Problem Set No. 4:
Quantum Mechanics in two and three dimensions
Due Date: November 3, 2017

1 Charged Particle in an Uniform Magnetic Field

In this problem you will consider a particle of mass $M$ and charge $e$, restricted to move on a large disk of radius $R$ in the presence of a magnetic field of strength $B$, perpendicular to the plane. The magnetic flux through the rectangle is $\Phi = \pi R^2 B$. You will work in the circular gauge in which the components of the vector potential, for $B > 0$, are

$$A_x = -\frac{1}{2} By, \quad A_y = \frac{1}{2} Bx$$

In this gauge it is convenient to work in polar coordinates

$$x = r \cos \varphi, \quad y = r \sin \varphi$$

where

$$0 \leq r < \infty, \quad \text{and} \quad 0 \leq \varphi < 2\pi$$

In polar coordinates the components of the vector potential become

$$A_r = 0, \quad \text{and} \quad A_\varphi = \frac{B}{2} r$$

You will work in the limit of a large disk, $i.e.$ $R \to \infty$, while keeping the total flux $\Phi$ fixed and finite.

The Hamiltonian for this problem is

$$\hat{H} = \frac{1}{2M} (\hat{\Pi}_x^2 + \hat{\Pi}_y^2)$$

where

$$\hat{\Pi}_x = \hat{P}_x - \frac{e}{c} A_x(\hat{X}, \hat{Y}), \quad \hat{\Pi}_y = \hat{P}_y - \frac{e}{c} A_y(\hat{X}, \hat{Y})$$

$$A_x(\hat{X}, \hat{Y}) = -\frac{B}{2} \hat{Y}, \quad A_y(\hat{X}, \hat{Y}) = \frac{B}{2} \hat{X}$$

The cartesian components of the canonical momentum $\hat{\mathbf{P}}$ and of the coordinates $\hat{\mathbf{X}}$ obey the canonical commutation relations:

$$[\hat{X}, \hat{P}_x] = [\hat{Y}, \hat{P}_y] = i\hbar, \quad [\hat{X}, \hat{P}_y] = [\hat{Y}, \hat{P}_x] = 0$$
1. Show that the $z$-component of the angular momentum $\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$ is conserved. Find an expression for $\hat{L}_z$ in polar coordinates. Derive the Schrödinger Equation for the eigenstates of energy $E$ in the polar coordinates $(r, \varphi)$.

2. Show that in polar coordinates the eigenstates of $\hat{H}$ have the form

$$\Psi(r, \varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} R_m(r)$$

Explain the physical meaning of the quantum number $m$ and find the allowed values of $m$. Find the Schrödinger Equation for the radial wave function $R_m(r)$ . Write this equation in terms of the cyclotron frequency $\omega_c = eB/Mc$ and of the magnetic length $\ell_0 = \sqrt{\hbar c/eB}$. Determine the boundary conditions for $R_m(r)$ as $r \to 0$ and $r \to \infty$.

3. The radial wave functions $R_m(r)$ of the allowed eigenstates have the form

$$R_m(r) = e^{-\frac{u^2}{2}} u^{\frac{|m|}{2}} F(u)$$

where $u = r^2/(2\ell_0^2)$ and $F(u) = F(\alpha, \gamma, u)$ is a confluent hypergeometric function

$$F(\alpha, \gamma, u) = 1 + \frac{\alpha u}{\gamma} + \frac{\alpha(\alpha + 1) u^2}{\gamma(\gamma + 1) 2!} + \ldots$$

which satisfies the differential equation

$$u \frac{d^2F}{du^2} + (\gamma - u) \frac{dF}{du} - \alpha F = 0$$

Show that for this problem the parameters $\alpha$ and $\gamma$ are

$$\alpha = \frac{1}{2} (-m + |m| + 1) - \frac{E}{\hbar \omega_c} \quad \text{and} \quad \gamma = |m| + 1$$

Show that these wave functions satisfy the boundary conditions at $r = 0$ and as $r \to \infty$ if the function $F(u)$ is a polynomial of degree $n$, with $n = 0, 1, 2, \ldots$; hence, $\alpha = -n$. Use these results to find the allowed energy levels (the Landau levels).

Note: these polynomials are the generalized Laguerre polynomials

$$F(-n, |m| + 1, u) = L_n^{m+1}(u) = (-1)^{|m|} |m|! \frac{u^{n+|m|}}{(n+|m|)!} e^{-u} u^{n+|m|}$$

How does the degeneracy of the Landau levels show up in this solution?

4. Consider now the wave functions of the states with $n = 0$ and $m \geq 0$. Calculate their energies. Plot their probability distributions as a function of $r$ and $\varphi$ for $m = 0$ and $m = +1$. Calculate the expectation value $\langle 0, m | \hat{X}^2 + \hat{Y}^2 | 0, m \rangle$. What is the r. m. s. value of the position vector in these states? How does it depend on $\ell_0$? How does in general depend on $m$?
5. Show that for a particle in a state at time \( t \) with wave function \( \Psi(x, y, t) \) the current density \( \vec{J}(x, y, t) \) and probability density \( |\Psi(x, y, t)|^2 \) satisfy the continuity equation

\[
\frac{\partial |\Psi|^2}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0
\]

where the current is

\[
\vec{J} = \frac{\hbar}{2M} \left( \Psi^* \vec{D} \Psi - \left( \vec{D} \Psi \right)^* \Psi \right)
\]

where

\[
\vec{D} = \vec{\nabla} + i \frac{e}{\hbar c} \vec{A}(x, y)
\]

is called the covariant derivative. Find a general expression for the radial and azimuthal components \( J_r(r, \phi) \) and \( J_\phi(r, \phi) \) of the current, and use this result to calculate \( J_r \) and \( J_\phi \) for the eigenstates \( |n, m\rangle = |0, 0\rangle, |0, 1\rangle \). Give a physical interpretation of your results.

2 Angular Momentum in Three Dimensions

In this problem you will work out a number of important properties of the angular momentum operators and of their eigenstates.

1. Construct the three 4 \( \times \) 4 matrices that represent the operators \( \hat{J}_x \), \( \hat{J}_y \) and \( \hat{J}_z \). Show that these matrices obey the commutation relations of the angular momentum operators. Construct the 4 \( \times \) 4 matrices which represent the operators \( \hat{J}_x \), \( \hat{J}_- \) and \( \hat{J}^2 \).

2. Consider now the general angular momentum eigenstates \( |j, m\rangle \). Show that

\[
\langle j, m| \hat{J}_x |j, m\rangle = \langle j, m| \hat{J}_y |j, m\rangle = 0
\]

and that

\[
\langle j, m| \hat{J}_z^2 |j, m\rangle = \langle j, m| \hat{J}_y^2 |j, m\rangle = \hbar^2 [j(j + 1) - m^2] / 2
\]

3. Use these results to show that \( \Delta J_x \Delta J_y \) satisfies an Uncertainty Principle. Show that the bound is saturated by the states \( |j, \pm j\rangle \).

4. Let us define a rotation \( R(\alpha, \beta, \gamma) \) where \( \alpha \), \( \beta \) and \( \gamma \) are the Euler Angles for three successive rotations:

\[
U[R(\alpha, \beta, \gamma)] = e^{-i\alpha \hat{J}_z / \hbar} e^{-i\beta \hat{J}_y / \hbar} e^{-i\gamma \hat{J}_z / \hbar}
\]

Construct the matrix \( D^{(1)}[R(\alpha, \beta, \gamma)] \) explicitly as a product of three matrices. Let \( \Psi = D^{(1)}|1, 1\rangle \). Show that

\[
\langle \Psi | \vec{J} | \Psi \rangle = \hbar (\sin \beta \cos \alpha \vec{e}_x + \sin \beta \sin \alpha \vec{e}_y + \cos \beta \vec{e}_z)
\]
where \( \vec{e}_x, \vec{e}_y \) and \( \vec{e}_z \) are unit vectors along the directions \( x, y \) and \( z \). Show that for no value of \( \alpha, \beta \) and \( \gamma \) it is possible to rotate \( |1,1\rangle \) into just \( |1,0\rangle \). Show that it is always possible to rotate \( |1,m\rangle \) into a linear combination involving \( |1,m'\rangle \), that is

\[
\langle 1,m| D^{(1)}(R(\alpha, \beta, \gamma))|1,m'\rangle \neq 0
\]

for some \( \alpha, \beta, \gamma \) and any \( m \) and \( m' \).

3 The Angular Momentum states in the Coordinate Basis

In what follows we will denote the wave functions of the angular momentum states in spherical coordinates as

\[
\langle \theta, \phi| \ell, m \rangle = Y^{m}_{\ell}(\theta, \phi)
\]

where \(-\pi \leq \theta \leq \pi \) and \(0 \leq \phi < 2\pi\). As usual, \( \ell = 0, 1, 2, \ldots \) and \( |m| \leq \ell\).

1. Show that in spherical coordinates, the operators \( \hat{L}^2, \hat{L}_z \) and \( \hat{L}_\pm \) are given by

\[
\hat{L}^2 = -\hbar^2 \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)
\]

\[
\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}
\]

\[
\hat{L}_\pm = \pm \hbar e^{\pm i\phi} \left( \frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \phi} \right)
\]

2. Show that the wave function \( Y^\ell_{\ell}(\theta, \phi) \) is the solution of the equation

\[
\left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) Y^\ell_{\ell}(\theta, \phi) = 0
\]

What condition on the ket \( |\ell, \ell\rangle \) does this equation represent?

3. Using the lowering operator \( \hat{L}_- \), construct the wave function for the ket \( |\ell, \ell - 2\rangle \), including the normalization constant.