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The two-body Problem and Central Forces in QM.

Let us consider a system of two particles of masses M_1 and M_2 interacting through a potential $U(\vec{r}_1, \vec{r}_2) = U(|\vec{r}_1 - \vec{r}_2|)$. The Hamiltonian is

$$\hat{H} = \frac{\hat{\vec{P}}_1^{\,2}}{2M_1} + \frac{\hat{\vec{P}}_2^{\,2}}{2M_2} + U(|\vec{r}_1 - \vec{r}_2|)$$

Let $\vec{r} = \vec{r}_2 - \vec{r}_1$ (relative coordinate), $\hat{\vec{P}}_i = -i\hbar \frac{\partial}{\partial \vec{r}_i}$

$$\vec{R} = \frac{M_1 \vec{r}_1 + M_2 \vec{r}_2}{M_1 + M_2} \qquad \left(\text{i.e. } \vec{R} \doteq \text{center of mass}\right)$$

$\Rightarrow$ In these coordinates we have

$$\hat{H} = -\frac{\hbar^2}{2M} \nabla^2_{\vec{R}} - \frac{\hbar^2}{2\mu} \nabla^2 + U(r)$$

where $M = M_1 + M_2$

$$\frac{1}{\mu} = \frac{1}{M_1} + \frac{1}{M_2} = \frac{1}{\text{reduced mass}}$$

$$\nabla^2 \equiv \nabla^2_{\vec{r}}$$

$\Rightarrow$ Just as in Classical Mechanics, the motion of the center of mass $\vec{R}$ decouples from the relative coordinate $\vec{r}$.

$\Rightarrow$ The wave function is factorized:

$$\Psi(\vec{r}_1, \vec{r}_2) \equiv \Psi(\vec{R}, \vec{r}) \equiv \underbrace{\phi(\vec{R})}_{\text{wave function for the center of mass}} \; \underbrace{\psi(\vec{r})}_{\text{wave function for the relative coordinate.}}$$

$\Rightarrow$ $\hat{H}\,\Psi = E\,\Psi$

where $E = \underbrace{E_{CM}}_{\text{center of mass}} + \underbrace{E_{relative}}_{\text{relative coordinate e.v.}}$

$\Rightarrow$ it is reduced to the solution of a problem of a particle of mass μ moving in the <u>central potential</u> $V(\vec{r}) \equiv V(|\vec{r}|)$

From now on E is the eigenvalue of the central potential problem

$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + V(r) \right] \psi(\vec{r}) = E\,\psi(\vec{r})$$

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In spherical coordinates it becomes

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\right.$$

$$\left.+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right)\right]$$

$$+\,V(|\vec{r}|)\,\psi(\vec{r}) = E\,\psi(\vec{r})$$

which we can write as

$$-\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\underbrace{\frac{\hat{\vec{L}}^2}{2\mu r^2}}_{\text{"centrifugal barrier"}}\psi + V(r)\psi = E\psi$$

Since $[\hat{\vec{L}}^2,\hat{H}] = [\hat{L}_z,\hat{H}] = 0 \Rightarrow$ they can be diagonalized simultaneously

$\Rightarrow$ $$\psi(r,\theta,\phi) = R(r)\, Y_\ell^m(\theta,\phi)$$

$$-\frac{\hbar^2}{2\mu r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}R + V(r)R = ER$$

Notice that $\underline{\underline{m}}$ does not enter here $\Rightarrow$ degeneracy

\# of degenerate states: $\underline{\underline{2\ell+1}}$ for each $\underline{\underline{\ell}}$

Let $R(r) = \dfrac{\chi(r)}{r}$

$$\Rightarrow \left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2} \right] \chi(r) = E\,\chi(r)$$

where $0 \le r < \infty$. The operator in brackets must be Hermitian on $0 \le r < \infty$.

What boundary conditions should we impose?

For the operator (call it $\hat{D}$) to be Hermitian we must have that if χ_1 and χ_2 are two solutions $\Rightarrow$

$$\int_0^\infty dr\ \chi_1^*(r)\ \hat{D}\ \chi_2(r) = \left(\int_0^\infty dr\ \chi_2^*(r) (\hat{D}\ \chi_1(r)) \right)^*$$

$$= \int_0^\infty dr\ \chi_2(r)\ (\hat{D}\ \chi_1)^*(r)$$

$$\Rightarrow \left(\chi_1^* \frac{d\chi_2}{dr} - \chi_2 \frac{d\chi_1}{dr} \right) \Big|_0^\infty = 0$$

A necessary condition for

$$\int_0^\infty dr\, r^2\, |R(r)|^2 = \int_0^\infty dr\ |\chi(r)|^2 \text{ to be}$$

normalizable (either to 1 for bound states or to the δ-function for un-bound states) is that $\quad \chi(r) \longrightarrow 0 \quad$ as $\quad r \to \infty$

or $\quad \chi(r) \longrightarrow e^{\pm ikr} \quad$ as $\quad r \to \infty$

(if $\quad U(r) \to 0 \quad$ as $\quad r \to \infty$)

$\Rightarrow$ the condition of Hermiticity is

$$\left(\chi_1^* \frac{d\chi_2}{dr} - \chi_2^* \frac{d\chi_1}{dr} \right)_{r=0} = 0$$

which is satisfied for $\quad \chi(r) \xrightarrow[r\to 0]{} \text{constant}$

$\Rightarrow \quad R(r) \longrightarrow \frac{c}{r} \quad$ as $\quad r \to 0$

and $\quad \psi(r,\theta,\phi) \to \frac{c}{r} Y_\ell^m(\theta,\phi)$

However since

$$\nabla^2 \frac{1}{r} = -4\pi\,\delta^3(\vec{r})$$

$\Rightarrow$ this solution is admissible only if $U(r)$

has a δ-function behavior as $r \to 0$

$\Rightarrow$ <u>unless</u> that is true, we must impose

that $c = 0 \Rightarrow \quad \chi(r) \longrightarrow 0 \quad$ as $\quad r \to 0$

as well.

This boundary condition holds for potentials $V(r)$ which are finite as $r \to 0$ and for potentials ~~that~~ which diverge as $r \to 0$ like $\frac{1}{r^\alpha}$ with $\alpha < 2$ (including the important case of $\alpha = 1$, the Coulomb potential).

Hence we reduced the radial problem to a one dimensional system with an effective potential $V_{eff}(r) = V(r) + \frac{\hbar^2 l(l+1)}{2\mu r^2}$ and boundary condition $\chi \to 0$ as $r \to 0$ and $\chi \to 0$ as $r \to \infty$ (bound states) or $\chi \sim e^{ikr}$ for scattering states.

Notice that $\chi \to 0$ as $r \to 0$ does not imply that the ~~probab~~ amplitude actually has a node at $\vec{r} = 0$ since $R = \frac{\chi}{r}$
$\Rightarrow$ if $\chi \sim r$ as $r \to 0$ $\Rightarrow$ $R \to$ constant as $r \to 0$.

The reduction to $d = 1$ (on a half line) means that, for bound states, the oscillation theorem holds (i.e. the theorem about the # of zeros of the wave function). However since $\chi(r)$ already has a zero at $r=0$ and at $r=\infty$ $\Rightarrow$ the ground state must have no other zeros for $0<r<\infty$

These bound states are labelled by an integer n_r, which we will call the radial quantum number. $\Rightarrow$ the full wave function is labelled by three integers: n_r, l and m

l: orbital quantum number
m: magnetic " "

Notation: $l = 0, 1, 2, 3, 4, 5, 6, \ldots$
s p d f g h i ...

How does $\chi(r)$ behave as $r \to 0$?

$$-\frac{\hbar^2}{2\mu}\frac{d^2\chi}{dr^2} + \left[\frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r)\right]\chi = E\chi$$

Consider first the case $l \neq 0$ and $\lim_{r\to 0} r^2 U(r) = 0$

$\Rightarrow$ as $r \to 0$ the radial Schrödinger Equ.

reduces to
$$\frac{d^2\chi}{dr^2} = \frac{l(l+1)}{r^2}\chi$$

$$\Rightarrow \quad \chi \sim r^{\lambda} \quad \text{with} \quad \lambda = \begin{cases} l+1 \\ -l \end{cases}$$

The solution $\chi \sim r^{-l}$ violates the boundary conditions $\Rightarrow$ $\chi \sim r^{l+1}$. It turns out that this also works for $l=0$ if $\lim_{r\to 0} r^2 U(r) = 0$

Note: $\chi \sim r^{l+1} \Rightarrow R \simeq \frac{\chi}{r} \sim r^l$

Behavior for $r \to \infty$: if $r \to \infty$ $\Rightarrow$ the ~~the~~ centrifugal barrier $\frac{\hbar^2 l(l+1)}{2\mu r^2}$ is negligible

$$\Rightarrow \quad \chi'' \simeq -k^2 \chi \qquad k^2 = \frac{2\mu E}{\hbar^2} > 0$$

$$\Rightarrow \quad \chi \sim e^{\pm ikr}$$

$$\Rightarrow R(r) \sim \frac{e^{\pm ikr}}{r} \quad \text{or} \; ; \; \frac{\sin kr}{r} \; ; \; \frac{\cos kr}{r}$$

for $r \to \infty$ and $k^2 > 0$, for $k^2 = -K^2 < 0$

$\Rightarrow$ $(E<0)$ $\quad \chi \sim e^{-Kr}$ and $R \sim e^{-Kr}/r$

Free Particle

For a free particle $U = 0$ and the radial equation is $\left(k^2 = \frac{2\mu E}{\hbar^2} > 0 \right)$

$$\frac{d^2 \chi}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2} \right) \chi = 0$$

or, alternatively, directly in terms of $R(r)$:

$$\frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(- \frac{\ell(\ell+1)}{r^2} + k^2 \right) R = 0$$

The solution is a <u>spherical</u> <u>Bessel</u> <u>function</u> $(x = kr)$

$$j_\ell(x) = \left(\frac{\pi}{2x} \right)^{1/2} J_{\ell+\frac{1}{2}}(x)$$

where $J_p(x)$ is a Bessel function.

In addition, there is a second solution, the Spherical Neumann function $n_\ell(x)$

$$n_\ell(x) = (-1)^{\ell+1} \left(\frac{\pi}{2x} \right)^{1/2} J_{-\ell-\frac{1}{2}}(x)$$

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{1}{x} \left(\frac{\sin x}{x} - \cos x \right)$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x$$

and $$n_0(x) = -\frac{\cos x}{x}$$

$$n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

For $x \to 0$ these solutions behave like

$$j_\ell(x) \to \frac{x^\ell}{(2\ell+1)!!} \qquad \left((2\ell+1)!! = (2\ell+1)(2\ell-1)\cdots\right)$$

$$n_\ell(x) \to -\frac{(2\ell-1)!!}{x^{\ell+1}}$$

while for $x \to \infty$

$$j_\ell(x) \to \frac{1}{x}\cos\left(x - (\ell+1)\frac{\pi}{2}\right) = \frac{1}{x}\sin\left(x - \ell\frac{\pi}{2}\right)$$

$$n_\ell(x) \to \frac{1}{x}\sin\left(x - (\ell+1)\frac{\pi}{2}\right) = -\frac{1}{x}\cos\left(x - \ell\frac{\pi}{2}\right)$$

Notice that $n_\ell(kr)$ diverges at $r=0$

$\Rightarrow$ if $U=0$ $\Rightarrow$ only $j_\ell(kr)$ are solutions.

In particular it is possible to expand a plane wave in Bessel functions:

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell \, Y_\ell^m(\theta_k,\phi_k)^* \, j_\ell(kr) \, Y_\ell^m(\theta,\phi)$$

where (θ_k, ϕ_k) are the angles of $\vec{k}$ and $(\theta,\phi) \leftrightarrow \vec{r}$

Let us use these result to discuss the eigenstates for

$$U(r) = \begin{cases} U_0 & r < a \\ 0 & r > a \end{cases}$$

for $U_0 < 0$ ("square well")

For $E > 0$, the solutions for $r < a$ must be just $j_\ell(kr)$ since the Neumann wave functions diverge as $r \to 0$. For $r > a$, we get both $j_\ell(kr)$ and $n_\ell(kr)$ as solutions. If we seek an <u>outgoing</u> wave as a solution $\Rightarrow$ the linear combination

$$h_\ell(kr) = j_\ell(kr) + i\, n_\ell(kr) \qquad \text{(Hankel function)}$$

behaves like $\frac{1}{x} e^{i(x - (\ell+1)\frac{\pi}{2})}$ and it is an outgoing wave, while $h_\ell^*(kr)$ is an <u>incoming</u> wave. For $E < 0$ and $r < a$, we still have only $j_\ell(kr)$ where $k^2 = \frac{2\mu}{\hbar^2}(|U_0| - |E|)$ as allowed solutions, For $r > a$ we write, as usual,

$$K = \sqrt{\frac{2\mu |E|}{\hbar^2}}$$ and we have two

Solutions : $j_\ell(iKr)$ and $n_\ell(iKr)$. Only the linear combination $h_\ell(iKr)$ behaves correctly as $r \to \infty$

$$h_\ell(iKr) \sim \frac{1}{Kr} e^{-Kr} \qquad \left[h_\ell(iz) = -i\frac{2}{\pi} e^{-\frac{i\pi\ell}{2}} \vec{K}_\ell(z) \right]$$

(s-wave bound state)

For example, for $\ell = 0$ we have

$$\psi(r, \theta, \phi) = R(r)\, Y_0^0(\theta, \phi) \equiv R(r)$$

where $R_-(r) = j_0(kr)$ $\qquad r < a$

and $R_+(r) = A\, h_0(iKr)$ $\qquad r > a$

$$j_0(kr) = \frac{\sin kr}{kr}$$

$$h_0(iKr) = -\frac{e^{-Kr}}{Kr}$$

$\Rightarrow$ $R_-(a) = R_+(a)$

$$\Rightarrow A_- \frac{\sin ka}{ka} = -\frac{A_+}{Ka} e^{-Ka}$$

and $\left.\frac{dR_-}{dr}\right|_a = \left.\frac{dR_+}{dr}\right|_a \Rightarrow$

$$A_-\left(\frac{\cos ka}{\cancel{a}} - \frac{\sin ka}{\cancel{k}a^{\cancel{2}}}\right) = +A_+\left[\cancel{\frac{1}{\kappa a^2}e^{-\kappa a}} + \frac{e^{-\kappa a}}{a}\right]$$

$$\Rightarrow \quad A_- \cos ka = A_+ e^{-\kappa a}$$

$$A_- \frac{\sin ka}{k} = -\frac{A_+}{\kappa} e^{-\kappa a}$$

$$\Rightarrow \quad u = ka \quad \Rightarrow$$

$$\kappa a = -ka \cot ka$$

$$\Rightarrow \quad \sqrt{\frac{2\mu}{\hbar^2}|U_0|a^2 - u^2} = -u \cot u$$

There is an s wave bound state if

$$\frac{2\mu}{\hbar^2}|U_0|a^2 > \left(\frac{\pi}{2}\right)^2$$

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The Hydrogen Atom

For the Hydrogen atom, the potential is the Coulomb potential

$$U(r) = -\frac{e^2}{r}$$

In this case the radial Schrödinger Equation is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + \left(\frac{2\mu E}{\hbar^2} + \frac{2\mu}{\hbar^2}\frac{e^2}{r}\right)R = 0$$

Let us scale $r = a_0 u$ and $E = E_0 \mathcal{E}$

with $a_0 = \dfrac{\hbar^2}{\mu e^2}$ = Bohr radius

$E_0 = \dfrac{\mu e^4}{\hbar^2}$ = 2 Rydbergs = 2 × 13.1 eV

$$\Rightarrow \frac{d^2R}{du^2} + \frac{2}{u}\frac{dR}{du} - \frac{\ell(\ell+1)}{u^2}R + 2\left(\mathcal{E} + \frac{1}{u}\right)R = 0$$

<u>Discrete Spectrum:</u> $E < 0$

define $n = \dfrac{1}{\sqrt{-2\mathcal{E}}} = \dfrac{1}{\sqrt{2|\mathcal{E}|}}$ and

$$\rho = \frac{2u}{n} = \frac{2}{n}\left(\frac{r}{a_0}\right)$$

$$\Rightarrow \frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[-\frac{1}{4} + \frac{n}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]R = 0$$

For $\rho \to 0$ $(r \to 0)$ the allowed solution $\sim \rho^\ell$

For $\rho \to \infty \Rightarrow R'' = +\frac{1}{4}R$

$$\Rightarrow R \sim e^{-\rho/2}$$

$$\Rightarrow R = \rho^\ell e^{-\rho/2} w(\rho)$$

$$\Rightarrow \quad \rho w'' + (2l+2-\rho)w' + (n-l-1)w = 0$$

Let $\quad \gamma = 2(l+1) \quad$ and $\quad \alpha = l+1-n$

The equation

$$\rho w'' + (\gamma - \rho)w' - \alpha w = 0$$

~~is known as the~~
has the solution

$$w = F(\alpha, \gamma, \rho) = 1 + \frac{\alpha}{\gamma}\frac{\rho}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{\rho^2}{2!} + \cdots$$

which is known as the confluent hypergeometric function. This solution is regular at $\rho = 0$. However it diverges exponentially fast as $\rho \to \infty$ unless it is a polynomial, i.e. unless the series terminates at some $\alpha = -N$
$(N = 0, 1, 2, 3, \ldots)$

$\Rightarrow$ the allowed wave functions must have $\quad n - l - 1 = +N$

$$\Rightarrow \quad n = N + l + 1 \quad \Rightarrow \quad n \in \mathbb{Z}^+$$

and $\quad l + 1 \leq n$

n: principal ~~radial~~ quantum number ~~(principal)~~ $\Rightarrow$ $\boxed{E_n = -\frac{\varepsilon_0}{2n^2}}$

$\Rightarrow$ the energy levels are

$$E_n = -\frac{\mathcal{E}_0}{2n^2} = -\frac{Ry}{n^2}$$

for each n, the allowed values of the orbital quantum number l ~~are~~

$1 \leq l+1 \leq n$ and $|m| \leq l$

$\Rightarrow$ $0 \leq l \leq n-1$ and $|m| \leq l$

$n=1 \Rightarrow l=0 \quad m=0 \Rightarrow 1S$ singlet

$n=2 \Rightarrow l=0, \quad m=0, \quad 2S$ singlet

$\qquad\qquad\; 1 \quad m=0,\pm1 \quad 2P$ triplet

$n=3, \; l=0 \quad m=0 \quad 3S$ singlet

$\qquad\qquad\; 1 \quad m=0,\pm1 \quad 3P$ triplet

$\qquad\qquad\; 2 \quad m=0,\pm1,\pm2 \quad 3D$ quintuplet

etc.

Degeneracy: the energy depends only on the principal quantum number and not on l and m: Coulomb degeneracy ("accidental degeneracy")

$$\# \text{ of degenerate states} = \sum_{l=0}^{n-1} (2l+1) = n^2$$

The wave function are

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi) \quad \text{with}$$

$$R_{n\ell}(\rho) = \text{const}\ \rho^{\ell}\, e^{-\rho/2}\, L^{2\ell+1}_{n-\ell-1}(\rho)$$

where $L^{k}_{p}(x)$ is the generalized Laguerre polynomial

$$L^{k}_{p}(x) = (-1)^k \frac{d^k}{dx^k} L^{0}_{p+k}(x)$$

$$L^{0}_{p}(x) = e^{x} \frac{d^p}{dx^p}\left(e^{-x} x^p\right) \qquad \text{(Laguerre polynomial)}$$

$$\Rightarrow R_{n\ell}(r) = \text{const} \left(\frac{r}{na_0}\right)^{\ell} e^{-\frac{r}{na_0}}\, L^{2\ell+1}_{n-\ell-1}\left(\frac{2r}{na_0}\right)$$

for $E_n = -\dfrac{\mathcal{E}_0}{2n^2}$

where $a_0 = \dfrac{\hbar^2}{\mu e^2}$ is the Bohr radius

and $\mathcal{E}_0 = \dfrac{\mu e^4}{\hbar^2} = 2$ Rydbergs $\Rightarrow E_n = -\dfrac{\text{Rydberg}}{n^2}$

In particular, since $\psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)\, Y^{m}_{\ell}(\theta,\phi)$

ground state $\Rightarrow$ $\psi_{1,0,0}(r,\theta,\phi) = \left(\dfrac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}$ 1S

$$\psi_{2,0,0}(r,\theta,\phi) = \left(\frac{1}{32\pi a_0^3}\right)^{1/2} \left(2-\frac{r}{a_0}\right) e^{-r/2a_0} \qquad 2S$$

$$\psi_{2,1,0}(r,\theta,\phi) = \left(\frac{1}{32\pi a_0^3}\right)^{1/2} \frac{r}{a_0}\, e^{-r/2a_0}\cos\theta \qquad 2P$$

$$\psi_{2,1,\pm1} = \mp \left(\frac{1}{64\pi a_0^3}\right)^{1/2} \frac{r}{a_0} e^{-r/2a_0} \sin\theta \; e^{\pm i\phi}$$

The degeneracy of the Hydrogen Spectrum

The eigenvalues of $\hat{H}$ for $U(r) = -\frac{e^2}{r}$ are degenerate, $E_{nlm} = - \frac{\varepsilon_0}{2n^2}$

and there is no dependence of l or m. Why is this so? The fact that E_{nlm} is independent of m is common to all central potentials: it is a consequence of rotational invariance. The degeneracy in l is "accidental", i.e. peculiar to $U(r) = -\frac{e^2}{r}$

Already in Classical Mechanics the $\frac{1}{r}$ potential is seen to be special: for central potentials the only restriction that exists is that angular momentum is conserved $\Rightarrow$ orbits are restricted to a plane $\perp$ to $\vec{L}$. But for $U(r) = -\frac{e^2}{r}$

the (planar) orbits are also closed, i.e. they do not precess. In Classical Mechanics this means that there must be an additional conserved vector, since the direction of the major axis of the elliptical orbit is a constant of motion. This direction is determined by the Runge - Lenz vector

$$\vec{R} = \frac{1}{\mu} \vec{p} \wedge \vec{L} - \frac{e^2}{r} \vec{r}$$

which is the additional conserved vector, i.e.

$$\{ R_i , H \}_{PB} = 0$$

In the quantum theory, up to an operator ordering prescription, $\vec{R}$ becomes an operator acting on the Hilbert space:

$$\hat{\vec{R}} \equiv \frac{1}{2\mu} \left(\hat{\vec{p}} \wedge \hat{\vec{L}} - \hat{\vec{L}} \wedge \hat{\vec{p}} \right) - \frac{e^2}{|\vec{r}|} \vec{r}$$

~~where~~ where $|\vec{r}| = \sqrt{\hat{x}^2 + \hat{y}^2 + \hat{z}^2}$

and $[\hat{H}, \hat{\vec{R}}] = 0$

Since $\hat{\vec{R}}$ is a vector $\Rightarrow$

$$[\hat{R}_i, \hat{L}_j] = i\hbar\, \varepsilon_{ijk}\, \hat{R}_k$$

Similarly we can show that

$$[\hat{R}_i, \hat{R}_j] = i\hbar \frac{(-2\hat{H})}{\mu} \varepsilon_{ijk} \hat{L}_k$$

Moreover the square of the Runge-Lenz vector is

$$\vec{R}^2 = e^4 + \frac{2\hat{H}}{\mu}(\vec{L}^2 + \hbar^2)$$

where $\hat{H} = \frac{\vec{P}^2}{2\mu} - \frac{e^2}{r}$

$\Rightarrow$ $\hat{H}$ can be written in terms of two constants of motion : $\vec{L}^2$ and $\vec{R}^2$

We are interested in the bound states of $\hat{H}$.

For those states, the operator

$$\vec{K} = \sqrt{\frac{-\mu}{2H}}\,\vec{R} \quad \text{is Hermitian.}$$

$$\Rightarrow \quad [K_i, K_j] = i\hbar\, \varepsilon_{ijk} L_k$$

$$[K_i, L_j] = i\hbar\, \varepsilon_{ijk} K_k$$

$$[L_i, L_j] = i\hbar\, \varepsilon_{ijk} L_k$$

$$\Rightarrow \quad \hat{H} = -\frac{\mu e^4}{2(K^2 + L^2 + \hbar^2)}$$

We will compute the eigenvalues and their degeneracies from these results.

Let $\quad \vec{M} = \frac{\vec{L} + \vec{K}}{2} \qquad \vec{N} = \frac{\vec{L} - \vec{K}}{2}$

$$\Rightarrow \quad [M_i, M_j] = i\hbar \varepsilon_{ijk} M_k$$

$$[N_i, N_j] = i\hbar \varepsilon_{ijk} N_k$$

$$[N_i, M_j] = 0$$

$\Rightarrow$ we get $\underline{\underline{\text{two}}}$ (commuting) angular momentum algebras!

$$H = -\frac{\mu e^4}{2(2M^2 + 2N^2 + \hbar^2)}$$

We can diagonalize simultaneously $\hat{M}^2, \hat{M}_z$, $\hat{N}^2$ and $\hat{N}_z$ with eigenvalues $m, n, \bar{\mu}, \bar{\nu}$

$$M^2 |m, n, \bar{\mu}, \bar{\nu}\rangle = \hbar^2 m(m+1) |m, n, \bar{\mu}, \bar{\nu}\rangle$$

$$N^2 |m, n, \bar{\mu}, \bar{\nu}\rangle = \hbar^2 n(n+1) |m, n, \bar{\mu}, \bar{\nu}\rangle$$

$$M_z |m, n, \bar{\mu}, \bar{\nu}\rangle = \hbar \bar{\mu} |m, n, \bar{\mu}, \bar{\nu}\rangle$$

$$N_z |m, n, \bar{\mu}, \bar{\nu}\rangle = \hbar \bar{\nu} |m, n, \bar{\mu}, \bar{\nu}\rangle$$

where $m, n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

and $|\bar{\mu}| \leq m$, $|\bar{\nu}| \leq n$

$\bar{\mu} = -m, -m+1, \ldots, m-1, m$

$\bar{\nu} = -n, -n+1, \ldots, n-1, n$

Since $\vec{R}.\vec{L} = 0 \Rightarrow \vec{K}.\vec{L} = 0$

$\Rightarrow M^2 = N^2 \Rightarrow m = n$ are the only relevant states

$\{|m, m, \bar{\mu}, \bar{\nu}\rangle\}$

$$H\,|m, m, \bar{\mu}, \bar{\nu}\rangle = -\frac{\mu e^4}{2\hbar^2(4m(m+1)+1)}\,|m, m, \bar{\mu}, \bar{\nu}\rangle$$

$$= -\frac{\mu e^4}{2\hbar^2(2m+1)^2}\,|m, m, \bar{\mu}, \bar{\nu}\rangle$$

where $\cancel{2m+1=}$ $2m+1 = 1, 2, 3, 4, \ldots$

We recognize that $2m+1 = n$ the principal quantum number!

and $E = -\dfrac{\mu e^4}{2\hbar^2 n^2}$ which are the correct energy levels!

degeneracy: for each fixed $m \to (2m+1)^2$ ~~decades~~ states

$\Rightarrow$ the degeneracy is $\boxed{n^2}$