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Potential Scattering

We will now discuss the physics of the states with $E > 0$, the continuum. In this regime the energy (for an ∞ system) is not quantized. ~~and~~ It is simpler to describe the nature of these states in terms of scattering processes. We will first discuss the case of short range potentials and later ~~leave~~ the ~~case~~ of Coulomb scattering.

Consider a fixed target located at the origin and a particle that "enters the system" at an (early) time t_0, still far away from the scatterer. We will imagine that at t_0, the particle is ~~...~~ in a wave packet state

$$\Psi(\vec{r}, t_0) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{r}} \, a(\vec{k})$$

and that, at t_0, the probability of finding the particle within the range of the scatterer is zero (this is the main difficulty in the Coulomb case)

$\Rightarrow$ $a(\vec{k})$ is peaked around $\vec{k} = \vec{k}_0$

$\Rightarrow$ the wave packet travels at a velocity $\vec{v}_0 = \frac{\hbar \vec{k}_0}{m}$ towards the scatterer. We will determine the wave function $\psi(\vec{r}, t)$ at a time t after the particle has interacted with the scatterer. We will use a method very similar to the one we discussed in $d=1$ (see Baym, Ch. 9)

Let $\psi_{\vec{k}}(\vec{r})$ be an exact eigenstate of the particle interacting with the scatterer with energy $E_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2\mu}$

$E_k = \frac{\hbar^2 \vec{k}^2}{2\mu}$. (There is no quantization condition on $E_{\vec{k}} > 0$.)

$$\Rightarrow \left(\frac{\hbar^2}{2\mu} \nabla^2 + E_{\vec{k}} \right) \psi_{\vec{k}}(\vec{r}) = U(\vec{r})\, \psi_{\vec{k}}(\vec{r})$$

where $U(\vec{r}) = U(r)$ $\qquad r = |\vec{r}|$

$\Rightarrow$ we expand $\psi(\vec{r}, t_0)$ in exact eigenstates

$$\psi(\vec{r}, t_0) = \int \frac{d^3k}{(2\pi)^3} \; \psi_{\vec{k}}(\vec{r}) \; A(\vec{k})$$

Note: only ~~the~~ states with $E > 0$ enter in this expansion since at t_0 the overlap between the wave packet and the bound states of the scatterer is zero. However, not ~~of~~ all the wave functions $\psi_{\vec{k}}(\vec{r})$ are present ~~also those~~: only those ~~with~~ representing the <u>incoming wave packet</u> and the <u>outgoing scattered wave</u>. (Recall what we did in $d=1$) $\Rightarrow$ we get $A(\vec{k})$.

Once the coefficients $A(\vec{k})$ are known, the wave packet evolves according to the Schrödinger eqn.:

$$\psi(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \; \psi_{\vec{k}}(\vec{r}) \; A(\vec{k}) \, e^{-i E_{\vec{k}} \frac{(t-t_0)}{\hbar}}$$

which is "the solution".

We first need to construct the eigenstates $\psi_{\vec{k}}(\vec{r})$. Let us define the Green function $G(\vec{r}, \vec{k})$

$$\left(\frac{\hbar^2}{2\mu} \nabla^2 + E_{\vec{k}} \right) G(\vec{r}, \vec{k}) = \delta^3(\vec{r})$$

$\Rightarrow$ the Schrödinger Eqn becomes the integral equation

$$\psi_{\vec{k}}(\vec{r}) = \varphi_0(\vec{r}) + \int d^3r' \; G(\vec{r}-\vec{r}', \vec{k}) \, U(r') \, \psi_{\vec{k}}(\vec{r}')$$

$\varphi_0(\vec{r})$: a solution of the free particle Schrödinger Eqn.

$\Rightarrow$ $\varphi_0(\vec{r})$: incident wave

rest = scattered wave $\Rightarrow$ We must use a Green function with outgoing boundary conditions.

The formal expression for $G(\vec{r}, \vec{k})$ is

$$G(\vec{r}, \vec{k}) = \int \frac{d^3k'}{(2\pi)^3} \; \frac{e^{i\vec{k}' \cdot \vec{r}}}{E_{\vec{k}} - E_{\vec{k}'}}$$

where $E_{\vec{k}'} = \dfrac{\hbar^2 \vec{k}'^2}{2\mu}$. As we did in $d=1$ we will specify the integration ~~contour~~ contour (i.e. how to go around the pole) to select only the outgoing waves:

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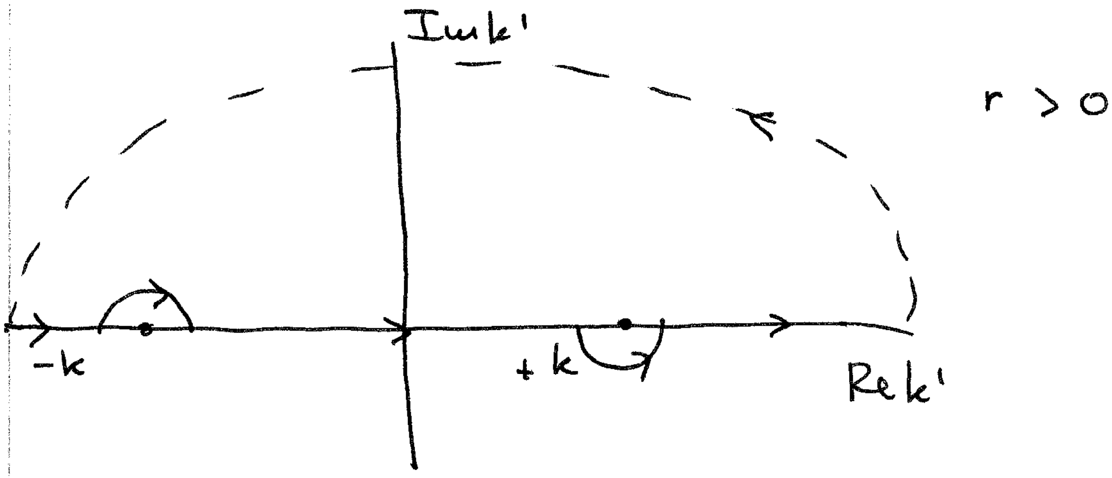
$$\int \frac{d^3k'}{(2\pi)^3}\,\frac{e^{i\vec{k}'\cdot\vec{r}}}{E_k-E_{k'}} = 2\pi\int_0^\infty \frac{dk'\,k'^2}{(2\pi)^3}\int_0^\pi d\theta\sin\theta\,\frac{e^{ik'r\cos\theta}}{E_k-E_{k'}}$$

$$= \int_0^\infty \frac{k'^{\cancel{2}}dk'}{(2\pi)^2}\,\frac{1}{E_k-E_{k'}}\,\frac{1}{i\cancel{k'}r}\left(e^{ik'r}+e^{-ik'r}\right)$$

$$= -\frac{1}{4\pi^2 i r}\,\frac{2\mu}{\hbar^2}\int_{-\infty}^{+\infty} dk'\,\frac{k'\,e^{ik'r}}{k'^2-k^2}$$

$$= -\frac{\mu}{2\pi^2 i r\hbar^2}\int_{-\infty}^{+\infty} dk'\,\frac{k'\,e^{ik'r}}{k'^2-k^2}$$



outgoing wave

$$= -\frac{\mu}{2\pi^2 i\hbar^2 r}\,\cancel{2}\pi i\,\frac{e^{ikr}}{\cancel{2}\cancel{k}}\cancel{k} = -\frac{\mu}{2\pi\hbar^2}\,\frac{e^{ikr}}{r} = G(\vec{r},k)$$

incoming plane wave

$$\Rightarrow \psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}.\vec{r}} - \frac{\mu}{2\pi\hbar^2}\int d^3r' \, \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} U(r')\psi_{\vec{k}}(\vec{r}')$$

$$\varphi_0(\vec{r}) = e^{i\vec{k}.\vec{r}}$$

outgoing ~~wave~~ scattered wave

This is an integral equation for $\psi_{\vec{k}}(\vec{r})$.

Note: $|\vec{k}|$ has not changed $\Rightarrow$ in elastic scattering there is no energy change.

We are interested in what happens far away from the scatterer (at the "detector") $(|\vec{r}| \gg |\vec{r}'|)$

~ range of U

$$\Rightarrow k|\vec{r}-\vec{r}'| \approx kr - \vec{q}.\vec{r}' + \dots$$

where $\vec{q} = k\frac{\vec{r}}{|\vec{r}|}$

$\Rightarrow$ we write an asymptotic solution ($|\vec{r}|$ large)

$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}.\vec{r}} + \frac{e^{ikr}}{r} f_{\vec{k}}(\Omega_r)$$

solid angle along $\vec{r}$

$$f_{\vec{k}}(\Omega_r) = -\frac{\mu}{2\pi\hbar^2}\int d^3r' \, e^{-i\vec{q}.\vec{r}'} U(r')\psi_{\vec{k}}(\vec{r}')$$

$f_{\vec{k}}(\Omega_r)$ = scattering amplitude.

$[f_k(\Omega_r)]$ = length

We must now expand $\psi(\vec{r}, t_0)$ in the eigenstates $\psi_{\vec{k}}(\vec{r})$:

$$\psi(\vec{r}, t_0) = \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \left[\psi_{\vec{k}}(\vec{r}) + \frac{\mu}{2\pi\hbar^2} \int d^3r' \, \frac{e^{i|\vec{k}||\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \, U(\vec{r}') \psi_{\vec{k}}(\vec{r}') \right]$$

Since $a(\vec{k})$ is peaked at $\vec{k}_0 \Rightarrow$ (except at a sharp scattering resonance) we can replace

$\psi_{\vec{k}}(\vec{r}') \rightarrow \psi_{\vec{k}_0}(\vec{r}')$ (i.e. $\psi_{\vec{k}}(\vec{r}')$ slowly varying in $\vec{k}$)

Since $\vec{k} \approx \vec{k}_0 \Rightarrow |\vec{k}| = k \approx \frac{\vec{k}\cdot\vec{k}_0}{|\vec{k}_0|} = \vec{k}\cdot\hat{k}_0$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \, e^{ik|\vec{r}-\vec{r}'|} \, \psi_{\vec{k}}(\vec{r}')$$

$$\approx \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \, e^{i\vec{k}\cdot\hat{k}_0|\vec{r}-\vec{r}'|} \, \psi_{\vec{k}_0}(\vec{r}') =$$

$$= \underbrace{\psi(\hat{k}_0|\vec{r}-\vec{r}'|, t_0) \, \psi_{\vec{k}_0}(\vec{r}')}_{\text{which have no overlap}}$$

$= 0$

$$\Rightarrow \psi(\vec{r}, t_0) = \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \, \psi_{\vec{k}}(\vec{r})$$

$$\Rightarrow \psi(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \, \psi_{\vec{k}}(r) \, e^{-i E_{\vec{k}} (t-t_0)/\hbar}$$

$\Rightarrow$ for $\vec{r}$ far from the scatterer we can use the asymptotic form to get

$$\psi(\vec{r}, t) = \psi_0(\vec{r}, t) + \int \frac{d^3k}{(2\pi)^3} \, \frac{e^{i(kr - E_{\vec{k}}(t-t_0)/\hbar)}}{r} \, f_{\vec{k}}(\Omega_r)$$

$$\psi_0(\vec{r}, t) = \int \frac{d^3k}{(2\pi)^3} \, a(\vec{k}) \, e^{i(\vec{k}\cdot\vec{r} - E_{\vec{k}}(t-t_0)/\hbar)}$$

evolution of the initial wave packet.

If $f_{\vec{k}}(\Omega_r)$ is slowly varying in $\vec{k}$ near $\vec{k}_0 \Rightarrow$

$$\psi(\vec{r}, t) = \underbrace{\psi_0(\vec{r}, t)}_{\text{evolution of initial wave packet}} + \underbrace{\frac{f_{\vec{k}_0}(\Omega_r)}{r} \, \psi_0(\hat{k}_0 r, t)}_{\text{scattering}}$$

This formula works away from resonances (i.e. no distortion of the wave packet) and for short ranged interactions.

Cross Sections

In classical Mechanics, scattering data is presented in the form of a {differential} cross section $\frac{d\sigma}{d\Omega}$. We imagine we have a beam (an ensemble) of particles scattered into a unit solid angle.

The cross section is the # of particles scattered into a unit solid angle per unit time relative to the # of particles in the beam crossing a unit area per unit time (i.e. a flux)

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\text{probability to be scattered into } \Omega_r}{\text{probability to cross the unit area in the beam}}$$

(wave packet is much broader than the target)

$\Rightarrow$ total probab. to be scattered into $d\Omega$ at $\vec{r}$

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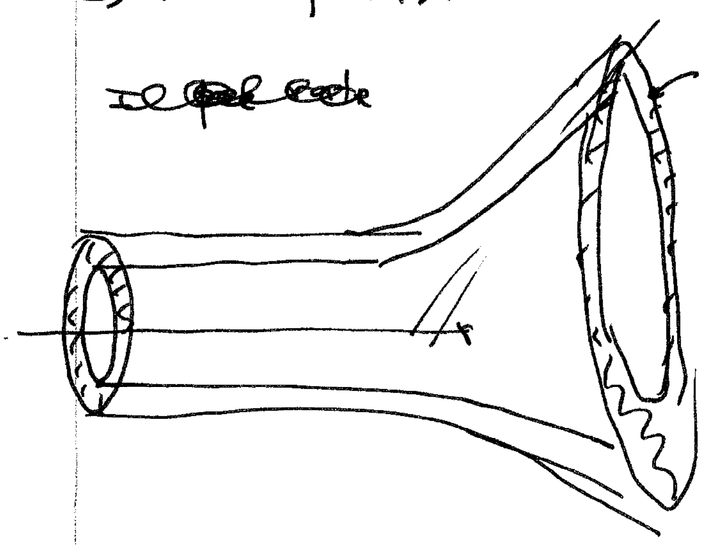

Probability to be scattered into $d\Omega$ at ? $= \int dt$ rate of strikes on area $r^2 d\Omega$

$$= \int_{-\infty}^{+\infty} \underbrace{dt}_{dt} \overset{\text{velocity}}{\frac{\hbar k_0}{\mu}} \times r^2 d\Omega \times \overbrace{\left|\frac{f_{\vec{k}_0}(\Omega_r)}{r}\right|^2}^{\text{square of the amplitude}} |\psi_0(\hat{k}_0 r, t)|^2$$

$\Rightarrow$ total probab. to be scattered into $d\Omega$ $= |f_{k_0}(\Omega_r)|^2 d\Omega \; \frac{\hbar k_0}{\mu} \int_{-\infty}^{+\infty} dt \, |\psi_0(\hat{k}_0 r, t)|^2$

probab. to cross a unit area at $\vec{r}_0$ in front of the beam $= \int dt$ flux $= \int_{-\infty}^{+\infty} dt \, \frac{\hbar k_0}{\mu} \, |\psi_0(\vec{r}_0, t)|^2$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = |f_{\vec{k}_0}(\Omega_r)|^2}$$

(neglecting the spread of the wave packet from $\vec{r}_0 \to \hat{k}_0 r$)

$\Rightarrow$ diff. cross section $=$ (scattering amplitude)^2

Total cross section $=$ total probability of scattering in all directions / unit flux

$$\boxed{\sigma = \int d\Omega \, |f_{\vec{k}_0}(\Omega_r)|^2}$$

Partial Wave Expansions

Since $V(\vec{r}) = V(|\vec{r}|) \Rightarrow$ rotational invariance.

$\Rightarrow$ it is natural to work out the eigenfunctions in the angular momentum basis.

Let us expand the plane wave

$$\Rightarrow e^{i\vec{k}\cdot\vec{r}} = \sum_{l=0}^{\infty} i^l (2l+1) \, P_l(\cos\theta) \, j_l(kr)$$

$$= \frac{1}{2} \sum_{l=0}^{\infty} i^l (2l+1) \, P_l(\cos\theta) \, (h_l(kr) + h_l^*(kr))$$

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} \; Y_{l,0}(\theta,\phi)$$

Legendre polynomials of order l.

Note: no $m \neq 0$ enter here

$\Rightarrow$ phase of $e^{i\vec{k}\cdot\vec{r}}$ is fixed as $\vec{r}$ rotates about $\vec{k}$.

$\Rightarrow$ scattered wave is also rotationally symmetric about $\vec{k}$

$\Rightarrow f_{\vec{k}}(\Omega_{\vec{r}})$ depends only on θ (not on ϕ) and E

Since $V(\vec{r}) = V(|\vec{r}|) \Longrightarrow$ if the incident wave ~~is~~ is in the state $(l,m) = (l,0)$

$\Rightarrow$ the scattered state is in the same state $(l, m) = (l, 0)$

$\Rightarrow$ we can study what each angular momentum state is doing separately.

$$\Rightarrow \psi_{\vec{k}}(\vec{r}) = \sum_{l=0}^{\infty} i^{l}(2l+1)\, P_l(\cos\theta)\, R_l(r)$$

where $\quad (r R_l = \chi_l)$

$$\frac{d^2(rR_l)}{dr^2} + k^2(rR_l) - \frac{l(l+1)}{r^2}(rR_l) = \frac{2\mu}{\hbar^2} V(r)(rR_l)$$

For $|\vec{r}|$ large (provided $\lim_{r\to\infty} r^2 V(r) = 0$)

$\Rightarrow$ it reduces to a Bessel equation.

$\Rightarrow$ in the far field

$$R_l(r) = B_l \left[h_l^*(kr) + S_l(E)\, h_l(kr) \right]$$

where h_l is the spherical Hankel function.

$$h_l = j_l + i n_l$$

we need B_l and $S_l(E)$

If $V = 0$ (strictly) $\Rightarrow$ $R_l = j_l = \frac{1}{2}(h_l + h_l^*)$

$\Rightarrow$ for $U = 0$ $\quad B_\ell = \frac{1}{2}$ and $S_\ell = 1$

h_ℓ^* : incoming spherical wave

h_ℓ : outgoing spherical wave

$\Rightarrow$ the potential only affects the outgoing wave.

$\Rightarrow$ even if $U \neq 0 \Rightarrow B_\ell = \frac{1}{2}$ but $S_\ell \neq 1$

Since the scattering is elastic $\Rightarrow |S_\ell(E)| = 1$

(This follows from

radial current $\rightarrow$ $$j_r(r) = \frac{\hbar}{2\mu i}\left(R_\ell^* \frac{\partial R_\ell}{\partial r} - R_\ell \frac{\partial R_\ell^*}{\partial r}\right) = 0$$

i.e. the potential is neither a source nor a sink of particles. But $j_r = 0$ only if $|S_\ell| = 1$)

$$\Rightarrow S_\ell = e^{i 2 \delta_\ell(E)}$$

$\uparrow$ phase shift: phase diff. between the outgoing wave function and the incoming plane wave.

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4/24/03 $\Rightarrow$ scattering process $\Leftrightarrow$ phase shifts.

$$\psi_{\vec{k}}(\vec{r}) = \frac{1}{2} \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) \left[h_l^*(kr) + \underbrace{e^{2i\delta_l(E)}}_{h_l(kr)} \right]$$

$$= \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \frac{1}{2} \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) \underbrace{(e^{2i\delta_l} - 1)\, h_l(kr)}_{\text{scattered}}$$

since for $kr \gg 1$ $\quad h_l(kr) \approx \frac{e^{i(kr - l\frac{\pi}{2})}}{ikr}$

Since the scattered wave $= f_{\vec{k}}(\Omega_r) \dfrac{e^{ikr}}{ikr}$

partial wave scatt. amp.

$$\Rightarrow f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) (e^{2i\delta_l} - 1)$$

$$\Rightarrow \boxed{f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta)\, e^{i\delta_l} \sin\delta_l}$$

is the form of the scattering amplitude ($kr \gg 1$)

Total cross section: $\int d\Omega \, |f(\theta)|^2$

since $\int_{-1}^{1} dx \, P_\ell(x) \, P_{\ell'}(x) = \frac{2\,\delta_{\ell\ell'}}{2\ell+1}$

$$\Rightarrow \boxed{\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell}$$

is the total cross section.

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell$$

$$\sigma_\ell = \frac{4\pi}{k^2} (2\ell+1) \sin^2\delta_\ell \leq \frac{4\pi}{k^2} (2\ell+1)$$

$\Rightarrow$ no interference between partial waves in σ

(although there is for $\frac{d\sigma}{d\Omega}$)

Optical Theorem

$$\mathrm{Im}\, f(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \, P_\ell(\cos\theta) \sin^2\delta_\ell$$

since $P_\ell(1) = 1$

$$\Rightarrow \boxed{\sigma = \frac{4\pi \, \mathrm{Im} f(0)}{k}}$$

Optical Thm.

total cross section $\propto$ Im $f(0)$ forward scattering.

The Born Approximation

In most cases we cannot compute exactly the radial wave functions. However if $U(r)$ is sufficiently short ranged $\Rightarrow$ it will have a small effect on a given partial wave.

Since

$$f_{\vec{k}}(\Omega_{\vec{r}}) = -\frac{\mu}{2\pi\hbar^2} \int d^3r' \, e^{-i\vec{k}' \cdot \vec{r}'} \, U(\vec{r}') \, \psi_{\vec{k}}(\vec{r}')$$

and

$$\psi_{\vec{k}}(\vec{r}) = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) R_l(r)$$

$$\Rightarrow f = -\frac{2\mu}{\hbar^2} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \int_0^\infty r^2 dr \, U(r) \, \underbrace{j_l(kr)}_{R_l(r)}$$

$$\Rightarrow e^{i\delta_l} \sin\delta_l = -\frac{2\mu k}{\hbar^2} \int_0^\infty dr \, r^2 \, U(r) \, j_l(kr) R_l(r)$$

If $U(r)$ has a "small" effect on $R_l(r)$

$\Rightarrow \delta_l$ will be small $\Rightarrow R_l \approx j_l(kr)$

$\Rightarrow O(\delta_l) \Rightarrow$ Born Approx. $\boxed{\delta_l \cong -\frac{2\mu k}{\hbar^2} \int_0^\infty r^2 dr \, U(r) \left(j_l(kr)\right)^2}$

If $U(r)$ has a small effect on all partial waves $\Rightarrow$

$$f_{\vec{k}}(\Omega_r) = -\frac{\mu}{2\pi\hbar^2}\int d^3r'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}\, U(\vec{r}\,')$$

$$= -\frac{\mu}{2\pi\hbar^2}\, U_{\vec{k}'-\vec{k}}$$

F.T. of $U(\vec{r})$ at $\vec{k}'-\vec{k}$

$$(\vec{k}-\vec{k}')^2 = k^2 + k'^2 - 2\vec{k}\cdot\vec{k}' = 2k^2(1-\cos\theta)$$

$$k^2 = k'^2 \qquad = \left(2k\sin\frac{\theta}{2}\right)^2$$

scatt. angle

e.g. for Yukawa $U(r) = \dfrac{a e^{-Kr}}{r}$

$$\Rightarrow U_{\vec{q}} = \frac{4\pi a}{q^2 + K^2} \Rightarrow \frac{d\sigma}{d\Omega} = \frac{a^2}{\left(4E_k \sin^2\frac{\theta}{2} + \frac{\hbar^2K^2}{2\mu}\right)^2}$$

Properties of the Scattering Amplitude

We will discuss some important general properties for potentials $U(r)$ w/ $\lim_{r\to\infty} r^2 U = 0$

For simplicity we will take $U(r) = 0 \quad r > b$

($b \to \infty$ if needed)

$\Rightarrow \quad r > b \qquad R_\ell(r) = \frac{1}{2}\left(h_\ell^*(kr) + e^{2i\delta_\ell} h_\ell(kr)\right)$

At $r = b \qquad R_\ell(r)$ and $\frac{dR_\ell}{dr}$ are continuous.

$$\Rightarrow \quad \left.\frac{\frac{\partial}{\partial r}\left(h_\ell^*(kr) + e^{i2\delta_\ell} h_\ell(kr)\right)}{h_\ell^*(kr) + e^{2i\delta_\ell} h_\ell(kr)}\right|_{r=b} =$$

$$= \left.\frac{1}{R_\ell(r)} \frac{dR_\ell(r)}{dr}\right|_{r=b} \equiv \alpha_\ell$$

$$\Rightarrow \quad e^{2i\delta_\ell} - 1 = 2\left(\frac{\frac{\partial j_\ell}{\partial r} - \alpha_\ell j_\ell}{\alpha_\ell h_\ell - \frac{\partial h_\ell}{\partial r}}\right)_{r=b}$$

$$\Rightarrow \quad \cot\delta_\ell = \left(\frac{\frac{\partial n_\ell}{\partial r} - \alpha_\ell n_\ell}{\frac{\partial j_\ell}{\partial r} - \alpha_\ell j_\ell}\right)_{r=b}$$

(using $e^{2i\delta} - 1 = \frac{2i}{\cot\delta - i}$)

e.g. Hard sphere: $V(r) = \begin{cases} \infty & r<b \\ 0 & r>b \end{cases}$

$\Rightarrow R_\ell(b) = 0 \quad \Rightarrow \quad \alpha_\ell = \infty$

$$\Rightarrow \cot\delta_\ell = \frac{n_\ell(kb)}{j_\ell(kb)}$$

$\Rightarrow$ s wave $\Rightarrow$ $\delta_0 = -kb$

<u>Small k behavior</u> (in general)

$$\cot\delta_\ell \approx \frac{1}{(kb)^{2\ell+1}} (2\ell-1)!!\,(2\ell+1)!!\ \frac{\ell+1+b\alpha_\ell}{\ell - b\alpha_\ell}$$

(using the form of j_ℓ and n_ℓ for $k\to 0$)

As $kb \to 0 \Rightarrow \alpha_\ell(E) \to \alpha_\ell(0)$

$$\Rightarrow \sin\delta_\ell \sim k^{2\ell+1}$$

$\Rightarrow$ higher partial waves have a small weight

$\Rightarrow$ for $kb \ll 1 \Rightarrow$ neglect all $\ell \geq 1$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2}\sin^2\delta_0 \qquad \text{for } kb \ll 1$$

$\Rightarrow$ all low energy scattering is s-wave.

* $e^{2i\delta_\ell} - 1$ has poles at the bound states of the potential (as an analytic function of the energy E) with angular momentum ℓ.

Recall that the wave functions for the <u>bound</u> <u>states</u> are, for $r > b$

$$R_\ell(r) \sim h_\ell(i\kappa r)$$

when $E = -\frac{\hbar^2\kappa^2}{2\mu}$

$$\Rightarrow \quad \alpha_\ell = \left[\frac{1}{h_\ell(i\kappa r)} \frac{\partial h_\ell(i\kappa r)}{\partial r}\right]_{r=b} \qquad \left(\begin{array}{c}\text{bound}\\\text{states}\end{array}\right)$$

But this can only be True if $\cot\delta_\ell(E) = i$

$\Rightarrow \quad e^{2i\delta_\ell(E)}$ has a pole at the bound state energies.

If $\kappa b \ll 1 \Rightarrow$ using the asymptotic expansion for $h_\ell(x)$ $(x \to 0)$

$$h_\ell(x) \sim \frac{x^\ell}{(2\ell+1)!!} - i\,\frac{(2\ell-1)!!}{x^{\ell+1}}$$

$$\Rightarrow \quad \alpha_\ell(E) = \left(\frac{1}{h_\ell(i\kappa r)} \frac{\partial}{\partial r} h_\ell(i\kappa r)\right)_{r=b} \simeq -\frac{\ell+1}{b}, \text{ or}$$

equivalently $\Rightarrow \ell + 1 + b\,\alpha_\ell(E) \simeq 0$

$\Rightarrow$ if there is a bound state with $|E| \ll \frac{\hbar^2}{2\mu b^2}$

$\Rightarrow$ by continuity $\quad l+1+b\,\alpha_l(0)$ will be small

$\Rightarrow$ $\cot\delta_l \downarrow$ and $\sin\delta_l \uparrow$ (over their values in the absence of a bound state)

What does this imply for low energy scattering?

First, we already know that at low energy all scattering is s-wave. $\quad (kb \ll 1)$

$$\Rightarrow \quad k\cot\delta_0 = -\frac{1+\cancel{b}\,\alpha_0(0)}{b^2\alpha_0(0)} \equiv -\frac{1}{a}$$

a: scattering length.

and σ at zero energy is

$$\boxed{\sigma = 4\pi a^2} \qquad (E \to 0)$$

i.e. a is an effective hard sphere radius.

$$\Rightarrow e^{2i\delta_0} - 1 = \frac{2i}{\cot\delta_0 - i} = \frac{2ka}{i - ka}$$

$(kb \ll 1)$ $\Rightarrow$ there is a pole at $k = \frac{i}{a}$

or $\quad K = -ik = \frac{1}{a} \quad (b \ll |a|)$

$\Rightarrow$ there is a bound state if $K > 0$ and small.

$\Rightarrow$ (1) if $a > 0$ and large $\Rightarrow \exists$ an s wave bound state near $E = 0$, with

$$E_b = -\frac{\hbar^2 K^2}{2M} = -\frac{\hbar^2}{2Ma^2}$$

Note: since $\psi \sim \frac{e^{-Kr}}{r} \Rightarrow a \sim$ size of the bound state.

low energy cross section:

$$\sigma = \frac{4\pi}{k^2} \frac{1}{\cot^2\delta_0 + 1} = \frac{2\pi\hbar^2/M}{E - E_b}$$

$\Rightarrow$ the bound state determines the behavior of the cross section (at low energy).

Low Energy Resonances

What happens at somewhat higher energies? (still in the regime $kb \ll 1$ where $\cot\delta_\ell$ has simple behavior). In this range

$$\cot\delta_\ell \sim \frac{1}{k^{2\ell+1}} \gg 1$$

$$\Rightarrow \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{1}{\cot^2\delta_\ell + 1} \sim k^{4\ell} \to 0$$

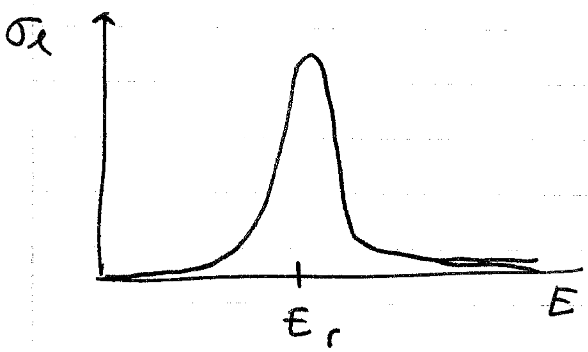
except for $\ell = 0$ (s wave)

But near $E = E_r$ where

$$\ell + 1 + b\,\alpha_\ell(E) \simeq 0$$

$\Rightarrow$ $\cot\delta_\ell \to 0$ and $\sigma_\ell \sim k^{-2} \gg 1$ instead

$\Rightarrow$ bump in σ_ℓ $\Rightarrow$ scattering resonance.



$(\ell \geq 1)$

Near E_r

$$\ell + 1 + b\,\alpha_\ell(E) \approx (E - E_r)\, b \left(\frac{\partial \alpha_\ell}{\partial E}\right)_{E_r}$$

$$\ell - b\,\alpha_\ell(E) \approx 2\ell + 1$$

$$\Rightarrow \cot\delta_\ell \approx -\frac{2(E - E_r)}{\Gamma_k}$$

$$\Gamma_k = -\frac{2k^{2\ell+1}\, b^{2\ell}}{[(2\ell-1)!!]^2 \left(\frac{\partial \alpha_\ell}{\partial E}\right)_{E_r}}$$

Since $\left(\frac{\partial \alpha_\ell}{\partial E}\right)_{E_r} < 0 \Rightarrow \Gamma_k > 0$

Near a resonance $(\ell \geq 1)$

$$\sigma_\ell = \frac{4\pi \Gamma_k^2 / k^2}{4(E-E_r)^2 + \Gamma_k^2}$$ ← Lorentzian lineshape

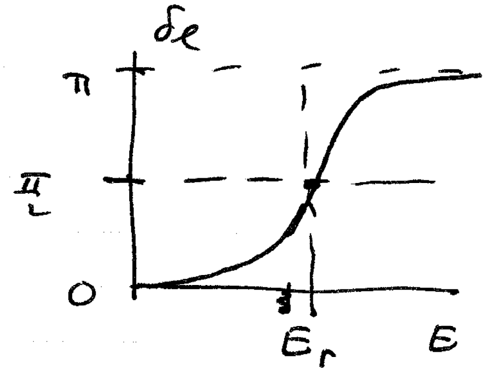
⇒ the more rapidly α_ℓ varies near E_r

⇒ the sharper the resonance.

A resonance occurs if $\cot \delta_\ell = 0 \Rightarrow \delta_\ell = (n+\frac{1}{2})\pi$

⇒ near a resonance:

$$\delta_\ell \approx \frac{\pi}{2} + \tan^{-1}\left(\frac{E-E_r}{\Gamma_k/2}\right)$$



Also, near a resonance.

$$e^{2i\delta_\ell} - 1 \approx - \frac{i\Gamma_k}{E - E_r + \frac{i\Gamma_k}{2}}$$

⇒ pole in the complex plane $E_r - i\frac{\Gamma}{2}$ (2nd Riemann sheet)