Physics 580: Quantum Mechanics I
Department of Physics, UIUC
Fall Semester 2017
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Problem Set No. 2:
Principles of Quantum Mechanics
Due Date: October 6, 2017

1 Wave Packets

Consider a quantum system of a single particle in one dimension. We will denote by $\hat{X}$ and $\hat{P}$ the coordinate and the momentum operators for this particle. Let us consider a state $|\Psi\rangle$, a wave packet, whose projection on the eigenstates $|p\rangle$ of the momentum operator is

$$\langle p|\Psi\rangle = A \; e^{-\frac{(p-p_0)^2}{4\sigma}} e^{\frac{ipa}{\hbar}}$$

where p_0, σ and a are parameters.

1. Find the value of the normalization constant A
2. Find the form of this state in the position representation, *i. e.* $\langle x|\Psi\rangle$. Recall that
$$\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$$
3. Find the expectation values $\langle\Psi|\hat{P}|\Psi\rangle$ and $\langle\Psi|\hat{X}|\Psi\rangle$. Give a physical interpretation of the parameters of this state.
4. Calculate the uncertainties $\Delta\hat{X}$ and $\Delta\hat{P}$ for this state. Discuss how is the Heisenberg Uncertainty Principle satisfied for this state.

2 States in a One-Dimensional Oscillator

Consider a one-dimensional linear harmonic oscillator of mass m and angular frequency ω. Let $\hat{H}$ be the Hamiltonian operator of this linear Harmonic oscillator

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{P}^2}{2m} + \frac{m}{2}\omega^2\hat{X}^2$$

Consider the quantum state $|\Phi\rangle$ whose position space representation is

$$\langle x|\Phi\rangle = \Phi(x) = \frac{e^{-\frac{x^2}{4a^2}}}{(2\pi a^2)^{1/4}}$$

1. Use the Uncertainty Principle to calculate the expectation value (the diagonal matrix element in this state) $\langle\Phi|\hat{H}|\Phi\rangle$ For what value of a is this expectation value minimized?

2. Consider again a state $|\Phi\rangle$ and its position space wave function $\Phi(x) = \langle x|\Phi\rangle$. Calculate the matrix element

$$\langle\Phi|[\hat{T},\hat{V}]|\Phi\rangle$$

3. Use this result to derive an uncertainty principle for the kinetic energy operator $\hat{T}$ and the potential energy operator $\hat{V}$ of a linear harmonic oscillator. Find an explicit form of the uncertainty principle if $|\Phi\rangle$ is the state considered in the previous item. What can you conclude about our a priori ability to measure simultaneously the kinetic and potential energies of a particle?

3 Equations of Motion of Operators

Consider once again a one dimensional system with coordinate and momentum operators $\hat{X}$ and $\hat{P}$, in the Schrödinger picture. Let

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X})$$

be the quantum mechanical Hamiltonian operator. Let $\hat{X}(t)$ and $\hat{P}(t)$ be the corresponding operators in the Heisenberg picture (or representation).

1. Write down the relation that exists between the operators in the Schrödinger and the Heisenberg pictures. What is the relation between the states in both pictures?

2. Derive the equation of motion for the Heisenberg operators $\hat{X}(t)$ and $\hat{P}(t)$ for the case of the forced linear harmonic oscillator of angular frequency ω, for which
$$V(\hat{X}) = \frac{m}{2}\omega^2\hat{X}^2 + f(t)\hat{X}$$
where $f(t)$ is a time-dependent classical external force.

4 Quantum Measurements with Photons

Consider an ensemble of photons prepared according to the density matrix $\hat{\rho}$

$$\hat{\rho} = \begin{pmatrix} a & c^* \\ c & b \end{pmatrix},$$

which is a hermitian matrix of trace 1. It is useful to define the following complete set of Hermitian 2×2 matrices

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. Verify that the operators $\hat{P}_x$, $\hat{P}_y$, $\hat{P}_R$ and $\hat{P}_L$, defined by

$$\hat{P}_x = \frac{\left(\hat{I} + \hat{\sigma}_3\right)}{2} \ , \ \hat{P}_y = \frac{\left(\hat{I} - \hat{\sigma}_3\right)}{2} \ , \ \hat{P}_R = \frac{\left(\hat{I} + \hat{\sigma}_2\right)}{2} \ , \ \hat{P}_L = \frac{\left(\hat{I} + \hat{\sigma}_2\right)}{2}$$

   are projection operators which project on states $|x\rangle$, $|y\rangle$, $|R\rangle$ and $|L\rangle$, *i. e.* x, y, R and L polarized photons respectively.

2. Calculate the probability to measure a photon with x, y, R and L polarization for an ensemble with the density matrix $\hat{\rho}$ given above.

3. Calculate the expectation value of the photon helicity $\hat{L}_z = \hbar\hat{\sigma}_2$ in this ensemble.

4. Consider a density matrix with

$$a = \cos^2\theta \ , \ b = \sin^2\theta \ , \ c = e^{i\phi}\sin\theta\cos\theta$$

   where $0 \leq \theta < 2\pi$ and $0 \leq \phi < 2\pi$. Show that with this choice of parameters this density matrix represents a *p*ure state. Which pure state? Calculate the expectation value of the photon helicity that would be measured in this ensemble.

5. The quantity

$$S = -\mathrm{tr}(\hat{\rho}\ln\hat{\rho}) \equiv -\sum_k(\lambda_k \ln\lambda_k)$$

   is known as the von Neumann (entanglement) entropy of the mixed state defined by the density matrix $\hat{\rho}$. Here $\{\lambda_k\}$ are the eigenvalues of the density matrix $\hat{\rho}$, and are real numbers in the interval $[0, 1]$.

   (a) Find an explicit expression for the von Neumann entanglement entropy for the density matrix as a function of the matrix elements a, b and c.

   (b) Show that for a pure state the von Neumann entanglement entropy is zero.