]  
\[ = \int_{-\infty}^{\infty} dp \frac{e^{-ipx/h}}{\sqrt{2\pi}} \sqrt{\frac{1}{2\pi\sigma}} e^{-\frac{(p-p_0)^2}{4\sigma}} e^{iap/h} \]  
\[ = \sqrt{\frac{1}{8\pi^3\hbar^2\sigma}} \int_{-\infty}^{\infty} dp e^{-\frac{(p-p_0)^2}{4\sigma}} e^{-\frac{a(x+a)}{\hbar}} \]  
\[ = \sqrt{\frac{\hbar^2}{8\pi^3\sigma}} e^{\frac{ia(x+a)}{\hbar}} \sqrt{\frac{4\pi\sigma}{\hbar^2}} e^{-\frac{a(x+a)^2}{\hbar^2}} \]  
\[ = \sqrt{\frac{2\pi}{\pi\hbar^2}} e^{-\frac{a(x+a)^2}{\hbar^2}} e^{\frac{ia(x+a)}{\hbar}} \]
1.3

We have

\[ \langle \Psi | \hat{P} | \Psi \rangle = \int_{-\infty}^{\infty} dp \ p \ | \langle p | \Psi \rangle |^2 \]
\[ = \sqrt{\frac{1}{2\pi\sigma}} \int_{-\infty}^{\infty} dp \ p e^{-\frac{(p-p_0)^2}{2\sigma}} \]
\[ = p_0 \]  
(11)

and similarly

\[ \langle \Psi | \hat{X} | \Psi \rangle = \int_{-\infty}^{\infty} dx \ x \ | \langle x | \Psi \rangle |^2 \]
\[ = \sqrt{\frac{2\sigma}{\pi\hbar^2}} \int_{-\infty}^{\infty} dx \ x e^{-\frac{2\sigma(x+a)^2}{\hbar^2}} \]
\[ = -a \]  
(12)

The state \( |\Psi\rangle \) represents a Gaussian wave packet centered at \( p_0 \) with spread \( \sigma \) in momentum space. Since the Fourier transformation of a Gaussian is still a Gaussian, it also represents a wave packet centered at \( -a \) and with spread \( \hbar^2 / \sigma \) in position space.

1.4

In order to compute the variance we need \( \langle \Psi | (\hat{P} - \bar{P})^2 | \Psi \rangle \) and \( \langle \Psi | (\hat{X} - \bar{X})^2 | \Psi \rangle \).

\[ \langle \Psi | (\hat{P} - \bar{P})^2 | \Psi \rangle = \sqrt{\frac{1}{2\pi\sigma}} \int_{-\infty}^{\infty} dp \ (p - p_0)^2 e^{-\frac{(p-p_0)^2}{2\sigma}} \]
\[ = \sigma \]  
(13)

\[ \langle \Psi | (\hat{X} - \bar{X})^2 | \Psi \rangle = \sqrt{\frac{2\sigma}{\pi\hbar^2}} \int_{-\infty}^{\infty} dx \ (x + a)^2 e^{-\frac{2\sigma(x+a)^2}{\hbar^2}} \]
\[ = \frac{\hbar^2}{4\sigma} \]  
(14)

Thus we can conclude that

\[ \Delta X \Delta P = \frac{\hbar}{2} \]  
(15)

That is, the Gaussian wave packet exactly satisfy the minimum of uncertainty principle.

2 States in a One-Dimensional Oscillator

2.1

Form problem 1 we see that Gaussian wave packet minimally satisfies the uncertainty principle. The uncertainty (variance) of position operator \( \hat{X} \) in this wave function is

\[ \Delta X = a \]  
(16)
We can infer from the uncertainty principle that the uncertainty of the momentum operator $\hat{P}$ has the lower bound
\[ \Delta P \geq \frac{\hbar}{2\Delta X} = \frac{\hbar}{2a} \]  
(23)

The expectation value of the Hamiltonian in this state is thus
\[ \langle \hat{H} \rangle = \langle \hat{P}^2 \rangle - \frac{m}{2} \omega^2 \langle \hat{X}^2 \rangle \]
\[ = \frac{\Delta P^2}{2m} + \frac{m}{2} \omega^2 \Delta X^2 \]  
(24)

\[ = \frac{\hbar^2}{8ma^2} + \frac{m\omega^2 a^2}{2} \]  
(25)

The energy is minimized when $a^2 = \frac{\hbar}{2m\omega}$, where
\[ E_{\text{min}} = \frac{1}{2} \hbar \omega \]  
(27)
is the ground state energy of a simple harmonic oscillator.

2.2

\[ [\hat{T}, \hat{V}] = \left[ \frac{\hat{P}^2}{2m} - \frac{m}{2} \omega^2 \hat{X}^2 \right] \]  
(28)

\[ = \frac{\omega^2}{4} [\hat{P}^2, \hat{X}^2] \]  
(29)

\[ = \frac{\omega^2}{4} (\hat{P}^2 \hat{X}^2 - \hat{X}^2 \hat{P}^2) \]  
(30)

Sandwich the commutator between state vector $|\Phi\rangle$ and insert two completeness relations, we obtain
\[ \langle \Phi | [\hat{T}, \hat{V}] | \Phi \rangle = \frac{\omega^2}{4} \langle \Phi | [\hat{P}^2, \hat{X}^2] | \Phi \rangle \]
\[ = \frac{\omega^2}{4} \langle \Phi | \hat{P}^2 \hat{X}^2 - \hat{X}^2 \hat{P}^2 | \Phi \rangle \]  
(31)

For position space wave functions we have
\[ \langle \Phi | [\hat{T}, \hat{V}] | \Phi \rangle = \frac{\omega^2}{4} \int dx \Phi^*(x)(-\hbar^2 \frac{d^2}{dx^2} x^2 + x^2 \hbar^2 \frac{d^2}{dx^2}) \Phi(x) \]
\[ = -\frac{\hbar^2 \omega^2}{4} \int dx \Phi^*(x)(2\Phi(x) + 4x \Phi'(x)) \]  
(33)

which shows that the commutator is non-zero.

2.3

The Robertson-Schödinger uncertainty relation of the kinetic energy $\hat{T}$ and of the potential energy $\hat{V}$ operators reads
\[ \Delta T \Delta V \geq \frac{1}{2} \left| \langle \Phi | [\hat{T}, \hat{V}] | \Phi \rangle \right| \]
\[ \geq \frac{\hbar^2 \omega^2}{8} \int dx \Phi^*(x)(2\Phi(x) + 4x \Phi'(x)) \]  
(35)

(36)
The right hand side provides a lower bound to the uncertainty relation.

For the wave function \( \Phi(x) = (2\pi a)^{-1/4}e^{-x^2/4a} \), we obtain

\[
\Delta T \Delta V \geq \frac{\hbar^2 \omega^2}{4\sqrt{2}\pi a} \int dx \ e^{-\frac{x^2}{2a}}(1 - \frac{x^2}{a}) \tag{37}
\]

\[
\geq \frac{\hbar^2 \omega^2}{4} - \frac{\hbar^2 \omega^2}{4} \tag{38}
\]

\[
\geq 0 \tag{39}
\]

The bound that we got here is zero because the wave function \( \Phi(x) \) is an optimal Gaussian wave packet in a Gaussian theory, the linear harmonic oscillator. In the case of this wave function the inequality does not give a useful lower bound since, obviously, the uncertainties are non-zero and, by definition, are positive.

### 3 Equations of Motion of Operators

#### 3.1

<table>
<thead>
<tr>
<th>Schrödinger picture</th>
<th>Heisenberg picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>state ket (</td>
<td>\psi_S(t)\rangle = e^{-iHt/\hbar}</td>
</tr>
<tr>
<td>observable stationary</td>
<td>( \hat{A}_H(t) = e^{iHt/\hbar} \hat{A} e^{-iHt/\hbar} )</td>
</tr>
<tr>
<td>base ket stationary</td>
<td>(</td>
</tr>
</tbody>
</table>

#### 3.2

Heisenburg’s equation of motion for some operator \( \hat{A} \) that has no explicit time dependence reads

\[
\frac{d}{dt} \hat{A}(t) = \frac{i}{\hbar} [\hat{H}, \hat{A}(t)] \tag{40}
\]

The Hamiltonian for the forced linear harmonic operator is

\[
\hat{H} = \hat{P}^2 \frac{2}{2m} + \frac{m\omega^2 \hat{X}^2}{2} + f(t)\hat{X} \tag{41}
\]

Hence we find the equation of motion of \( \hat{X} \) and \( \hat{P} \) to be

\[
\frac{d}{dt} \hat{X}(t) = \frac{\hat{P}(t)}{m} \tag{42}
\]

and

\[
\frac{d}{dt} \hat{P}(t) = -m\omega^2 \hat{X}(t) + f(t) \tag{43}
\]

They take exactly the same form as in classical mechanics.

### 4 Quantum Measurements with Photons

#### 4.1

The state vectors of these polarizations are

\[
|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \ |L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{44}
\]
Proceed to find the corresponding projection operator

\[ \hat{P}_x = |x\rangle \langle x| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{P}_R = |R\rangle \langle R| = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \] (45)

\[ \hat{P}_y = |y\rangle \langle y| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{P}_L = |L\rangle \langle L| = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \] (46)

Which match the definitions given earlier.

4.2

The probabilities of finding the system in a given pure state \(|\psi\rangle\) is

\[ \text{Prob}(|\psi\rangle) = \text{Tr}(\hat{P}_\psi \hat{\rho}) \] (47)

where \(\hat{P}_\psi\) is the projection operator onto \(|\psi\rangle\).

Applying this to the projection operators defined in 1., we find

\[ \text{Prob}(|x\rangle) = a, \quad \text{Prob}(|y\rangle) = b, \quad \text{Prob}(|R\rangle) = \frac{a + b + i(c^* - c)}{2}, \quad \text{Prob}(|L\rangle) = \frac{a + b + i(c - c^*)}{2} \] (48)

4.3

\[ \langle \hat{L}_z \rangle \hat{\rho} = \text{Tr}(\hat{\rho} \hat{L}_z) = \text{Tr}(\hbar \hat{\sigma}_2 \hat{\rho}) = \hbar (c^* - c) \] (50)

(51)

4.4

Notice that

\[ \left( \begin{array}{cc} \cos^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \sin^2 \theta \end{array} \right) = \left( \begin{array}{cc} e^{-i\phi/2} \cos \theta & e^{i\phi/2} \cos \theta \\ e^{i\phi/2} \sin \theta & e^{-i\phi/2} \sin \theta \end{array} \right) \] (52)

\[ = |\psi\rangle \langle \psi| \] (53)

where the pure state \(|\psi\rangle = \left( \begin{array}{c} e^{i\phi/2} \cos \theta \\ e^{-i\phi/2} \sin \theta \end{array} \right)\) is some elliptical polarized state. The expectation value of the helicity can be computed

\[ \langle \hat{L}_z \rangle \hat{\rho} = \hbar (c^* - c) = 2\hbar \sin \phi \sin \theta \cos \theta \] (54)

4.5

Solving for the eigenvalues of \(\hat{\rho}\) we find

\[ \lambda_\pm = \frac{1 \pm \sqrt{1 - 4(ab - cc^*)}}{2} \] (55)
We can compute the value of von Neumann entropy

\[ S = -\lambda_+ \ln \lambda_+ - \lambda_- \ln \lambda_- \quad (56) \]
\[ = -\frac{1 + \sqrt{D}}{2} \ln \left( \frac{1 + \sqrt{D}}{2} \right) - \frac{1 - \sqrt{D}}{2} \ln \left( \frac{1 - \sqrt{D}}{2} \right) \quad (57) \]

where \( D = 1 - 4(ab - cc^*) \)

For pure states, the density matrix has only one unity eigenvalue and others are all zero. Following the definition of von Neumann entropy we can show that

\[ S = -\ln 1 = 0 \quad (58) \]