The two-body Problem and Central Forces in QM.

Let us consider a system of two particles of masses $M_1$ and $M_2$ interacting through a potential $U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|)$. The Hamiltonian is

$$\hat{H} = \frac{\hat{\mathbf{P}}_1^2}{2M_1} + \frac{\hat{\mathbf{P}}_2^2}{2M_2} + U(\mathbf{r}_1 - \mathbf{r}_2)$$

Let

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

(relative coordinate)

$$\hat{\mathbf{r}} = -i \hbar \frac{\partial}{\partial \mathbf{r}}$$

$$\hat{\mathbf{r}}^2 = \frac{M_1 \hat{\mathbf{r}}_1 + M_2 \hat{\mathbf{r}}_2}{M_1 + M_2}$$

(i.e. $\mathbf{r}$ center of mass)

In these coordinates, we have

$$\hat{H} = -\frac{\hbar^2}{2\mu} \nabla^2_{\mathbf{r}} - \frac{\hbar^2}{2\mu} \nabla^2_{\mathbf{r}} + U(r)$$

where $\mu = \frac{M_1 M_2}{M_1 + M_2}$

$$\frac{1}{\mu} = \frac{1}{M_1} + \frac{1}{M_2} = \frac{1}{\text{reduced mass}}$$

$$\nabla^2 = \nabla^2_{\mathbf{r}}$$

Just as in classical mechanics, the motion
of the center of mass \( \vec{R} \) decouples from the relative coordinates \( \vec{r} \).

\[
\psi(\vec{R}, \vec{r}) \equiv \psi(\vec{R}, \vec{r}) = \phi(\vec{R}) \chi(\vec{r})
\]

\( \phi(\vec{R}) \) wave function for the center of mass coordinates.

\( \chi(\vec{r}) \) wave function for the relative coordinates.

\[
\hat{H} \psi = E \psi
\]

Where \( E = E_{\text{CM}} + E_{\text{rel}} \) -

\[
\hat{H} = \begin{pmatrix}
\hat{H}_{\text{center}} & \hat{H}_{\text{rel}} \\
\hat{H}_{\text{rel}} & \hat{H}_{\text{center}}
\end{pmatrix}
\]

\( \hat{H}_{\text{center}} \) wave operator for the center of mass.

\( \hat{H}_{\text{rel}} \) wave operator for the relative coordinates.

\( E \) is reduced to the solution of a problem of a particle of mass \( \mu \) moving with central potential \( U(\vec{r}) \equiv U(1/\vec{r}) \).

From now on \( \psi \) is the eigenvalue of the central potential problem

\[
\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + U(\vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r})
\]
In spherical coordinates it becomes

\[-\frac{\hbar^2}{2\mu r^2} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right) + V(r) \psi(r) = E \psi(r) \]

which we can write as

\[-\frac{\hbar^2}{2\mu r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\hbar^2}{2\mu} \frac{L_z^2}{r^2} + V(r) \psi = E \psi \]

Since \( [L_z, \hat{H}] = \hat{L}_z^2 \) \( \text{"centrifugal barrier"} \)

\[ = 0 \Rightarrow \text{they can be diagonalized simultaneously} \]

\[ \psi(r, \phi, \phi) = R(r) Y^m_l(\theta, \phi) \]

Notice that \( m \) does not enter here \( \Rightarrow \)

\[ \text{Number of degenerate states: } \frac{2l+1}{l} \text{ for each } l \]
Let \( R(r) = \frac{\chi(r)}{r} \)

\[
\Rightarrow \left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \tilde{V}(r) + \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} \right] \chi(r) = E \chi(r)
\]

where \( 0 \leq r < \infty \). The operator in brackets must be Hermitian on \( 0 \leq r < \infty \).

What boundary conditions should we impose?

For the operator \( \hat{O} \) to be Hermitian, we must have that if \( \chi_1 \) and \( \chi_2 \) are two solutions,

\[
\int_0^\infty dr \; \chi_1^*(r) \hat{O} \chi_2(r) = \left( \int_0^\infty dr \; \chi_2^*(r) (\hat{O} \chi_1(r)) \right)^*
\]

\[
= \int_0^\infty dr \; \chi_1^*(r) \hat{O} \chi_2(r)
\]

\[
\Rightarrow \left( \chi_1^* \frac{d\chi_2}{dr} - \chi_2^* \frac{d\chi_1}{dr} \right) \bigg|_0^\infty = 0
\]

A necessary condition for

\[
\int_0^\infty dr \; r^2 \left| R(r) \right|^2 = \int_0^\infty dr \; \left| \chi(r) \right|^2
\]

to be normalizable (either to 1 for bound states or to the \( \delta \)-function for un-bound states) is
That \( \chi(r) \rightarrow 0 \) as \( r \rightarrow \infty \)
or \( \chi(r) \rightarrow e^{-kr} \) as \( r \rightarrow \infty \)

(if \( U(r) \rightarrow 0 \) as \( r \rightarrow \infty \))

\( \Rightarrow \) the condition of non-singularity is

\[
\left( \chi_1^* \frac{d\chi_1}{dr} - \chi_2^* \frac{d\chi_2}{dr} \right)_{r=0} = 0
\]

which is satisfied for \( \chi(r) \rightarrow \text{constant} \) as \( r \rightarrow 0 \)

\( \Rightarrow R(r) \rightarrow \frac{c}{r} \) as \( r \rightarrow 0 \)

and \( \psi(r,0,\phi) \rightarrow \frac{c}{r} Y_{l+\frac{1}{2}}(\phi) \)

However since

\[
\nabla^2 \frac{1}{r} = -4\pi \delta^3(\mathbf{r})
\]

\( \Rightarrow \) this solution is admissible only if \( \psi(r) \)

has a \( \delta \)-function behavior as \( r \rightarrow 0 \)

\( \Rightarrow \) unless that is true, we must impose

that \( c=0 \Rightarrow \chi(r) \rightarrow 0 \) as \( r \rightarrow 0 \)

c, as well.
This boundary condition holds for potentials $U(r)$ which are finite as $r \to 0$ and for potentials which diverge as $r \to 0$ like $\frac{1}{r^\alpha}$ with $\alpha < 2$ (including the important case of $\alpha = 1$, Coulomb's potential).

Hence we reduced the problem to a one-dimensional system with an effective potential $U_{\text{eff}}(r) = U(r) + \frac{\hbar^2(k^2+1)}{2m r^2}$

and boundary condition $X \to 0$ as $r \to 0$

and $X \to 0$ as $r \to \infty$ (bound state),

or $X = \text{e}^{\text{ikr}}$ for scattering states.

Notice that $X \to 0$ as $r \to 0$ does not imply that the amplitude actually has a node at $r = 0$ since $R = \frac{X}{r}$.

$\Rightarrow$ if $X \sim r$ as $r \to 0 \Rightarrow R \to \text{constant}$ as $r \to 0$.

The reduction to $\alpha = 1$ (on a half line)
means that, in bound states, the oscillation theorem holds (i.e., the theorem about the
# of zeros of the wave function). However, since \( \psi(r) \) already has a zero at \( r=0 \)
and at \( r=\infty \) in the ground state
must have no other zeros in \( 0<r<\infty \). These bound states are labelled by an
integer \( n_r \), which we will call the radial
quantum number. \( \Rightarrow \) The full wave function
is labelled by three integers: \( n_r, l \), and \( m \)
\( l \): orbital quantum number
\( m \): magnetic \\
Notation: \( l=0, 1, 2, 3, 4, 5, 6, \ldots \)
\( s, p, d, f, g, h, \ldots \)

How does \( \psi(r) \) behave as \( r \to 0 \)?

\[-\frac{\hbar^2}{2\mu} \frac{d^2 \psi}{dr^2} + \left( \frac{\hbar^2 l(l+1)}{2\mu r^2} + U(r) \right) \psi = E \psi \]
Consider first the case \( l \neq 0 \) and \( \mu i \sqrt{r^2 U(r)} = 0 \).

\[ \Rightarrow \text{as } r \to 0 \text{ the radial Schrödinger Eqn. reduces to} \]

\[ \frac{d^2 \chi}{dr^2} = \frac{l(l+1)}{r^2} \chi \]

\[ \Rightarrow \chi \sim r^\lambda \quad \text{with} \quad \lambda = \frac{l+1}{2} \]

The solution \( \chi \sim r^{-\lambda} \) violates the boundary condition \( \chi \sim r^{l+1} \). It turns out that this also works for \( l = 0 \) if \( \mu i \sqrt{r^2 U(r)} = 0 \).

**Note:** \( \chi \sim r^{l+1} \Rightarrow R = \frac{\chi}{r} \)

**Behavior for** \( r \to 0 \): \( \chi \sim r^{l+1} \text{ centrifugal barrier} \)

\[ k^2 \frac{l(l+1)}{2} \text{ is negligible} \]

\[ \Rightarrow \chi'' \sim -k^2 \chi \]

\[ k^2 = \frac{\mu E}{\hbar^2} > 0 \]

\[ \Rightarrow \chi \sim e^{\pm i k r} \]

\[ \Rightarrow R(r) \sim \frac{e^{\pm i k r}}{r} \]

for \( r \to 0 \), and \( k^2 > 0 \), \( \Rightarrow \chi \sim e^{-k r} \quad \text{for } \mu i \sqrt{r^2 U(r)} \to 0 \)

\[ \Rightarrow (E < 0) \quad \chi \sim e^{-k r} \quad \text{and} \quad R \sim e^{\frac{\mu i k r}{r}} \]
Free Particle

For a free particle, \( U = 0 \) and the radial equation is

\[
\frac{d^2 \chi}{dr^2} + \left( k^2 - \frac{l(l+1)}{r^2} \right) \chi = 0
\]

or, alternatively, directly in terms of \( R(r) \):

\[
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left( -\frac{l(l+1)}{r^2} + k^2 \right) R = 0
\]

The solution is a spherical Bessel function

\[
\chi = kr
\]

\[
J_l(kr) = \left( \frac{\pi}{2kr} \right)^{1/2} J_{l+\frac{1}{2}}(kr)
\]

where \( J_l(kr) \) is a Bessel function.

In addition, there is a second solution, the spherical Neumann function \( N_l(kr) \)

\[
N_l(kr) = (-1)^{l+1} \left( \frac{\pi}{2kr} \right)^{1/2} J_{-l-1}(kr)
\]

\[
\tilde{\delta}_0(x) = \frac{\sin x}{x}
\]

\[
\tilde{\delta}_1(x) = \frac{1}{x} \left( \frac{\sin x}{x} - \cos x \right)
\]

\[
\tilde{\delta}_2(x) = \left( \frac{3}{x^2} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x
\]
and \( n_0(x) = -\frac{\cos x}{x} \)
\[ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \]
\[ n_2(x) = -\left( \frac{3}{x^3} - \frac{1}{x} \right) \cos x - \frac{3}{x^2} \sin x \]

For \( x \to 0 \) these solutions behave like,
\[ j_k(x) \to \frac{x^k}{(2k+1)!!} \]  
\[ n_k(x) \to -\frac{(2k-1)!!}{x^{k+1}} \]  

where for \( x \to \infty \)

\[ j_k(x) \to \frac{1}{x} \cos \left( x - (k+1) \frac{\pi}{2} \right) = \frac{1}{x} \sin \left( x - \frac{\pi}{2} \right) \]  
\[ n_k(x) \to \frac{1}{x} \sin \left( x - (k+1) \frac{\pi}{2} \right) = -\frac{1}{x} \cos \left( x - \frac{\pi}{2} \right) \]  

Notice that \( n_k(kr) \) diverges at \( r=0 \)

\[ \Rightarrow \text{if } U=0 \Rightarrow \text{only } j_k(kr) \text{ are solutions.} \]

In particular it is possible to expand a plane wave in Bessel functions:

\[ e^{i k \cdot r} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \ i^l \ Y_l^m(\theta \phi) \ * j_l(kr) \ Y_l^m(\theta \phi) \]

where \((\theta_k, \phi_k)\) are the angles of \( k \) and \((\theta, \phi)\) of \( r \).
let us use these results to discuss the eigenstates for

\[ U(r) = \begin{cases} U_0 & r \leq a \\ 0 & r > a \end{cases} \]

for \( U_0 < 0 \) ("square well")

For \( E > 0 \), the solution for \( r < a \) must be just \( J_0(kr) \) since the Neumann wave function diverges as \( r \to 0 \). For \( r > a \), we get both \( J_0(kr) \) and \( N_0(kr) \) as solutions. If we seek an outgoing wave as a solution \( \Rightarrow \) the linear combination

\[ h_e(kr) = J_0(kr) + iN_0(kr) \]

behaves like \( \frac{1}{r} e^{i(k-r+1)\frac{\pi}{2}} \) and it is an outgoing wave, while \( h^*(kr) \) is an incoming wave. For \( E < 0 \) and \( r < a \), we still have only \( J_0(kr) \) as allowed solutions, where \( k^2 = \frac{mE}{\hbar^2} (\mathbf{U}_0 - 1E) \).
For $r > a$ we write, as usual,

$$k = \sqrt{\frac{2\mu E}{\hbar^2}}$$

and we have two solutions: $\phi_+ (i\epsilon kr)$ and $\phi_0 (i\epsilon kr)$. Only the linear combination $\phi_0 (i\epsilon kr)$ behaves correctly as $r \to \infty$

$$\phi_0 (i\epsilon kr) \sim \frac{1}{kr} e^{-kr}$$

(5-wave bound state)

For example, if $\phi = 0$ we have

$$\psi (r, \theta, \phi) = R (r) Y_0^0 (\theta, \phi) \equiv R (r)$$

where

$$R (r) = \phi_0 (kr) \quad r < a$$

and

$$R (r) = A_+ \phi_0 (c i \epsilon kr) \quad r > a$$

$$\phi_0 (kr) = \frac{\sin kr}{kr}$$

$$\phi_0 (i\epsilon kr) = -\frac{e^{-kr}}{kr}$$

$$\Rightarrow \quad R_- (a) = R_+ (a)$$

$$\Rightarrow \quad A_- \frac{\sin ka}{ka} = -A_+ \frac{e^{-ka}}{ka}$$

and

$$\frac{dR_-}{dr} \bigg|_a = \frac{dR_+}{dr} \bigg|_a \Rightarrow$$
\[ A_+ \left( \frac{\cos ka}{k^2} - \frac{\sin ka}{k^2} \right) = A_- \left[ \frac{1}{k^2}e^{-ka} + \frac{1}{k}e^{-ka} \right] \]

\[ \Rightarrow \quad A_- \cos ka = A_+ e^{-ka} \]
\[ A_- \frac{\sin ka}{k} = -A_+ \frac{e^{-ka}}{k} \]

\[ \Rightarrow \quad u = ka \quad \Rightarrow \quad ka = -ka \cot ka \]

\[ \Rightarrow \quad \sqrt{\frac{2\mu |V_0| a^2 - u^2}{\hbar^2}} = -u \cot u \]

There is an s-wave bound state if

\[ \frac{2\mu |V_0| a^2}{\hbar^2} > \left( \frac{\pi}{2} \right)^2 \]

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The Hydrogen Atom

For the Hydrogen atom, the potential is the Coulomb potential

\[ U(r) = -\frac{e^2}{r} \]

In this case the radial Schrödinger equation is
\[
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} - \frac{\ell (\ell+1)}{r^2} R + \left(\frac{2\mu E}{\hbar^2} + \frac{2\mu}{\hbar^2} \frac{e^2}{r}\right) R = 0
\]

Let us scale \( r = a_0 u \) and \( E = E_0 \varepsilon \)

with \( a_0 = \frac{\hbar^2}{\mu e^2} = \text{Bohr radius} \)

\[ E_0 = \frac{\mu e^4}{\hbar^2} = 2 \text{ Rydbergs} = 2 \times 13.1 \text{ eV} \]

\[
\Rightarrow \frac{d^2 R}{du^2} + \frac{2}{u} \frac{dR}{du} - \frac{\ell (\ell+1)}{u^2} R + 2 \left( \frac{E + \frac{1}{u^2}}{u^2} \right) R = 0
\]

**Discrete Spectrum:** \( E < 0 \)

Define \( n = \frac{1}{\sqrt{-2E}} = \frac{1}{\sqrt{2161}} \) and \( \rho = \frac{2 \beta u}{n} = \frac{2}{n} \left( \frac{r}{a_0} \right) \)

\[
\Rightarrow \frac{d^2 R^*}{d\rho^2} + \frac{2}{\rho} \frac{dR^*}{d\rho} + \left[ -\frac{1}{4} + \frac{n}{\rho} + \frac{\ell (\ell+1)}{\rho^2} \right] R^* = 0
\]

For \( \rho \to 0 \) \( (r \to 0) \) the allowed solution \( \sim \rho^\ell \)

For \( \rho \to \infty \Rightarrow R''^* \frac{\rho^{1+\ell}}{a_0^{1+\ell}} = \frac{1}{4} R^* \)

\[
\Rightarrow R \sim e^{-\rho/2}
\]

\[
\Rightarrow R = \rho^\ell e^{-\rho/2} w(\rho)
\]
\[ \Phi \Psi'' + (2\ell + 1 - \gamma) \Psi' + (\gamma - \ell - 1) \Psi = 0 \]

Let \( \gamma = 2(\ell + 1) \) and \( \alpha = \ell + 1 - \gamma \)

The equation

\[ \Phi \Psi'' + (\gamma - \ell) \Psi' - \alpha \Psi = 0 \]

it is known as the confluent hypergeometric function. This solution is regular at \( \ell = 0 \).

However, it diverges exponentially fast as \( \ell \to \infty \) unless it is a polynomial, i.e., unless the series terminates at some \( \ell = -N \) \((N = 1, 2, \ldots)\).

Thus, the allowed wave functions must have

\[ n \beta = \ell + 1 = \ell + 1 \]

\[ = \] \[ n = N + \ell + 1 \Rightarrow n \in \mathbb{Z}^+ \]

And

\[ l + 1 \leq N \]

\[ n: \text{principal quantum number} \Rightarrow E_n = -\frac{\varepsilon_0}{2n^2} \]
\[ E_n = - \frac{E_0}{2n^2} = - \frac{Ry}{n^2} \]

for each \( n \), the allowed values of the \textit{orbital quantum number} \( l \) are

\[ 1 \leq l + 1 \leq n \quad \text{and} \quad |m| \leq l \]

\[ 0 \leq l \leq n - 1 \quad \text{and} \quad |m| \leq l \]

\( n=1 \) \quad \Rightarrow \quad l=0 \quad m=0 \quad \Rightarrow \quad 1S \text{ singlet} \]

\( n=2 \) \quad \Rightarrow \quad l=0,1 \quad m=0,1 \quad 2S \text{ singlet} \]

\[ \quad m=0, \pm 1 \quad 2P \text{ triplet} \]

\( n=3 \), \( l=0 \) \quad m=0 \quad 3S \text{ singlet} \]

\[ \quad m=0, \pm 1 \quad 3P \text{ triplet} \]

\[ \quad m=0, \pm 1, \pm 2 \quad 3D \text{ quintplet} \]

\textit{etc.}

\textbf{Degeneracy:} the energy depends only on the principal quantum number and not on \( l \) and \( m \): \textit{Column 5 degeneracy} ("accidental degeneracy")

\[ \# \text{ of degenerate states} = \sum_{l=0}^{n-1} (2l+1) = n^2 \]

The wave function are

\[ \Psi_{n\ell m} (r, \theta, \phi) = R_{n\ell} (r) \ Y_{\ell}^m (\theta, \phi) \quad \text{with} \]
\[ R_{nl}(s) = \text{const} \int_0^s e^{-s/2} L_{n-l-1}^{2l+1}(s) \]

where \( L_p^k(x) \) is the generalized Laguerre polynomial:
\[
L_p^k(x) = (-1)^k \frac{d^k}{dx^k} L_p(x) \\
L_p^0(x) = e^x \frac{d^p}{dx^p} \left( e^{-x} x^p \right) \quad \text{(Laguerre polynomial)}
\]

\[ \Rightarrow \quad R_{nl}(r) = \text{const} \left( \frac{r}{na_0} \right)^l e^{-\frac{r}{na_0}} L_{n-l-1}^{2l+1} \left( \frac{2r}{na_0} \right) \]

for \( E_n = -\frac{E_0}{2n^2} \)

where \( a_0 = \frac{\hbar^2}{\mu e^2} \) is the Bohr radius.

and \( E_0 = \frac{\mu e^4}{\hbar^2} = 2 \text{ Rydberg} \Rightarrow \quad E_n = -\frac{\text{Rydberg}}{n^2} \)

In particular, since
\[ \psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi) \]

\[ \psi_{1,0,0}(r, \theta, \phi) = \left( \frac{1}{\pi a_0^3} \right)^{\frac{1}{2}} e^{-r/\alpha_0} \quad \text{1S (1s)} \]
\[ \psi_{2,0,0}(r, \theta, \phi) = \left( \frac{1}{3\pi a_0^3} \right)^{\frac{1}{2}} \left( e^{-\frac{r}{\alpha_0}} \right)^2 \quad \text{2S} \]
\[ \psi_{2,1,0}(r, \theta, \phi) = \left( \frac{1}{32\pi a_0^3} \right)^{\frac{1}{2}} \frac{r}{\alpha_0} e^{-r/2\alpha_0} \cos \theta \quad \text{2P} \]

ground state
\[ y_{2,1,\pm 1} = \pm \left( \frac{1}{64\pi a_0^3} \right)^{\frac{1}{2}} \frac{r}{a_0} e^{-r/2a_0} \sin \theta \, e^{\pm \phi} \]

The degeneracy of the Hydrogen Spectrum

The eigenvalues of \( \hat{H} \) for \( U(r) = -\frac{\epsilon}{r} \) are degenerate, \( E_{n\ell m} = \frac{\epsilon}{2n^2} - \frac{E_0}{2n^2} \)

and there is no dependence of \( \ell \) or \( m \). Why is this so? The fact that \( E_{n\ell m} \) is independent of \( m \) is common to all central potentials: it is a consequence of rotational invariance. The degeneracy in \( \ell \) is "accidental," e.g., peculiar to \( U(r) = -\frac{\epsilon^2}{r} \)

Already in Classical Mechanics the \( \frac{1}{r} \) potential is seen to be special: for central potentials the only restrictions that exist is that angular momentum is conserved \( \Rightarrow \) orbits are restricted to a plane \( z = \text{constant} \), but for \( U(r) = -\frac{\epsilon^2}{r} \)
the (planar) orbits are also closed, i.e., they do not precess. In Classical Mechanics, this means that there must be an additional conserved vector, since the direction of the major axis of the elliptical orbit is a constant of motion. This direction is determined by the Runge-Lenz vector

$$\vec{R} = \frac{1}{\mu} \vec{p} \times \vec{L} - \frac{e^2}{r} \hat{r}$$

which is the additional conserved vector, i.e.,

$$\langle R^i, H \rangle_{PB} = 0$$

In the quantum theory, up to an ordering prescription, $\vec{R}$ becomes an operator on the Hilbert space:

$$\hat{R} \equiv \frac{1}{2\mu} (\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}) - \frac{e^2}{|\vec{r}|}$$

where $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

and $[\hat{A}, \hat{R}] = 0$
Since $\hat{R}$ is a vector, we have
\[ [\hat{R}_i, \hat{L}_j] = i \hbar \epsilon_{ijk} \hat{R}_k \]

Similarly, we can show that
\[ [\hat{R}_i, \hat{R}_j] = i \hbar \left( -\frac{2\hat{H}}{\mu} \right) \epsilon_{ijk} \hat{L}_k \]

Moreover, the square of the Runge-Lenz vector is
\[ \hat{R}^2 = e^4 + \frac{2\hat{H}}{\mu} (\hat{L}^2 + \hat{h}^2) \]

where \( \hat{H} = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{r} \)

\( \Rightarrow \hat{H} \) can be written in terms of two constants of motion: \( \hat{L}^2 \) and \( \hat{R}^2 \).

We are interested in the bound states of \( \hat{H} \).

For these states, the operator
\[ \sqrt{-\mu \hat{K}} = \sqrt{\frac{-\mu}{2\hat{H}}} \hat{R} \] is Hermitian.

\( \Rightarrow \) \([\hat{K}_c, \hat{K}_j] = i \hbar \epsilon_{cjk} \hat{L}_k \]
\([\hat{K}_c, \hat{L}_j] = i \hbar \epsilon_{cjk} \hat{K}_k \]
\([\hat{L}_c, \hat{L}_j] = i \hbar \epsilon_{cjk} \hat{L}_k \]
\[ H = -\frac{\mu e^4}{2(K^2 + L^2 + h^2)} \]

We will compute the eigenvalues and their degeneracies from these results.

Let \[ M = \frac{1 + \kappa}{2} \quad N = \frac{1 - \kappa}{2} \]

\[ [M_i, M_j] = i\hbar \varepsilon_{ijk} M_k \]
\[ [N_i, N_j] = i\hbar \varepsilon_{ijk} N_k \]
\[ [N_i, M_j] = 0 \]

\[ = \text{two commuting angular momentum algebras!} \]

\[ H = -\frac{\mu e^4}{2(M^2 + 2N^2 + h^2)} \]

We can diagonalize simultaneously \( \hat{M}_x, \hat{M}_y, \hat{N}_x \) and \( \hat{N}_z \) with eigenvalues \( m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z \)

\[ M_x |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle = \hbar^2 m |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle \]
\[ N_x |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle = \hbar^2 n |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle \]
\[ M_z |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle = \hbar \bar{m} |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle \]
\[ N_z |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle = \hbar \bar{n} |m, n, \bar{m}, \bar{n}, \bar{N}_x, \bar{N}_z\rangle \]

When \( M, \bar{M} = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \)
and \( |\bar{m}| \leq M, \quad |\bar{n}| \leq \bar{m} \)

\[
\bar{m} = -M, -M+1, \ldots, M-1, M
\]

\[
\bar{n} = -n, -n+1, \ldots, n-1, n
\]

\( \text{Since } \bar{\mathbf{r}} \cdot \bar{\mathbf{L}} = 0 \Rightarrow \bar{\mathbf{r}} \cdot \bar{\mathbf{L}} = 0 \)

\( \Rightarrow M^2 = \bar{N}^2 \Rightarrow M = \bar{N} \) are the only relevant states

\[
\{ |M, M, \bar{m}, \bar{n} \rangle \}
\]

\[
H \cdot |M, M, \bar{m}, \bar{n} \rangle = -\frac{\mu e^4}{2\hbar^2 (4M(M+1)+1)} |M, M, \bar{m}, \bar{n} \rangle
\]

\[= -\frac{\mu e^4}{2\hbar^2 (2M+1)^2} |M, M, \bar{m}, \bar{n} \rangle
\]

Where \( 2M+1 = 2, 3, 4, \ldots \)

We recognize that \( 2M+1 = n \) the principal quantum number!

And \( E = -\frac{\mu e^4}{2\hbar^2 n^2} \) which are the correct energy levels!

Degeneracy: for each fixed \( M \Rightarrow (2M+1)^2 \) states

\[\Rightarrow \text{the degeneracy is } n^2\]