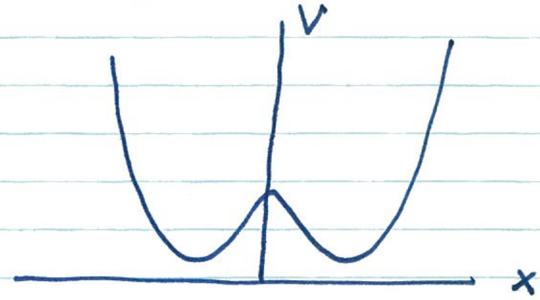
Instantons and the double-well potential

Let's consider a quantum mechanical double well problem



$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{\mu^2}{2}x^2 - \frac{\lambda}{4}x^4$$

(λ small)

Classically there are two degenerate ground states

$$x_\pm = \pm\left(\frac{\mu^2}{\lambda}\right)^{1/2}$$

and the $\pm$ symmetry is (classically) broken.

In p.thy. $\quad x = +\left(\frac{\mu^2}{\lambda}\right)^{1/2} + y \quad$ (y small)

and nothing dramatic happens (the degeneracy is not lifted).

$$\Rightarrow \quad \langle x(0)\; x(t)\rangle \xrightarrow[t\to\infty]{} \text{constant}.$$

Wick rotation $\quad \tau = it$

$$\langle x(0)\; x(\tau)\rangle = \sum_n \underbrace{|x_{n0}|^2}_{|\langle n|x|0\rangle|^2} e^{-(E_n - E_0)\tau} \xrightarrow[\tau\to\infty]{} e^{-\Delta E\,\tau}$$

ΔE: splitting.

$$\langle x(0)\, x(\tau) \rangle = \frac{\int \mathcal{D}x(\tau)\, e^{-\mathcal{E}[x]}\, x(0)\, x(\tau)}{\int \mathcal{D}x(\tau)\, e^{-\mathcal{E}[x]}}$$

$$\mathcal{E}[x] = \int_{-\infty}^{+\infty} d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - \frac{\mu^2 x^2}{2} + \frac{\lambda}{4} x^4 \right]$$

Q.M. $\leftrightarrow$ 1 D c.S.M.

Extrema of $\mathcal{E}[x]$: $\delta\mathcal{E} = 0 \quad \Rightarrow \bar{x}(\tau)$

$$\mathcal{E}(x) = \mathcal{E}(\bar{x}(\tau)) + \int \left.\frac{\delta\mathcal{E}}{\delta x}\right|_{\bar{x}} \delta x + \frac{1}{2} \int \frac{\delta^2 \mathcal{E}}{\delta x\, \delta x'}\, \delta x\, \delta x'$$

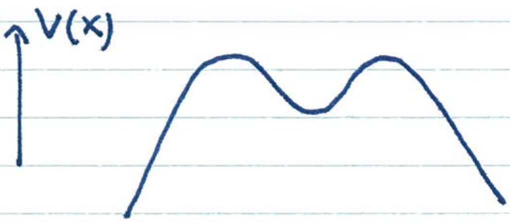
$$\left.\frac{\delta\mathcal{E}}{\delta x}\right|_{x=\bar{x}} = 0 \qquad \delta x = x(\tau) - \bar{x}(\tau)$$

$$0 = \frac{\delta\mathcal{E}}{\delta\bar{x}} - \frac{d}{d\tau}\frac{\delta\mathcal{E}}{\delta\dot{\bar{x}}} \qquad \text{(Euler-Lagrange)}$$

$$\frac{\delta\mathcal{E}}{\delta\dot{\bar{x}}} = \dot{\bar{x}} \qquad \frac{\delta\mathcal{E}}{\delta\bar{x}} = -\mu^2\bar{x} + \lambda\bar{x}^3$$

$$\frac{d^2\bar{x}}{d\tau^2} = -\mu^2\bar{x} + \lambda\bar{x}^3$$

Eq. of. Mot. of Classical particle in pot. $V(x)$



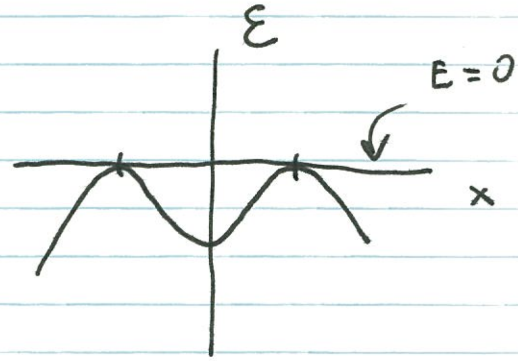
$$V(x) = \frac{\mu^2}{2}\bar{x} - \frac{\lambda}{4}\bar{x}^4 = -U(x)$$

$E = \frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2 - V(\bar{x})$ is "conserved"

(1) $\bar{x} = \pm\left(\frac{\mu^2}{\lambda}\right)^{1/2}$ is a pair of deg. solutions.

$$\mathcal{E}(\bar{x}) = T\left(\frac{\lambda}{4}\bar{x}^4 - \frac{\mu^2}{2}\bar{x}^2\right) = T\left[\frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^2}{2}\frac{\mu^2}{\lambda}\right]$$

$$\mathcal{E}(\bar{x}) = -\frac{\mu^4}{4\lambda}T$$



$\boxed{30}$ (2) $E=0$ $\quad \frac{d\bar{x}}{d\tau} = \sqrt{2V(\bar{x})}$

$$\bar{x}_c(\tau) = \pm\left(\frac{\mu^2}{\lambda}\right)^{1/2} \tanh\frac{\mu(\tau-a)}{\sqrt{2}}$$

$$\mathcal{E}(\bar{x}_c(\tau)) - \mathcal{E}\left(\left(\frac{\mu^2}{\lambda}\right)^{1/2}\right) = 2\sqrt{2}\,\frac{\mu^3}{\lambda} \quad \text{finite !}$$

$\Rightarrow$ the solution $\bar{x}_c(\tau)$ has a contrib. $\sim e^{-2\sqrt{2}\frac{\mu^3}{\lambda}}\,\tau$

$\Rightarrow$ it is important if $\tau > e^{+2\sqrt{2}\mu^3/\lambda}$

$$\text{Stability} = \frac{1}{2}\int d\tau \int d\tau' \left.\frac{\delta^2\mathcal{E}}{\delta x(\tau)\,\delta x(\tau')}\right|_{\bar{x}} (x(\tau)-\bar{x}(\tau))\,(x(\tau')-\bar{x}(\tau'))$$

$$\left.\frac{\delta^2\mathcal{E}}{\delta x(\tau)\,\delta x(\tau')}\right|_{\bar{x}_c} = -\frac{d^2}{d\tau^2}\delta(\tau-\tau') + (-\mu^2 + 3\lambda\,\bar{x}_c^2(\tau))\,\delta(\tau-\tau')$$

$$\equiv \left[-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda\,\bar{x}_c^2(\tau))\right]\delta(\tau-\tau')$$

Let $y_n(\tau)$ be a complete set of eigenstates of the operator $-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda \bar{x}_c^2(\tau))$ satisfying

$$y(-\infty) = y(+\infty) = 0$$

$$\boxed{3\lambda \bar{x}_c^2 - \mu^2 = 3\mu^2 - \frac{4\mu^2}{\cosh^2\left(\frac{\mu(\tau-a)}{\sqrt{2}}\right)}}$$ Pöschl-Teller

$$\left[-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda \bar{x}_c^2(\tau))\right] y_n(\tau) = \omega_n^2\, y_n(\tau)$$

↳ eigenvalues.

$\Rightarrow \bar{x}_c(\tau)$ is stable iff $\omega_n^2 \geq 0$

Try

$$x(\tau) = \bar{x}_c(\tau) + \sum_n \xi_n\, y_n(\tau) \qquad \xi_n \text{ arbitrary}$$

$$\Rightarrow \mathcal{D}x(\tau) = N \prod_n d\xi_n$$

1 kink ↓ no kink ↓

$$\langle x(0)\, x(\tau)\rangle \cong \frac{e^{-\mathcal{E}[\bar{x}_c(\tau)]}\, \bar{x}_c(0)\, \bar{x}_c(\tau) \prod_n \omega_n^{-1}(\bar{x}_c) + \frac{\mu^2}{\lambda}}{1 + e^{-\mathcal{E}[\bar{x}_c(\tau)]} \prod_n \omega_n^{-1}(\bar{x}_c)}$$

problem: $\det\left(-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda \bar{x}_c^2(\tau))\right)^{1/2} = \prod_n \omega_n(\bar{x}_c)$

Vanishes because $\omega_0 = 0$!

Symmetry: $\bar{x}_c(\tau)$ has a zero-mode : a (origin)

$\mathcal{E}[\bar{x}_c]$ is indep. of a

$$\Rightarrow \frac{\delta \mathcal{E}(\bar{x}_c)}{\delta a} = 0$$

$$\Rightarrow \frac{\delta}{\delta \bar{x}_c(\tau')} \frac{\delta \mathcal{E}}{\delta a} = 0$$

$$\frac{\delta^2 \mathcal{E}}{\delta \bar{x}_c(\tau') \delta \bar{x}_c(\tau)} \frac{d\bar{x}_c(\tau)}{da} = 0 \Rightarrow \quad \frac{d\bar{x}(\tau)}{da} \text{ is a zero-energy mode.}$$

$$y_0 = \frac{d\bar{x}_c(\tau)}{da} \Leftrightarrow \omega_0^2 = 0 \quad \text{no restoring force.}$$

a: Collective Coordinate. Need to deal with it exactly.

But $\begin{cases} \mathcal{E}[x] = \mathcal{E}[x_a] & x(\tau) \mapsto x_a \equiv x(\tau + a) \text{ translation} \\ \mathcal{D}x = \mathcal{D}x_a & \text{(invariant measure)} \end{cases}$

Introduce "1" ; Let $F(x)$ be a function

Jacobian!

$$1 = \int dF \, \delta(F(x_a)) = \int_{-\infty}^{+\infty} da \, \delta(F(x_a)) \frac{\partial F}{\partial a}(x_a)$$

$$\Rightarrow Z = \int \mathcal{D}x \, e^{-\mathcal{E}[x]} \equiv \int \mathcal{D}x \int_{-\infty}^{+\infty} da \, \delta(F(x_a)) \frac{\partial F}{\partial a}(x_a) \, e^{-\mathcal{E}[x]}$$

Change of variables: $x \to x_{-a}$ or $x(\tau) \to x(\tau - a)$

$$\frac{\partial F}{\partial a}(x_a) = D[x, a]$$

$$Z = \int \mathcal{D}x_{-a} \int_{-\infty}^{+\infty} da \, \delta(F(x)) \, D[x_{-a}, a] \, e^{-\mathcal{E}[x_a]}$$

$$(x_a)_{-a} = x_0 \equiv x$$

Using translation invariance and the invariance of the measure $\Rightarrow$

$$Z = \int \mathcal{D}x \int_{-\infty}^{+\infty} da \; D(x_{-a}, a) \; \delta(F(x)) \; e^{-\mathcal{E}[x]}$$

Let's calculate D

$$D = D[x_{-a}, a]$$

$$\cancel{D[x_a, a] = \frac{\partial F[x_a]}{\partial a} = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0} (x}$$

$$F[x_a] = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0} \left(x(\tau+a) - x_{cl}(\tau)\big|_{a=0} \right)$$

$$\frac{\partial F}{\partial a}[x_a] = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0} \frac{\partial x(\tau+a)}{\partial a} = D[x_a, a]$$

$$D = D[x_{-a}, a] = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0}(\tau) \; \frac{\partial x}{\partial a}\Big|_{a=0}(\tau)$$

$$x(\tau) = x_{cl}(\tau, a) + \sum_{n \neq 0} \xi_n \, y_n(\tau - a)$$

$$x_{cl}(\tau, a) \equiv x_{cl}(\tau - a)$$

$$\frac{\partial x_{cl}}{\partial a} \equiv - \frac{\partial x_{cl}}{\partial \tau}$$

$$\frac{\partial y_n(\tau - a)}{\partial a} = - \frac{\partial y_n}{\partial \tau}$$

$$\Rightarrow \frac{\partial x(\tau)}{\partial a}\Big|_{a=0} = - \left[\frac{\partial x_{cl}(\tau)}{\partial \tau} + \sum_{n\neq 0} \xi_n \frac{\partial y_n(\tau)}{\partial \tau} \right]_{a=0}$$

$$\Rightarrow D = \int_{-\infty}^{+\infty} d\tau \left(\frac{\partial x_{cl}}{\partial \tau}\right)^2 + \sum_{n\neq 0} \xi_n \int_{-\infty}^{+\infty} d\tau \left.\frac{\partial x_{cl}}{\partial \tau}\right|_{a=0}(\tau) \left.\frac{\partial y_n}{\partial \tau}\right|_{a=0}(\tau)$$

$$A = \int_{-\infty}^{+\infty} d\tau \; \dot{x}_{cl}^2 \qquad \dot{x}_{cl} = \left.\frac{\partial x_{cl}}{\partial \tau}\right|_{a=0}$$

$$r_n = \int_{-\infty}^{+\infty} d\tau \; \dot{x}_{cl}\, \dot{y}_n \qquad \dot{y}_n = \left.\frac{\partial y_n}{\partial \tau}\right|_{a=0}$$

$$D = A + \sum_{n\neq 0} \xi_n r_n$$

Hence

$$Z = \int \mathcal{D}x \int_{-\infty}^{+\infty} da \; \delta[F(x)] \; \left[A + \sum_{n\neq 0} \xi_n r_n\right] e^{-\mathcal{E}[x]}$$

$$\equiv \int \prod_n d\xi_n \int_{-\infty}^{+\infty} da \; \left[A + \sum_{n\neq 0} \xi_n r_n\right] e^{-\mathcal{E}[x_{cl}]} \, e^{-\left[\int d\tau d\tau' \frac{1}{2} \left.\frac{\delta^2 \mathcal{E}}{\delta x(\tau)\delta x(\tau')}\right|_{x_{cl}} \cdot \tilde{x}(\tau)\, \tilde{x}(\tau')\right]}$$

$$\tilde{x} = x(\tau) - x_{cl}(\tau, a) \equiv \sum_{n\neq 0}^{\infty} \xi_n \, y_n(\tau - a)$$

Note that y_n's are the e. functions with non-vanishing e. values. These are necessarily $\xi > 0$ for stability. Also note that since $x_{cl}(\tau, a)$ is monotonic $\Rightarrow \frac{\partial x_{cl}(\tau, a)}{\partial a}$ has no zeros $\Rightarrow$ (nodeless) $\Rightarrow$ is the gnd. state w.f. $\Rightarrow$ it is the lowest energy state $= 0$

Now $F(x)=0$ is automatically satisfied.

$$x_{a=0}(\tau) = x_{cl}(\tau,a)\Big|_{a=0} + \sum_{n\neq 0} \xi_n y_n(\tau) \quad ; \quad y_0(\tau) = N \frac{\partial x_{cl}}{\partial a}\Big|_{a=}$$

$$F[x] = \int_{-\infty}^{+\infty} d\tau \; y_0(\tau) \sum_{n\neq 0} \xi_n y_n(\tau) =$$

$$= \sum_{n\neq 0} \xi_n \int_{-\infty}^{+\infty} d\tau \, y_0(\tau) y_n(\tau) = 0 \qquad \text{(orthogonality)}.$$

$$\Rightarrow Z = \int_{-\infty}^{+\infty} \prod_n d\xi_n \int_{-\infty}^{+\infty} da \left(A + \sum_{n\neq 0} \xi_n r_n\right) e^{-\mathcal{E}[x_{cl}]} \; e^{-S_{eff}[\xi_n]}$$

to leading order we can neglect $\sum_{n\neq 0} \xi_n r_n$

$$Z \simeq \underbrace{\left[\int_{-\infty}^{+\infty} da \; A\right]}_{\text{inv. measure}} e^{-\mathcal{E}[x_{cl}(a)]} \left[\prod_{n\neq 0} \omega_n^{-1}\right] \text{const.}$$

and

$$\langle x(0) \, x(\tau) \rangle \simeq A \int_{-\infty}^{+\infty} da \; x_{cl}(0,a) \, x_{cl}(\tau,a) \; e^{-\mathcal{E}[x_{cl}]} \prod_n \omega_n^{-1}$$

$$Z \approx A \left[\int_{-\infty}^{+\infty} da\right] e^{-\mathcal{E}[x_{cl}]} \prod_n \omega_n^{-1}$$

$\Rightarrow$ Sum

$$\langle x(0), x(\tau) \rangle \simeq \frac{\frac{\mu^2}{\lambda} \prod_{n\neq 0} \omega_{n,0}^{-1} + A\left[\int_{-\infty}^{+\infty} da \; x_{cl}(0) \, x_{cl}(\tau)\right] e^{-\mathcal{E}[x_{cl}]} \prod_{n\neq 0} \omega_n^{-1}}{\prod_{n\neq 0} \omega_{n,0}^{-1} + A\left(\int_{-\infty}^{+\infty} da\right) e^{-\mathcal{E}[x_{cl}]} \prod_{n\neq 0} \omega_n^{-1}}$$

$$\langle x(0)\, x(\tau)\rangle \simeq \frac{\mu^2}{\lambda} + A\, e^{-\mathcal{E}[x_{cl}]} \int_{-\infty}^{+\infty} da \left[x_{cl}(0;a)\, x_{cl}(\tau,a) - \frac{\mu^2}{\lambda} \right] \cdot$$

$$\cdot \left[\frac{\prod_{n\neq 0} \omega_n^{-1}}{\prod_{n\neq 0} \omega_{n,0}^{-1}} \right]$$

$$\prod_{n\neq 0} \omega_n^{-1} = \det{}' \left[-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda\, x_{cl}^2(\tau)) \right]^{-1/2}$$ without zero modes

$$\prod_{n\neq 0} \omega_{n,0}^{-1} = \det{}' \left[-\frac{d^2}{d\tau^2} + (2\mu^2) \right]^{-1/2}$$

This ratio (call it K) can be calculated in terms of phase shifts.

$$\int_{-\infty}^{+\infty} da \left[x_{cl}(0,a)\; x_{cl}(\tau,a) - \frac{\mu^2}{\lambda} \right] = -\frac{\mu^2}{\lambda}\tau \frac{2}{\tanh \frac{\mu\tau}{\sqrt{2}}}$$

$$\underset{\tau\to\infty}{\sim} -\frac{2\mu^2}{\lambda}\tau$$

$$f(\tau) \underset{\tau\to\infty}{\sim} \frac{\mu^2}{\lambda} - K\, A\, e^{-\mathcal{E}[x_{cl}]} \left(\frac{2\mu^2}{\lambda}\right)\tau$$

this approx. fails for $\tau \gtrsim \frac{e^{\mathcal{E}[x_{cl}]}}{2KA} = \frac{e^{2\sqrt{2}\mu^3/\lambda}}{2KA}$

$$A = \int \dot{x}^2\, d\tau \equiv \mathcal{E}[x_{cl}] = \frac{2\sqrt{2}\,\mu^3}{\lambda}$$

$\Rightarrow$ we have problems for $\tau \to \infty$

The width of the kink (or instanton) is $\sim \frac{1}{\mu}$

If λ is small $\Rightarrow e^{2\sqrt{2}\mu^3/\lambda} \gg \frac{1}{\mu}$

Hence we can sum over an ensemble of weakly interacting instantons (exponentially weak!)

Let $\quad x_{cl}(\tau) \simeq \sqrt{\frac{\mu^2}{\lambda}} \prod_{j=1}^{N} \mathrm{sgn}(\tau - a_j)$

a_j: location of the j-th instanton.

$\mathcal{E}_{cl}$ for N instantons $\simeq N \frac{2\sqrt{2}\mu^3}{\lambda}$ + exp. small terms

$$\Rightarrow \quad \langle x(0)\, x(\tau) \rangle = \frac{\frac{\mu^2}{\lambda}\left[\sum'_{N=0}^{\infty} c^N e^{-\frac{2\sqrt{2}\mu^3}{\lambda}} \int_{a_1 < \dots < a_N} da_1 \dots da_N \prod_{j=1}^{N} \mathrm{sgn}(\tau - a_j)\right.}{\sum_{N=0}^{\infty} c^N e^{-2\sqrt{2}\frac{\mu^3 N}{\lambda}} \int_{a_1 < a_2 < \dots < a_N} da_1 \dots da_N}$$

$$c = \frac{AK\, 2\tau}{\tanh\left(\frac{\mu\tau}{\sqrt{2}}\right)} \approx 2AK\tau \qquad (\tau \to \infty)$$

$$\int_{a_1 < \dots < a_N} da_1 \dots da_N = \frac{T^N}{N!}$$

$$\Rightarrow \langle x(0)\, x(\tau) \rangle = \frac{\mu^2}{\lambda} e^{-(\Delta E)\cdot \tau}$$

$$\Delta E \cong 2AK\, e^{-\frac{2\sqrt{2}\mu^3}{\lambda}}$$

$\Rightarrow$ The ground state and the 1st excited state are split by an amount

$$\sim 2AK\, e^{-A}$$

$$A = \frac{2\sqrt{2}\,\mu^3}{\lambda}$$

which has an essential singularity in λ.

References:

- Aspects of Symmetry (Coleman)
- Solitons and Instantons (Rajaraman)
- Gauge Fields and Strings (Polyakov)