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The Linear Harmonic Oscillator

We will discuss now the Quantum Mechanics of the LHO. This is a very important system.

For a LHO the potential is

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

where m is the mass and ω the angular frequency of the oscillator. The eigenvalue problem is

$$\hat{H}\psi = E\psi$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi}$$

We will find the spectrum of eigenvalues and their wavefunctions. We can write this equation in the form

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2} m\omega^2 x^2\right)\psi = 0$$

Let us scale x by introducing the length scale ℓ

$$x = \ell y \Rightarrow y \text{ is } \underline{\text{dimensionless}}$$

$$\frac{d^2\psi}{dy^2} + \frac{2mE}{\hbar^2}\ell^2\psi - \ell^4\frac{m^2\omega^2}{\hbar^2}y^2\psi = 0$$

$$\Rightarrow \quad \ell = \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

and $\quad \varepsilon = \frac{mE}{\hbar^2}\ell^2 = \frac{E}{\hbar\omega} \quad$ is the "dimensionless energy"

$$\Rightarrow \boxed{\frac{d^2\psi}{dy^2} + (2\varepsilon - y^2)\psi = 0}$$

$\Rightarrow$ we learned two things

① there is a natural length scale $\quad \ell = \left(\frac{\hbar}{m\omega}\right)^{1/2}$

② there is a natural energy scale : $\hbar\omega$

Q: How do the wave functions $\psi(y)$ behave for large $|y| \to \infty$? Clearly if $|y| \gg \sqrt{2\varepsilon}$

$\Rightarrow \quad \psi'' - y^2\psi = 0 \quad$ is our equation

(note: the spectrum must be positive!)

Let us try $\quad \psi(y) \simeq A\, y^n\, e^{\pm y^2/2}$

$$\Rightarrow \psi' = n A y^{n-1} e^{-y^2/2} \pm A y^{n+1} e^{-y^2/2}$$

$$\psi'' = A\, y^{n+2}\, e^{\pm y^2/2}\left[1 \pm \frac{2n+1}{y} + \frac{n(n-1)}{y^2}\right]$$

Hence, for $|y| \gg \sqrt{2E}$, the equation is

$$\psi'' \approx A\, y^{n+2}\, e^{\pm y^2/2} + O\left(\frac{1}{y}\right)$$

$\Rightarrow$ $\psi'' = y^2 \psi$ is satisfied.

Notice that we have two possible solutions, $e^{\pm y^2/2}$, but only one is normalizable in the ~~for~~ whole line. $\Rightarrow$ $\psi \sim A\, y^n\, e^{-y^2/2}$

Let us now examine the opposite regime $y \to 0$.

For $|y| \ll \sqrt{2E}$ the equation becomes

$$\psi'' + 2E\,\psi = 0$$

$$\Rightarrow \psi = A \cos(\sqrt{2E}\, y) + B \sin(\sqrt{2E}\, y)$$

For $|y| \ll \sqrt{2E}$ we get

$$\psi(y) \longrightarrow A + (B\sqrt{2E})\, y + O(y^2)$$

$\Rightarrow$ $\psi(y) = u(y)\, e^{-y^2/2}$ should work.

$\Rightarrow$ we get ~~$-2yu'$~~ $u'' - 2yu' + (2E-1)\, u = 0$

~~$u'' + (2E-1)u = 0$~~

We need solutions of this equation which behave "nicely" at $|y| \to \infty$, i.e. as $|y| \to \infty$ $u \to y^n$ where n is a number. If this condition is not met we will not reproduce the correct $|y| \to \infty$ behavior.

Hence, $u(y)$ should grow <u>at most</u> like a power, y^n, as $y \to \infty$ (we don't know as of now if n is an integer or not). Thus equation can be solved in terms of a power series

$$u(y) = \sum_{\ell=0}^{\infty} c_\ell \, y^\ell$$

$\Rightarrow$ we get

$$u'' - 2yu' + (2\varepsilon - 1)\, u = 0 \Rightarrow \sum_{\ell=0}^{\infty} c_\ell \left[\ell(\ell-1)\, y^{\ell-2} - 2\ell y^\ell + (2\varepsilon-1)\, y^\ell \right] = 0$$

$$\sum_{\ell=0}^{\infty} \ell(\ell-1)\, c_\ell \, y^{\ell-2} \equiv \sum_{\ell=2}^{\infty} \ell(\ell-1)\, c_\ell \, y^{\ell-2} = \sum_{\ell=0}^{\infty} (\ell+2)\ell \, c_{\ell+2}\, y^\ell$$

$$\Rightarrow \sum_{\ell=0}^{\infty} \left[\ell(\ell+2)\, c_{\ell+2} - 2\ell \, c_\ell + (2\varepsilon - 1)\, c_\ell \right] y^\ell = 0$$

Since it must be obeyed $\forall y \Rightarrow$

$$\ell(\ell+2)\, c_{\ell+2} + (2\varepsilon - 1 - 2\ell)\, c_\ell = 0$$

or

$$c_{\ell+2} = c_\ell \, \frac{(2\ell + 1 - 2\varepsilon)}{\ell(\ell+2)}$$

Notes: if we set a value for $c_0 \Rightarrow$ we get all even terms c_ℓ with ℓ even, and if we set a value for c_1 we get c_ℓ with ℓ odd.

Since $\psi(y) \sim y^n e^{\pm y^2/2}$ $(|y| \to \infty)$ and

$$\psi(y) = u(y)\, e^{-y^2/2} \Rightarrow$$

$$u(y) \sim \begin{cases} y^n & \text{or} \quad y^n e^{+y^2} \end{cases} \qquad |y| \to \infty$$

$\Rightarrow$ we only want the solution $u(y) \sim y^n$

$$\frac{c_{\ell+2}}{c_\ell} = \frac{2\ell+1-2\varepsilon}{(\ell+2)\,\ell} \xrightarrow[\ell \to \infty]{\text{(if it does not vanish)}} \frac{2\ell}{\ell^2} = \frac{2}{\ell}$$

$\Rightarrow$ For $\ell \gg 1$ $\qquad c_{\ell+2} \simeq \frac{2}{\ell}\, c_\ell + \cdots$

and $$y^n e^{y^2} = \sum_{\ell=0}^{\infty} \frac{y^{2\ell+n}}{\ell!} \equiv \sum_m c_m y^m$$

$$m = 2\ell + n$$

$$\ell = \frac{m-n}{2}$$

$$\Rightarrow \quad c_m = \frac{1}{\ell!} = \frac{1}{\left(\frac{m-n}{2}\right)!}$$

$$\frac{c_{m+2}}{c_m} = \frac{1}{\left(\frac{m+2-n}{2}\right)!} \left(\frac{m-n}{2}\right)! = \frac{1}{\frac{m-n}{2}+1} \sim \frac{2}{m}$$

$\Rightarrow$ if the series does not terminate, $u(y) \sim y^n e^{y^2}$

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$\Rightarrow$ the series must terminate and this can only happen if for some integer $l=n$, $c_l=0$

Notice that we ~~can~~ construct two linearly independent solutions by setting $C_1=0$ or $C_0=0$.

If $C_1=0 \Rightarrow$ all odd powers vanish $\Rightarrow u(y)$ is an even function. If $C_0=0 \Rightarrow$ all even powers vanish and $u(y)$ is an odd function.

This is in agreement with the general theorem.

Let us look first at the even solutions, with $\begin{cases} C_0=1 \\ C_1=0 \end{cases}$.

Let $l=n$ the highest coefficient $c_l \neq 0$, i.e.

$$C_n \neq 0 \quad \text{and} \quad C_{n+2}=0 \Rightarrow$$

$$C_{n+2}=0=C_n \frac{(2n+1-2\varepsilon)}{n(n+2)}$$

$$\Rightarrow \boxed{\varepsilon = n+\tfrac{1}{2}}$$

only these energies (n even) are allowed

Likewise, for the odd sector, we set $c_0=0$ and $C_1=2$ (convention) $\Rightarrow \exists\, l=n$ odd /

$$c_{l+2}=0 \Rightarrow \varepsilon = n+\tfrac{1}{2}, \; n \text{ odd}$$

$\Rightarrow$ The allowed energies are $E_n = (n+\frac{1}{2})\hbar\omega$

The eigenfunctions are $\psi(y) = u(y)\, e^{-y^2/2}$

where $u(y)$ are polynomials of degree n, for the solution with energy E_n. These polynomials are ~~called~~ known as the Hermite Polynomials, $H_n(y)$.

$H_0(y) = 1$ nodeless

$H_1(y) = 2y$ one zero

$H_2(y) = -2(1-2y^2)$ 2 zeros

$H_3(y) = -12(y - \frac{2}{3}y^3)$ 3 zeros

$H_4(y) = 12(1-4y^2+\frac{4}{3}y^4)$ 4 zeros.

$\vdots$

$$\Rightarrow \psi_n(x) = \underbrace{\left(\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right)^{1/4}}_{\text{normalization.}} H_n\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

The Hermite Polynomials are Orthonormal:

$$\int_{-\infty}^{+\infty} H_n(y)\, H_{n'}(y)\, e^{-y^2}\, dy = \delta_{nn'}\, \sqrt{\pi}\, 2^n\, n!$$

The Oscillator in the Energy Basis

Recall that if $|E\rangle$ is an eigenstate of $\hat{H}$

$$\hat{H}|E\rangle = E|E\rangle$$

$\Rightarrow$ for an oscillator we have

$$\left(\frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2 \hat{X}^2\right)|E\rangle = E|E\rangle$$

where $[\hat{X}, \hat{P}] = i\hbar\,\hat{I} \equiv i\hbar$

Let us define the <u>creation</u> and <u>annihilation</u> operators

$$\hat{a}^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{X} - i\frac{1}{(2m\omega\hbar)^{1/2}}\hat{P}$$

$$\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{X} + i\frac{1}{(2m\omega\hbar)^{1/2}}\hat{P}$$

$$\hat{X} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{P} = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(\hat{a}^\dagger - \hat{a})$$

where $\hat{X}^\dagger = \hat{X}$ and $\hat{P}^\dagger = \hat{P}$. <u>Note</u>: $\hat{a}^\dagger$ and $\hat{a}$ are adjoints of each other.

$$\Rightarrow [\hat{a}, \hat{a}^\dagger] = \hat{I} \quad (\text{or } 1)$$

and

$$\hat{a}^\dagger\hat{a} = \frac{m\omega}{2\hbar}X^2 + \frac{1}{2m\omega\hbar}P^2 + \frac{i}{2\hbar}\underbrace{[\hat{X}, \hat{P}]}_{i\hbar}$$

$$\hat{a}^\dagger\hat{a} + \frac{1}{2} = \frac{1}{\hbar\omega}\left(\frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2\right)$$

$$\Rightarrow \hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2$$

$$\Rightarrow \hat{H} = \hbar\omega\left(\hat{a}^+ \hat{a} + \frac{1}{2}\right)$$

Let $|0\rangle$ be a vector anihilated by $\hat{a}$;

$$\hat{a}|0\rangle = 0$$

$$\Rightarrow \hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle$$

Q: what is the wave function of this state ?

$$\hat{a}|0\rangle = 0$$

$$\left(\left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + \frac{i}{(2m\omega\hbar)^{1/2}} \hat{P}\right)|0\rangle = 0$$

$$\Rightarrow \langle x|\hat{a}|0\rangle = 0 = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x \langle x|0\rangle + \frac{i}{(2m\omega\hbar)^{1/2}} \langle x|\hat{P}|0\rangle = 0$$

$$\langle x|\hat{P}|0\rangle = -i\hbar \frac{d}{dx}\langle x|0\rangle$$

$\Rightarrow \langle x|0\rangle = \psi_0(x)$ satisfies

$$\left(\frac{m\omega}{2\hbar}\right)^{1/2} x\, \psi_0 + \frac{\hbar}{(2m\omega\hbar)^{1/2}} \frac{d\psi_0}{dx} = 0$$

$$\Rightarrow \quad \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} \, x \, \psi_0$$

$$\Rightarrow \quad \frac{d \ln \psi_0}{dx^2} = -\frac{1}{2} \frac{m\omega}{\hbar}$$

$$\Rightarrow \ln \psi_0 = -\frac{1}{2} \frac{m\omega}{\hbar} x^2 + \ln C_0$$

$$\psi_0 = C_0 \, e^{-\frac{1}{2}\frac{m\omega}{\hbar} x^2} \equiv C_0 \, e^{-x^2/2\ell^2}$$

$$\int_{-\infty}^{+\infty} dx \, |\psi_0(x)|^2 = C_0^2 \int_{-\infty}^{+\infty} dx \, e^{-\frac{x^2}{\ell^2}} =$$

$$= C_0^2 \ell \sqrt{\pi} = 1$$

$$\Rightarrow \quad C_0 = \frac{1}{(\pi \ell^2)^{1/4}}$$

$$\Rightarrow \quad \psi_0(x) = \frac{1}{(\pi \ell^2)^{1/4}} \, e^{-x^2/2\ell^2}$$

which is node-less $\Rightarrow$ it is the gnd. state!

with e.v. $\frac{\hbar\omega}{2}$.

What about the excited states? To solve this problem let us first observe that

$$[\hat{a}, \hat{H}] = \hbar\omega \, [\hat{a}, \hat{a}^+ \hat{a}]$$

$$[\hat{a}, \hat{a}^+\hat{a}] = \hat{a}\,\hat{a}^+\hat{a} - \hat{a}^+\hat{a}\hat{a} = \overbrace{[\hat{a}, \hat{a}^+]}^{=1} \hat{a} + \cancel{\hat{a}^+\hat{a}\hat{a}} - \cancel{\hat{a}^+\hat{a}\hat{a}}$$

$$\Rightarrow \quad [\hat{a}, \hat{a}^+\hat{a}] = \hat{a} \Rightarrow [\hat{a}, \hat{H}] = \hbar\omega\, \hat{a}$$

and

$$[\hat{a}^+, \hat{H}] = \hbar\omega\, [\hat{a}^+, \hat{a}^+\hat{a}]$$

$$[\hat{a}^+, \hat{a}^+\hat{a}] = -\,[\hat{a}, \hat{a}^+\hat{a}]^+ = -\,\hat{a}^+$$

$$\Rightarrow \quad [\hat{a}^+, \hat{H}] = -\,\hbar\omega\, \hat{a}^+$$

Let $|\alpha\rangle$ be the eigenvectors of $\hat{a}^+\hat{a}$ (which is hermitian)

$$\Rightarrow \quad \hat{a}^+\hat{a}\,|\alpha\rangle = \alpha|\alpha\rangle \qquad (\langle\alpha|\alpha\rangle = 1)$$

$$\Rightarrow \langle\alpha|\,\hat{a}^+\hat{a}\,|\alpha\rangle = \alpha = \|\hat{a}\,|\alpha\rangle\|^2 \geq 0$$

Since $[\hat{a}^+\hat{a}, \hat{a}] = -\,\hat{a}$

$$[\hat{a}^+\hat{a}, \hat{a}^+] = +\,\hat{a}^+$$

$$\Rightarrow (\hat{a}^+\hat{a})\,\hat{a} = \hat{a}\,(\hat{a}^+\hat{a} - 1)$$

$$(\hat{a}^+\hat{a})\,\hat{a}^+ = \hat{a}^+(\hat{a}^+\hat{a} + 1)$$

$$\Rightarrow \quad (\hat{a}^+\hat{a})\,\hat{a}|\alpha\rangle = \hat{a}\,(\hat{a}^+\hat{a} - 1)|\alpha\rangle = \hat{a}\,(\alpha - 1)|\alpha\rangle$$
$$= (\alpha - 1)\,\hat{a}|\alpha\rangle$$

$\Rightarrow$ $\hat{a}|\alpha\rangle$ is an eigenvector of $\hat{a}^+\hat{a}$ with e.v. $\alpha - 1$

<u>unless</u> $\hat{a}|\alpha\rangle = 0$, and $\hat{a}^+|\alpha\rangle$ is an

eigenvector of $\hat{a}^+\hat{a}$ with eigenvalue $\alpha+1$ unless

$\hat{a}^+|\alpha\rangle = 0$

The norm of $\hat{a}|\alpha\rangle$ is

$$\| \hat{a}|\alpha\rangle\|^2 = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha$$

$$\Rightarrow \|\hat{a}|\alpha\rangle\| = \sqrt{\alpha}$$

and $\|\hat{a}^+|\alpha\rangle\| = \sqrt{\alpha+1}$

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$[\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+$

let us ~~observe that~~ compute the commutators

$[\hat{a}^+\hat{a}, \hat{a}^n]$ and $[\hat{a}^+\hat{a}, \hat{a}^{+n}]$

we will use the identity

$$[A, BC] = [A, B]C + B[A, C]$$

$$\Rightarrow [\hat{a}^+\hat{a}, \hat{a}^n] = [\hat{a}^+\hat{a}, \hat{a}\,\hat{a}^{n-1}]$$

$$= [\hat{a}^+\hat{a}, \hat{a}]\,\hat{a}^{n-1} + \hat{a}\,[\hat{a}^+\hat{a}, \hat{a}^{n-1}]$$

$$= -\hat{a}\,\hat{a}^{n-1} + \hat{a}\,[\hat{a}^+\hat{a}, \hat{a}^{n-1}]$$

$$\Rightarrow [\hat{a}^+\hat{a}, \hat{a}^n] = -\hat{a}^n + \hat{a}\,[\hat{a}^+\hat{a}, \hat{a}^{n-1}]$$

$$= -\hat{a}^n + \hat{a}\left\{-\hat{a}^{n-1} + \hat{a}[\hat{a}^+\hat{a}, \hat{a}^{n-2}]\right\}$$

$$\Rightarrow [\hat{a}^+\hat{a}, \hat{a}^n] = -2\hat{a}^n + \hat{a}^2\,[\hat{a}^+\hat{a}, \hat{a}^{n-2}]$$

$$\Rightarrow \quad [\hat{a}^+ \hat{a}, \hat{a}^n] = -k\hat{a}^n + \hat{a}^k [\hat{a}^+\hat{a}, \hat{a}^{n-k}]$$

Let us choose $n-k=1 \Rightarrow k = n-1$

$$\Rightarrow \quad [\hat{a}^+ \hat{a}, \hat{a}^n] = -n\hat{a}^n$$

Likewise

$$[\hat{a}^+ \hat{a}, \hat{a}^{+n}] = +n\hat{a}^{+n}$$

Let us consider now the state $(\hat{a})^n|\alpha\rangle$

and write

$$\begin{aligned}
\hat{a}^+ \hat{a}\ \hat{a}^n|\alpha\rangle &= \left([\hat{a}^+\hat{a}, \hat{a}^n] + \hat{a}^n \hat{a}^+\hat{a}\right)|\alpha\rangle \\
&= (-n\hat{a}^n + \hat{a}^n\hat{a}^+\hat{a})|\alpha\rangle \\
&= -n\hat{a}^n|\alpha\rangle + \alpha\hat{a}^n|\alpha\rangle
\end{aligned}$$

$$\Rightarrow (\hat{a}^+\hat{a})\hat{a}^n|\alpha\rangle = (\alpha - n)\hat{a}^n|\alpha\rangle$$

$\Rightarrow$ $\hat{a}^n|\alpha\rangle$ is an eigenvector of $\hat{a}^+\hat{a}$ with

e.v. $\alpha - n$

But if $|\beta\rangle = \hat{a}^n|\alpha\rangle$ is an eigenstate of $\hat{a}^+\hat{a}$

with e.v. $\alpha - n \Rightarrow \alpha - n = \|\hat{a}^{n+1}|\alpha\rangle\|^2 \geq 0$

$\Rightarrow \alpha - n \geq 0$

But for n large enough $\alpha - n < 0$

<u>unless</u> $\alpha \in \mathbb{Z} \Rightarrow \exists\ \hat{n} = \alpha \ /\ \alpha = \hat{n}$

and $\Rightarrow$ the e.v.'s of $\hat{a}^+\hat{a}$ are non-negative integers $\Rightarrow |0\rangle$ is the ground state of the Hermitian operator $\hat{a}^+\hat{a}$ (i.e. the eigenvector of the lowest eigenvalue of its spectrum)

Let $|n\rangle$ be ~~an~~ the eigenstate of $\hat{a}^+\hat{a}$ with eigenvalue $n \geq 0 \Rightarrow$

$$|n\rangle = \frac{(\hat{a}^+)^n|0\rangle}{\| \hat{a}^{+n}|0\rangle\|}$$

But $|n-1\rangle = \frac{1}{\sqrt{n}}\hat{a}\,|n\rangle$ has e.v. $n-1$

and $\hat{a}^+\,|n-1\rangle = \frac{1}{\sqrt{n}}\hat{a}^+\hat{a}\,|n\rangle = \sqrt{n}\,\,|n\rangle$

$$\Rightarrow |n\rangle = \frac{1}{\sqrt{n}}\hat{a}^+\,|n-1\rangle = \frac{1}{\sqrt{n(n-1)}}\hat{a}^{+2}|n-2\rangle$$

$$= \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle$$

and we also have

$$\hat{a}^+\,|n\rangle = \sqrt{n+1}\,\,|n+1\rangle$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,\,|n-1\rangle$$

$$\hat{a}^+\hat{a}\,|n\rangle = n\,\,|n\rangle$$

These identities show that

$$\langle m|\hat{a}^+|n\rangle = \sqrt{n+1}\;\delta_{m,n+1}$$

$$\langle m|\hat{a}\,|n\rangle = \sqrt{n}\;\delta_{m,n-1}$$

Other useful identities:

$$[\hat{a}, (\hat{a}^+)^n] = n\,(\hat{a}^+)^{n-1}$$

If $\quad f(\hat{a}^+) \equiv \sum_{n=0}^{\infty} f_n\, \hat{a}^{+\,n}$

$$\Rightarrow \quad [\hat{a}, f(\hat{a}^+)] = f'(\hat{a}^+)$$

<u>Wave functions</u>

This analysis shows that the spectrum of

$\hat{H} \equiv \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})$ is $E_n = \hbar\omega(n + \frac{1}{2})$

The ground state is $|0\rangle$ with $E_0 = \frac{\hbar\omega}{2}$

and $n=0$

What is the wave function of the state $|n\rangle$?

$$\langle x|n\rangle = \langle x|\frac{\hat{a}^{+\,n}|0\rangle}{\sqrt{n!}}$$

since $\quad \langle x|\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\langle x|(\hat{x} - i\frac{\hat{p}}{m\omega})$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\langle x|$$

$$\Rightarrow \quad \langle x|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n \langle x|0\rangle$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n e^{-\frac{m\omega x^2}{2\hbar}}$$

Note:

$$\langle x|n\rangle = \psi_n\left(x = \sqrt{\frac{\hbar}{m\omega}}\, y\right) = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)\right]^n \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{y^2}{2}}$$

$$\Rightarrow \quad H_n(y) = e^{\frac{y^2}{2}}\left(y - \frac{d}{dy}\right)^n e^{-\frac{y^2}{2}}$$ are the Hermite polynomials.

Hence, we have constructed the full spectrum of eigenstates of $\hat{H}$ and found their associated wave functions.