1 Semi-classical Motion in a Central Potential

In this problem you will apply the WKB approximation that was discussed in class to the case of the motion of a particle of mass $M$ moving in three dimensions in a central potential $U(r)$. In class we discussed extensively the one-dimensional case. Here you will have to generalize those ideas to three dimensions.

1. The Schrödinger Equation for a particle of mass $M$ in a central potential $U(r)$ is separable in spherical coordinates $r, \phi, \theta$, where $0 \leq r < \infty$ is the radial coordinate, $0 \leq \phi < 2\pi$ is the azimuthal angle and $0 \leq \theta \leq \pi$ is the colatitude. Show that the wave function of an arbitrary stationary state $|\Psi\rangle$ in these coordinates factorizes, i.e.

$$\langle r, \theta, \phi | \Psi \rangle \equiv \Psi(r, \theta, \phi) = R(r)\Phi(\theta, \phi)$$

(1)

Show that it is possible to choose $|\Phi\rangle$ to be an eigenstate of the angular momentum operators $L^2$ with eigenvalue $\hbar^2 \ell(\ell + 1)$, and $L_z$, with eigenvalue $\hbar m$, where $\ell = 0, 1, 2, \ldots$ and $|m| = 0, 1, \ldots, \ell$. Write $\Phi(\theta, \phi)$ in terms of the spherical harmonics $Y_{\ell,m}(\theta, \phi)$. 
2. Show that the function \( \chi(r) \), defined by
\[
\chi(r) = rR(r)
\] (2)
obeyrs an effective one-dimensional Schrödinger Equation for \( 0 \leq r < \infty \) and find the form of the effective potential \( U_{\text{eff}}(r) \) when the angular wave function is \( \Phi(\theta, \phi) = A_{\ell,m} Y_{\ell,m}(\theta, \phi) \). Find the value of the normalization constant \( A_{\ell,m} \). What boundary condition does the wave function \( \chi(r) \) satisfy as \( r \to 0 \)?

3. Find the form of the semi-classical wave function for a bound state of a general attractive potential \( U(r) \). Discuss in detail the form of the wave function in both the classically allowed and forbidden regions. Find a set of consistent connection formulas, or boundary conditions, to match the wave functions between both regions.

4. Rederive the Bohr-Sommerfeld quantization formula for a bound state in a general central potential \( U(r) \). Use this formula to find a condition that the potential \( U(r) \) must satisfy to have an \( infinite \) number of bound states. What condition can you impose on the potential to have \( no \) bound states at all? (at least in the semi-classical approximation). Compare this answer with the one-dimensional case. How different is the result? Why?

2 Radioactive \( \alpha \)-decay and Tunneling

The first theory of radioactive \( \alpha \)-decay, by Gamow and by Gurney and Condon, consisted of the following. The \( \alpha \) particle is regarded as a point particle of mass \( m \) and charge \( q = +2e \) (recall that an \( \alpha \) particle is just a the nucleus of a Helium atom, i.e. stripped of its electrons). Inside the nucleus of a large atom, \( r < r_0 \), the \( \alpha \) particle feels a constant potential \( -U_0 \), while outside the \( \alpha \) particle feels the Coulomb repulsion of the nucleus. The resulting potential is
\[
U(r) = \begin{cases} 
-U_0, & \text{for } r \leq r_0 \\
\frac{g}{r}, & \text{for } r > r_0 
\end{cases}
\] (3)
where \( g \) measures the strength of the Coulomb repulsion. The energy \( E \) of the \( \alpha \) particle is large enough so that it is in the well but not in a bound state.

1. Find the allowed range of energies, in terms of \( g, m \) and \( r_0 \), for the particle to be in this regime. Discuss the physical meaning of both the lower and upper limits.

2. Use the WKB approximation to calculate the probability \( W \) for the particle to escape from the well. Find the form of \( W \) for \( E \) very close both to the lower and upper ends of the allowed range of energies.
Hint: The integral can be done exactly. You may have to use the substitution \( r = \sin^2 \theta \), where \( \theta \) is an angle.
3 Path Integral for the Forced Harmonic Oscillator

In this problem you will consider a forced linear harmonic oscillator of mass \( M \), and angular frequency \( \omega_0 \). The Hamiltonian of the oscillator is

\[
H = \frac{P^2}{2M} + \frac{1}{2}M\omega_0^2Q^2 - J(t)Q
\]

where \( P \) and \( Q \) are the momentum and position operators of the oscillator. The external force, which we denoted by \( J(t) \), will be assumed to be absent in the remote past, \( t_i \to -\infty \), turned on slowly (“adiabatically”) and turned off (slowly again) in the future such that it vanishes at \( t \to +\infty \). The form of the time dependence of the external (driving) force will be specified below. We will assume that in the initial state the system is at the origin, i.e. is an eigenstate of the coordinate operator \( Q \) with eigenvalue zero, \( |0, t_i\rangle \) (with \( t_i \to -\infty \)) and that at \( t \to +\infty \) it is measured at an eigenstate of the coordinate \( Q \) with eigenvalue 0, i.e. \( |0, t_f\rangle \), with \( t_f \to +\infty \).

1. Rederive the expression shown in class for the amplitude \( \langle q_f, t_f|q_i, t_i\rangle \) as a Feynman path integral. Show that for the case at hand the amplitude \( \langle 0, +\infty|0, -\infty\rangle_J \) in the presence of the external force \( J \) has the form

\[
\langle 0, +\infty|0, -\infty\rangle_J = e^{i\frac{\hbar}{\hbar}S_{cl}(J)}\langle 0, +\infty|0, -\infty\rangle_J = 0
\]

where

\[
S_{cl}(J) = \frac{1}{2M} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' J(t)G(t - t')J(t')
\]

Here \( G(t - t') \) is the (Green function) solution of the differential equation

\[
\left( \frac{d^2}{dt^2} + \omega_0^2 \right) G(t - t') = \delta(t - t')
\]

with vanishing boundary conditions at \( t \to \pm \infty \). Which Green function should be used in this expression? Justify your answer. Find the explicit form of the Green function \( G(t, t') \) in real time and in frequency \( \omega \). Be careful to make your prescription for how to around the singularities of \( \tilde{G}(\omega) \) explicit.

2. Show that for a general force \( J(t) \), \( S_{cl}(J) \) is given by

\[
S_{cl}(J) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} |\tilde{J}(\omega)|^2 \tilde{G}(\omega)
\]

where \( \tilde{J}(\omega) \) and \( \tilde{G}(\omega) \) are, respectively, the Fourier transforms of the force \( J(t) \) and of the Green function \( G(t - t') \),

\[
\tilde{J}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} J(t), \quad \tilde{G}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} G(t)
\]
3. Consider now the case of an oscillatory driving force \( J(t) \) of strength \( J_0 \) and frequency \( \Omega \),

\[
J(t) = J_0 \cos(\Omega t) e^{-|t|/\gamma} e^{-|t|/\gamma}
\]  

(10)

where \( \gamma \) represents the adiabatic turning on and off of the driving force \( J(t) \). Find an explicit expression for \( S_{c1} \) in terms of \( J_0 \), \( \Omega \) and \( \gamma \) (and of the mass \( M \) and \( \omega_0 \)). Use this result to find an expression for the ratio of amplitudes

\[
R(J_0, \Omega) = \frac{\langle 0, +\infty | 0, -\infty \rangle_J}{\langle 0, +\infty | 0, -\infty \rangle_{J=0}}
\]  

(11)

Show that at resonance, \( \Omega = \omega_0 \), \( R(J_0, \Omega) \to 0 \) as \( \gamma \to 0 \). Hint: the simplest way to do this part is to find the Fourier transform of \( J(t) \) and to compute the integral of Eq.(8) using residues.