Solution to Problem Set No. 4:
Quantum Mechanics in two and three Dimensions

Simon Lin
November 13, 2017

1 Charged Particle in an Uniform Magnetic Field

1.1

First note that the Hamiltonian can be written as

$$\hat{H} = \frac{1}{2M} \left( \hat{\Pi}^2_x + \hat{\Pi}^2_y \right)$$  \hspace{1cm} (1)

$$= \frac{1}{2M} \left( \left( \hat{P}_x + \frac{eB}{2c} \hat{Y} \right)^2 + \left( \hat{P}_y - \frac{eB}{2c} \hat{X} \right)^2 \right)$$  \hspace{1cm} (2)

$$= \frac{1}{2M} \left[ (\hat{P}_x^2 + \hat{P}_y^2) + \left( \frac{eB}{2c} \right)^2 (\hat{X}^2 + \hat{Y}^2) + \frac{eB}{c}(\hat{P}_x \hat{Y} - \hat{P}_y \hat{X}) \right]$$  \hspace{1cm} (3)

$$= \frac{1}{2M} \left[ (\hat{P}_x^2 + \hat{P}_y^2) + \left( \frac{eB}{2c} \right)^2 (\hat{X}^2 + \hat{Y}^2) - \frac{eB}{c} \hat{L}_z \right]$$  \hspace{1cm} (4)

We will prove $\hat{L}_z$ is a conserved quantity by showing it commutes with the Hamiltonian.

$$[\hat{L}_z, \hat{H}] = \frac{1}{2M} \left( [\hat{L}_z, \hat{P}_x^2 + \hat{P}_y^2] + \left( \frac{eB}{2c} \right)^2 [\hat{L}_z, \hat{X}^2 + \hat{Y}^2] \right)$$  \hspace{1cm} (5)

Note that

$$[\hat{L}_z, \hat{X}^2 + \hat{Y}^2] = [\hat{X} \hat{P}_y - \hat{Y} \hat{P}_x, \hat{X}^2 + \hat{Y}^2]$$  \hspace{1cm} (6)

$$= \hat{X} [\hat{P}_y, \hat{Y}^2] - \hat{Y} [\hat{P}_x, \hat{X}^2]$$  \hspace{1cm} (7)

$$= 2i\hbar (\hat{X} \hat{Y} - \hat{Y} \hat{X})$$  \hspace{1cm} (8)

$$= 0$$  \hspace{1cm} (9)

Similarly we can also show that

$$[\hat{L}_z, \hat{P}_x^2 + \hat{P}_y^2] = 0$$  \hspace{1cm} (10)

Hence both terms in the commutators vanish. In other words,

$$[\hat{L}_z, \hat{H}] = 0$$  \hspace{1cm} (11)
So \( \hat{L}_z \) is a conserved quantity. Now we write \( \hat{L}_z \) in polar coordinates

\[
\hat{L}_z = \hat{X} \hat{P}_y - \hat{Y} \hat{P}_x
\]

\[
= -i\hbar (r \cos \varphi \partial_y - r \sin \varphi \partial_x)
\]

\[
= -i\hbar \left[ r \cos \varphi \left( -\sin \varphi \partial_r + \frac{\cos \varphi}{2r} \partial_\varphi \right) - r \sin \varphi \left( -\cos \varphi \partial_r - \frac{\sin \varphi}{2r} \partial_\varphi \right) \right]
\]

\[
= -i\hbar \partial_\varphi
\]

(12)

(13)

(14)

(15)

The Hamiltonian in polar coordinates easily follows:

\[
\hat{H} = \frac{1}{2M} \left[ (\hat{P}_x^2 + \hat{P}_y^2) + \left( \frac{eB}{2c} \right)^2 (\hat{X}^2 + \hat{Y}^2) - \frac{eB}{c} \hat{L}_z \right]
\]

\[
= \frac{1}{2M} \left[ -\hbar^2 \nabla^2 + \left( \frac{eB}{2c} \right)^2 r^2 + i\hbar \frac{eB}{c} \partial_\varphi \right]
\]

(16)

(17)

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}
\]

(18)

is the Laplacian operator in polar coordinates. The stationary Schrödinger equation in polar coordinates is thus

\[
\frac{1}{2M} \left[ -\hbar^2 \nabla^2 + \left( \frac{eB}{2c} \right)^2 r^2 + i\hbar \frac{eB}{c} \partial_\varphi \right] \Psi(r, \varphi) = E \Psi(r, \varphi)
\]

(19)

1.2

We know \( \hat{L}_z \) is conserved, i.e. if \( \Psi \) is the solution to the wave equation, then it is also an eigenstate of \( \hat{L}_z \). Based on this fact we demand that

\[
\hat{L}_z \Psi = -i\hbar \frac{\partial}{\partial \varphi} \Psi(r, \varphi) = m\hbar \Psi(r, \varphi)
\]

(20)

Solving for this equation immediately gives

\[
\Psi(r, \varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} R_m(r)
\]

(21)

The factor \( 1/\sqrt{2\pi} \) is just a convinent normalization factor which comes from requiring the radial and angular wave functions be seperately normalized. The \( m \) appearing in the exponent stands for the angular momentum of this state. This wavefunction must be continuous across \( \varphi = 0 \) and \( \varphi = 2\pi \), so \( m \) can only be an integer.

Now plugging this solution back into our polar Schrödinger equation, we find

\[
\frac{1}{2M} \left[ -\hbar^2 \frac{\partial}{\partial r} \left( r \frac{\partial R_m(r)}{\partial r} \right) + m^2 \hbar^2 R_m(r) \frac{r^2}{2} + \left( \frac{eB}{2c} \right)^2 r^2 R_m(r) - m\hbar \frac{eB}{c} R_m(r) \right] = E_m R_m(r)
\]

(22)

or expressed in terms \( \omega_c \) and \( l_0 \):

\[
\frac{\omega_c^2 l_0^2}{2\hbar} \left[ -\hbar^2 \frac{\partial}{\partial r} \left( r \frac{\partial R_m(r)}{\partial r} \right) + m^2 \hbar^2 R_m(r) \frac{r^2}{2} + \frac{\hbar^2}{4l_0^2} r^2 R_m(r) - \frac{m\hbar^2}{l_0^2} R_m(r) \right] = E_m R_m(r)
\]

(23)
As for the boundary conditions, the wave function $\Psi(r, \varphi)$ must be continuous and integrable at $r = 0$, so we have

$$R_m(r) \sim r^{|m|} \quad \text{as } r \to 0$$

(24)

and we also need $R_m(r)$ to go to zero faster than $1/r^2$ to ensure normalizability, so the boundary condition at infinity reads

$$|R_m(r)| < \frac{1}{r^2} \quad \text{as } r \to \infty$$

(25)

1.3

By making a change of variables

$$R_m(r) = e^{-\frac{u}{2}} u^{|m|} F(u), \quad u = \frac{r^2}{2l_0^2}$$

(26)

The differentials transforms as

$$\frac{d}{dr} = \frac{\sqrt{2u} \; d}{l_0 \; du}$$

(27)

Now we substitute all these new definitions to the radial Schrödinger equation. After a bit of algebra, we can show that it becomes

$$u F''(u) + (|m| - u - 1) F'(u) + \left[ \frac{1}{2}(|m| - m + 1) - \frac{E_m}{\hbar \omega_c} \right] F(u) = 0$$

(28)

By comparing with the form of the differential equation satisfied by the confluent hypergeometric functions we find that

$$\alpha = \frac{1}{2}(|m| - m + 1) - \frac{E_m}{\hbar \omega_c}, \quad \gamma = |m| + 1$$

(29)

And the confluent hypergeometric functions $F(\alpha, \gamma, u)$ is given by

$$F(\alpha, \gamma, u) = 1 + \frac{\alpha u}{\gamma 1!} + \frac{\alpha(\alpha + 1) u^2}{\gamma(\gamma + 1) 2!} + \cdots$$

(30)

We see that our solution already satisfies the boundary condition at the origin since $R_m(r) \sim r^{|m|}$ as $r \to 0$. However we also need the radial wave function $R_m(r) \to 0$ fast enough as $r \to \infty$. Therefore $F(u)$ must not grow exponentially as $r \to \infty$. We need $F(u)$ to be terminate after finite terms, which is to say that $\alpha$ must be a negative integer. More specifically, if $F(u)$ is a polynomial of degree $n$, then

$$\alpha = \frac{1}{2}(|m| - m + 1) - \frac{E_m}{\hbar \omega_c} = -n$$

(31)

$$\Rightarrow E_m = \hbar \omega_c \left[ \frac{1}{2}(|m| - m + 1) + n \right]$$

(32)

The allowed energy values are called Landau levels. For $m \geq 0$ states, the electron rotates in “wrong classical direction”. The energy is now independent of $m$. We have huge degeneracies for each level since $m$ can now take any positive integral values.

$$E_m = \hbar \omega_c \left( n + \frac{1}{2} \right)$$

(33)
The $m < 0$ states are actually redundant. If we carefully examine the definitions of $\alpha$ and $\gamma$ again, we would find that picking a negative $m$ is actually equivalent to a shift $n \rightarrow n + |m|$. As a result we only need to consider the case of $m \geq 0$.

It may seem that we now have a infinite number of degeneracies here. However it is not the case. As we shall see in the following part of this problem, the quantum number $n$ controls the spread while $m$ controls the expectation value $\langle r \rangle$ of the wave function. Therefore the allowed number of electrons in a given state is approximately contoled by the area (or more precisely, the **magnetic flux**) of the system.

### 1.4

For $n = 0$ states the all the higher order terms hypergeometric function $F(u)$ vanish. Our wave function is simply (with correct normalization)

$$
\Psi(u, \varphi) = \frac{1}{\sqrt{2\pi}} e^{i m \varphi} \frac{1}{l_0 \sqrt{m!}} e^{-\frac{u^2}{2}} u^{\frac{m}{2}}
$$

We plot the radial probability distribution function for $m = 0$ and $m = 1$ below (setting $l_0^2 = 1$):

![Radial Probability Distribution Function](image)

The expectation value of $r^2$ is

$$
\langle 0, m | r^2 | 0, m \rangle = \frac{1}{l_0^2 m!} \int_0^\infty e^{-\frac{r^2}{2l_0^2}} \left( \frac{r^2}{2l_0^2} \right)^m r^3 dr
\tag{35}
$$

$$
= 2l_0^2 (1 + m)
\tag{36}
$$

The mean square root value of the position vector is thus

$$
\sqrt{\langle r^2 \rangle} = l_0 \sqrt{2(1 + m)}
\tag{37}
$$

With this expression we can work out the actual number of degenercies in the $n = 0$ Landau level. We know that the expectation value $\langle r \rangle$ of the wave function cannot be greater than the radius of the disk $R$. The maximally allowed $m$ in this level is thus

$$
m_{\text{max}} \approx \frac{R^2}{2l_0^2} = \frac{eBR^2}{2hc} = \frac{e\Phi}{\hbar c}
\tag{38}
$$

where $\Phi = 2\pi R^2 B$ is the magnetic flux through the disk.
1.5

The Schrödinger equation in the presence of vector potential \( \vec{A} \) is

\[
\frac{i \hbar}{\partial t} \Psi = \frac{1}{2M} \left( \hat{\Pi}_x^2 + \hat{\Pi}_y^2 \right) \Psi
\]

\[= \frac{1}{2M} \left( \frac{\hbar}{i} \nabla + \frac{e}{c} \vec{A} \right)^2 \Psi \]  

\[= \frac{-\hbar^2}{2M} \vec{D}^2 \Psi \]

Taking the complex conjugate of the last equation we obtain

\[
-i \frac{\hbar}{\partial t} \Psi^* = -\frac{\hbar^2}{2M} \vec{D}^* \Psi^*
\]

where \( \vec{D}^* \) is the conjugate of the covariant derivative.

Multiply (41) by \( \Psi^* \) and subtract (42) multiplied by \( \Psi \):

\[
i \frac{\hbar}{\partial t} \left( \Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \right) = \frac{\hbar^2}{2M} (\Psi^* \vec{D}^2 \Psi - \Psi \vec{D}^* \Psi^*)
\]

\[
i \frac{\hbar}{\partial t} |\Psi|^2 = \frac{\hbar^2}{2M} (\Psi^* \vec{D}^2 \Psi - \Psi \vec{D}^* \Psi^*)
\]

The RHS of this equation can be written as

\[
\Psi^* \vec{D}^2 \Psi - \Psi \vec{D}^* \Psi^* = \Psi^* (\nabla + i \frac{e}{\hbar c} \vec{A})^2 \Psi - \Psi (\nabla - i \frac{e}{\hbar c} \vec{A})^2 \Psi^*
\]

\[= i \frac{e}{\hbar c} \left[ \Psi^* (\nabla^2 + \nabla \cdot \vec{A} + \vec{A} \cdot \nabla) \Psi + \Psi (\nabla^2 + \nabla \cdot \vec{A} + \vec{A} \cdot \nabla) \Psi^* \right]
\]

\[= \nabla \cdot (\Psi^* \vec{D} \Psi - \Psi \vec{D}^* \Psi^*)
\]

Hence we got the probability continuity equation

\[
\frac{\partial}{\partial t} |\Psi|^2 + \nabla \cdot \vec{J} = 0
\]

where

\[
\vec{J} = \frac{\hbar}{2Mi} (\Psi^* \vec{D} \Psi - \Psi \vec{D}^* \Psi^*)
\]

Now we need the expression for the current in polar coordinates. Let us first start with the covariant derivative:

\[
\vec{D} = \nabla + i \frac{e}{\hbar c} \vec{A}
\]

\[= \left( \frac{1}{r} \frac{\partial}{\partial r} (r \vec{A}_r) \right) \vec{e}_r + \left( \frac{1}{r} \frac{\partial}{\partial \varphi} + i \frac{e}{\hbar c} A_\varphi \right) \vec{e}_\varphi
\]

We can obtain the radial and the azimuthal components of the current

\[
J_r = \frac{\hbar}{2Mi} \left( \frac{1}{r} \left[ \Psi^* \frac{\partial}{\partial r} (r \Psi) - \Psi \frac{\partial}{\partial r} (r \Psi^*) \right] + \frac{2e}{\hbar c} A_r |\Psi|^2 \right)
\]

\[
J_\varphi = \frac{\hbar}{2Mi} \left( \frac{1}{r} \left[ \Psi^* \frac{\partial}{\partial \varphi} \Psi - \Psi \frac{\partial}{\partial \varphi} \Psi^* \right] + \frac{2e}{\hbar c} A_\varphi |\Psi|^2 \right)
\]
For Landau level wave function $\Psi = e^{im\varphi} F(u)$ we have (using the fact that $F(u)$ is always real):

\[
J_r = \frac{\hbar A_r}{M} |\Psi|^2 = 0 \quad (54)
\]

\[
J_\varphi = \frac{\hbar}{M} \left( \frac{m}{r} + \frac{e}{\hbar c} A_\varphi \right) |\Psi|^2 = \frac{\hbar}{M} \left( \frac{m}{r} + \frac{eB}{2c} r \right) |\Psi|^2 \quad (55)
\]

Specifically, for $m = 0$ and $m = 1$:

\[
J_\varphi(m = 0) = \frac{eB}{2cM} r e^{-u} \quad (56)
\]

\[
J_\varphi(m = 1) = \frac{\hbar}{M} \left( \frac{1}{r} + \frac{eB}{2c} r \right) e^{-u} u \quad (57)
\]

We found that no matter what number $m$ is, the radial component of $\vec{J}$ vanishes. This is not surprising since all the Landau level wave function we solved are stationary eigenstates. However the azimuthal component is nonzero and has a part that scales with $m$, which is a result from the fact that it is a eigenstate of angular momentum $m\hbar$. The other part that scales with $Br$ can be viewed as the current generated by the classical magnetic field, in analogy of an electron circulating in a constant magnetic field. Note that despite the angular part of $\vec{J}$ is non-zero, the divergence of the current is indeed vanishing, as we are dealing with energy eigenstates.

2 Angular Momentum in Three Dimensions

2.1

We first note that the rotation generators satisfy the commutation relation

\[
[J_i, J_j] = \epsilon_{ijk} J_k \quad (58)
\]

And the definition of total angular momentum operator and the raising/lowering operators

\[
J^2 = \sum_i J_i^2, \quad J_\pm = J_x \pm i J_y \quad (59)
\]

One can check that $J^2$ commute with all the $J_i$’s. As in usual we choose our basis $|j, m\rangle$ to be the eigenbasis of $J^2$ and $J_z$:

\[
J^2 |l, m\rangle = j(j+1)\hbar^2 |j, m\rangle, \quad J_z |j, m\rangle = m\hbar |j, m\rangle \quad (60)
\]

To find the matrix elements of $J_x$ and $J_y$, we first focus on the matrix elements of $J_\pm$:

\[
\langle j, m | J_+ | j, m \rangle = \langle j, m | (J_x - iJ_y)(J_x + iJ_y) | j, m \rangle \quad (61)
\]

\[
= \langle j, m | J^2 - J_z^2 - \hbar J_z | j, m \rangle \quad (62)
\]

\[
= \hbar^2 [j(j+1) - m^2 - m] \quad (63)
\]

\[
= \hbar^2 (j - m)(j + m + 1) \quad (64)
\]

Since the operation of $J_+$ is to increment $m$ by one, we have

\[
J_+ |j, m\rangle = \hbar \sqrt{(j - m)(j + m + 1)} |j, m + 1\rangle \quad (65)
\]
And using the orthogonality of the basis elements we obtain

\[ \langle j', m' | J_+ | j, m \rangle = \hbar \sqrt{(j - m)(j + m + 1)} \delta_{j'j} \delta_{m'm+1} \] (66)

Following the similar procedure we can also show

\[ \langle j', m' | J_- | j, m \rangle = \hbar \sqrt{(j + m)(j - m + 1)} \delta_{j'j} \delta_{m'm-1} \] (67)

In a 4x4 \((j = \frac{3}{2})\) basis they read

\[
J_+ = \hbar \begin{pmatrix}
0 & \sqrt{3}/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{3}/2 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
J_- = \hbar \begin{pmatrix}
0 & 0 & 0 & 0 \\
\sqrt{3}/2 & 0 & 1 & 0 \\
0 & 1 & 0 & \sqrt{3}/2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (68)

\[
J_x = \frac{1}{2} (J_+ + J_-) = \hbar \begin{pmatrix}
0 & \sqrt{3}/2 & 0 & 0 \\
\sqrt{3}/2 & 0 & 1 & 0 \\
0 & 1 & 0 & \sqrt{3}/2 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
J_y = \frac{1}{2i} (J_+ - J_-) = \hbar \begin{pmatrix}
0 & -\sqrt{3}i/2 & 0 & 0 \\
\sqrt{3}i/2 & 0 & -i & 0 \\
0 & i & 0 & -\sqrt{3}i/2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
J_z = \hbar \begin{pmatrix}
3/2 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 \\
0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & -3/2
\end{pmatrix}
\] (70)

The \(J^2\) operator is just \(\hbar^2(3/2)(1+3/2)I_{4x4}\). Showing that these matrices satisfy the commutation relation (58) is a trivial task and we will skip it here.

2.2

It is easier to work with \(J_+\) and \(J_-\).

\[
\langle j, m | J_x | j, m \rangle = \frac{1}{2} \langle j, m | J_+ + J_- | j, m \rangle = 0 
\] (71)

\[
\langle j, m | J_y | j, m \rangle = \frac{1}{2i} \langle j, m | J_+ - J_- | j, m \rangle = 0 
\] (72)

\[
\langle j, m | J_+^2 | j, m \rangle = \frac{1}{4} \langle j, m | J_+^2 + J_-^2 + 2 J_+ J_- | j, m \rangle 
\]

\[
= \frac{\hbar^2}{2} \sqrt{(j - m)(j + m + 1)} \sqrt{(j + m)(j - m + 1)} 
\]

\[
= \frac{\hbar^2}{2} [j(j + 1) - m^2] 
\] (74)

\[
\langle j, m | J_-^2 | j, m \rangle = \frac{1}{4} \langle j, m | J_-^2 + J_+^2 - 2 J_+ J_- | j, m \rangle 
\]

\[
= \frac{\hbar^2}{2} [j(j + 1) - m^2] 
\] (75)

\[
\langle j, m | J_g^2 | j, m \rangle = -\frac{1}{4} \langle j, m | J_+^2 - J_-^2 + 2 J_- J_+ | j, m \rangle 
\]

\[
= \frac{\hbar^2}{2} [j(j + 1) - m^2] 
\] (76)
\[ \Delta J_x \Delta J_y = \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2} \sqrt{\langle J_y^2 \rangle - \langle J_y \rangle^2} = \frac{\hbar^2}{2} (j(j+1) - m^2) \] (78)

We see that the uncertainties is minimized when \( m = \pm j \), where

\[ \Delta J_x \Delta J_y = \frac{\hbar^2 j}{2} \] (79)

This is exactly the lower bound implied from the commutation relation (58) for state \( |j, \pm j\rangle \). For any other states \( |j, m\rangle \) we can show that (58) implies \( \Delta J_x \Delta J_y \geq \frac{\hbar^2}{2} |m| / 2 \), which is smaller than the actual value given by (78).

2.4

We need to work with spin-1 (3 \( \times \) 3) angular momentum matrices. One can show in a similar manner that 3 \( \times \) 3 generators \( J_i \) are

\[
J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (80)
\]

\[
J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (81)
\]

\[
J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (82)
\]

Upon exponentiating we obtain

\[
e^{-i\theta J_x / \hbar} = \begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \quad (83)
\]

\[
e^{-i\theta J_y / \hbar} = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \quad (84)
\]

\[
e^{-i\theta J_z / \hbar} = \begin{pmatrix} e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 \end{pmatrix} \quad (85)
\]

\[
\Rightarrow D^{(1)} [R(\alpha, \beta, \gamma)] = e^{-i\alpha J_x / \hbar} e^{-i\beta J_y / \hbar} e^{-i\gamma J_z / \hbar}
\]

\[
= \begin{pmatrix} e^{-i(\alpha+\gamma)} \cos \frac{\beta}{2} & -e^{-i\alpha \sin \beta} & e^{i(-\alpha+\gamma)} \sin \frac{\beta}{2} \\ e^{i\gamma \sin \beta} & \cos \beta & -e^{i\gamma \sin \beta} \\ e^{i(\alpha-\gamma)} \sin \frac{\beta}{2} & e^{i\alpha \sin \beta} & e^{i(\alpha+\gamma)} \cos \frac{\beta}{2} \end{pmatrix} \quad (86)
\]
Now we act this finite rotation operator on the state $\Psi$:

$$|\Psi\rangle = D^{(1)} |1,1\rangle $$  \hspace{1cm} (88)

$$= \begin{pmatrix}
e^{-i(\alpha+\gamma)} \cos \frac{\beta}{2} & -e^{-i\alpha} \sin \frac{\beta}{\sqrt{2}} & e^{i(\alpha+\gamma)} \sin \frac{\beta}{2} \\
e^{-i\gamma} \sin \frac{\beta}{\sqrt{2}} & \cos \beta & -e^{i\gamma} \sin \frac{\beta}{\sqrt{2}} \\
e^{i(\alpha-\gamma)} \sin \frac{\beta}{2} & e^{i\alpha} \sin \frac{\beta}{\sqrt{2}} & e^{i(\alpha-\gamma)} \cos \frac{\beta}{2}
\end{pmatrix} \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} $$  \hspace{1cm} (89)

$$= \begin{pmatrix}
e^{-i(\alpha+\gamma)} \cos \frac{\beta}{2} \\
e^{i\gamma} \sin \frac{\beta}{\sqrt{2}} \\
e^{i(\alpha-\gamma)} \sin \frac{\beta}{2}
\end{pmatrix} $$  \hspace{1cm} (90)

So the expectation value of the angular momentum is

$$\langle \Psi | \vec{J} | \Psi \rangle = \sum_i \langle \Psi | J_i | \Psi \rangle \vec{e}_i $$  \hspace{1cm} (91)

$$= (\cos \alpha \sin \beta)\vec{e}_x + (\sin \alpha \sin \beta)\vec{e}_y + (\cos \beta)\vec{e}_z $$  \hspace{1cm} (92)

We note that it is impossible to pick $(\alpha, \beta)$ such that the expectation value vanish in each component. Since $\langle 1,0 | \vec{J} | 1,0 \rangle = 0$, it is impossible to rotate $|1,1\rangle$ to just $|1,0\rangle$. However we can show that

$$\langle 1,0 | \vec{J} | \Psi \rangle = e^{-i\gamma} \left[ (\cos \alpha - i \cos \beta \sin \alpha)\vec{e}_x + (\sin \alpha + i \cos \alpha \cos \beta)\vec{e}_y \right] $$  \hspace{1cm} (93)

and

$$\langle 1,-1 | \vec{J} | \Psi \rangle = \left[ (e^{-i\gamma} \sin \beta/2)\vec{e}_x + (e^{i\gamma} \sin \beta)\vec{e}_y + (-e^{i(\alpha-\gamma)} \sin(\beta/2))\vec{e}_z \right] $$  \hspace{1cm} (94)

So it is possible to rotate $|1,1\rangle$ to a linear combination of two other states. By reflection symmetry around the $x-y$ plane we can argue that the same holds for $|1,-1\rangle$ state. Therefore it is always possible to rotate $|1,m\rangle$ into a linear combination of other states involving $|1,m'\rangle$.

3 The Angular Momentum States in the Coordinate Basis

3.1

Recall the definition of orbital angular momenta operator

$$\hat{\mathbf{L}}_k = \epsilon_{ijk} \hat{X}_i \hat{P}_j $$  \hspace{1cm} (95)

In spherical coordinates we have the following relations

$$\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta \\
\partial_x &= \frac{1}{r} (\sin \theta \cos \phi \partial_r + \cos \theta \cos \phi \partial_\theta - \csc \theta \sin \phi \partial_\phi) \\
\partial_y &= \frac{1}{r} (\sin \theta \sin \phi \partial_r + \cos \theta \sin \phi \partial_\theta + \csc \theta \cos \phi \partial_\phi) \\
\partial_z &= \frac{1}{r} (\cos \theta \partial_r - \sin \theta \partial_\phi)
\end{align*} $$  \hspace{1cm} (96)
Let us unwrap the definition of $L_i$ and re-express in spherical coordinates (I’ll drop the hat in all operators for convinience):

\[ L_x = Y P_z - Z P_y \]
\[ = -\hbar \left[ (\sin \theta \sin \phi)(\cos \theta \partial_r - \sin \theta \partial_\phi) 
- \cos \theta (\sin \theta \sin \phi \partial_r + \cos \theta \sin \phi \partial_\theta + \csc \theta \cos \phi \partial_\phi) \right] \]
\[ = -\i \hbar \left[ -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi \right] \]

\[ L_y = X P_z - Z P_x \]
\[ = -\i \hbar \left[ (\sin \theta \cos \phi)(\cos \theta \partial_r - \sin \theta \partial_\phi) 
- \cos \theta (\sin \theta \cos \phi \partial_r + \cos \theta \cos \phi \partial_\theta - \csc \theta \sin \phi \partial_\phi) \right] \]
\[ = -\i \hbar \left[ \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi \right] \]

\[ L_z = X P_y - Y P_x \]
\[ = -\i \hbar \left[ \sin \theta \cos \phi (\sin \theta \sin \phi \partial_r + \cos \theta \sin \phi \partial_\theta + \csc \theta \cos \phi \partial_\phi) 
- \sin \theta \sin \phi (\sin \theta \cos \phi \partial_r + \cos \theta \cos \phi \partial_\theta - \csc \theta \sin \phi \partial_\phi) \right] \]
\[ = -\i \hbar \partial_\phi \]

With these in hand we can now work out the form of $L_{\pm}$ and $L^2$:

\[ L_{\pm} = L_x \pm i L_y \]
\[ = -\i \hbar e^{\pm i \phi} (\pm i \partial_\theta - \cot \theta \partial_\phi) \]
\[ L^2 = L^2_x + L^2_y + L^2_z \]
\[ = -\hbar^2 (\csc \theta \partial_\theta (\sin \theta \partial_\theta) + \csc^2 \theta \partial^2_\phi) \]

### 3.2

The spherical harmonics all satisfy the eigen equation

\[ l(l + 1)\hbar^2 Y^m_l = L^2 Y^m_l \]
\[ = (L^2_x + L^2_y + L^2_z) Y^m_l \]
\[ = \left[ \frac{1}{2}(L_+ L_- + L_- L_+) + L^2_z \right] Y^m_l \]
\[ = (L_- L_+ + \hbar L_z + L^2_z) Y^m_l \]

The highest z-angular momentum function $Y^l_l$ satisfies the $L_z$ eigen equation $\hat{L}_z Y^l_l = l \hbar Y^l_l$. So (111) becomes

\[ (L_- L_+ + \hat{l}^2 + l) Y^l_l = l(l + 1) Y^l_l \]
\[ \Rightarrow L_- L_+ Y^l_l = 0 \]

Similarly we can show for $Y^{-l}_l$ that

\[ L_+ L_- Y^{-l}_l = 0 \]
For the two equations to be simultaneously true we must require

\[ L_+ Y_l^t = L_- Y_{-l}^t = 0 \]  \hspace{1cm} (118)

In coordinate language the \( L_+ \) equation reads

\[ \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) Y_l^t(\theta, \phi) = 0 \] \hspace{1cm} (119)

Which says that the highest angular momentum state is annihilated by the raising operator \( L_+ \).

### 3.3

Equation (119) can be easily solved by separation of variables. The solution (with correct normalization) is

\[ Y_l^t(\theta, \phi) = (-1)^l \sqrt{\frac{(2l+1)}{4\pi}} \frac{1}{(2l)!} e^{i\phi} \sin^l \theta \] \hspace{1cm} (120)

Act the lowering operator \( L_- \) twice on this function we can obtain \( Y_{-l}^{t-2}(\theta, \phi) \): (assuming \( l > 1 \))

\[ Y_{l}^{t-2}(\theta, \phi) = \frac{1}{\hbar^2 \sqrt{2l} \sqrt{2(2l-1)}} L_+^2 Y_l^t(\theta, \phi) \] \hspace{1cm} (121)

\[ = \frac{1}{\sqrt{2l} \sqrt{2(2l-1)}} \left[ -ie^{i\phi} \left( -i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) \right]^2 Y_l^t(\theta, \phi) \] \hspace{1cm} (122)

\[ = (-1)^{l-2} \sqrt{\frac{(2l+1)}{4\pi}} \frac{2}{(2l-2)!} e^{i(l-2)\phi} [-\sin^l \theta + 2(l-1) \sin^{l-2} \theta \cos^2 \theta] \] \hspace{1cm} (123)