1 Time Evolution of a Spin in an External Magnetic Field and Spin Resonance

Throughout this solution I will use boldface characters to denote vectors (e.g. $\mathbf{S}$) or vector operators (e.g. $\mathbf{\hat{S}}$) to avoid the awkward symbol $\mathbf{\hat{S}}$.

1.1

$$\frac{\partial}{\partial t} \mathbf{\hat{S}}(t) = \frac{i}{\hbar} [\hat{H}, \mathbf{\hat{S}}]$$  \hspace{1cm} (1)

$$= -\frac{ige}{2\hbar mc} [B_i S_i, S_je_j]$$  \hspace{1cm} (2)

$$= \frac{ge}{2mc} \epsilon_{ijk} B_i S_k e_j$$  \hspace{1cm} (3)

$$= -\frac{ge}{2m_c} \mathbf{B} \times \mathbf{\hat{S}}$$  \hspace{1cm} (4)

In terms of the components of $\mathbf{B}$ field:

$$\frac{\partial}{\partial t} \mathbf{\hat{S}}(t) = -\frac{ge}{2mc} \left[ B_1 (\sin(\omega t) \hat{S}_y - \cos(\omega t) \hat{S}_x) e_z + (B_1 \sin(\omega t) \hat{S}_z - B_0 \hat{S}_y) e_x + (B_1 \sin(\omega t) \hat{S}_z - B_0 \hat{S}_x) e_y \right]$$  \hspace{1cm} (5)

1.2

Setting $B_1 = 0$, the Hamiltonian of the system is simply

$$\hat{H} = -\frac{geB_0}{2mc} \hat{S}_z$$  \hspace{1cm} (6)

The unitary time evolution operator in Schrödinger picture is

$$\hat{U}(t) = e^{-\frac{i}{\hbar} \hat{H} t}$$  \hspace{1cm} (7)

$$= e^{\frac{igeB_0}{4mc} \hat{S}_z t}$$  \hspace{1cm} (8)

Using that $\hat{S}_z = \hbar/2 \hat{\sigma}_z = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we find that

$$\hat{U}(t) = \begin{pmatrix} e^{\frac{igeB_0}{4mc} t} & 0 \\ 0 & e^{-\frac{igeB_0}{4mc} t} \end{pmatrix}$$  \hspace{1cm} (9)
Since $S_z/\hbar$ is the rotation generator with respect to $z$ axis in $SU(2)$ spin space, $\hat{U}(t)$ generates a finite rotation of angle

$$\varphi = -\frac{geB_0 t}{2mc}$$  \hspace{1cm} (10)

1.3

$$\hat{S} \cdot \mathbf{n} = n_z \hat{S}_z$$  \hspace{1cm} (11)

$$= \frac{\hbar n_z}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (12)

The normalized eigenvectors of this operator are

$$\begin{cases} 
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ with eigenvalue } \hbar n_z/2 \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with eigenvalue } -\hbar n_z/2
\end{cases}$$  \hspace{1cm} (13)

We see that in order for the eigenvalue to be $\hbar/2$, we need $n = e_z$. Comparing the first eigenvector with the form of $|\psi(0)\rangle$ we can deduce that $\theta = 0$ and $\phi$ is in fact arbitrary, and $|\psi(0)\rangle$ represent a spinor pointing in $e_z$. Now, acting the time evolution operator $\hat{U}(t)$ on the general ket, we would obtain

$$|\psi(t)\rangle = \hat{U}(t) |\psi(0)\rangle$$  \hspace{1cm} (14)

$$= \left( e^{i\frac{g e B_0 t}{4mc} t} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{g e B_0 t}{4mc} t} \end{pmatrix} \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \right) \hspace{1cm} (15)

$$= \begin{pmatrix} \cos(\theta/2)e^{-\frac{i}{2}(\phi - \frac{geB_0 t}{2mc})} \\ \sin(\theta/2)e^{\frac{i}{2}(\phi - \frac{geB_0 t}{2mc})} \end{pmatrix} \hspace{1cm} (16)

We now see more clearly that $\hat{U}(t)$ induces a rotation of $\phi \rightarrow \phi - \frac{geB_0}{2mc}$.  

1.4

We first find the Schrödinger equation of the rotated ket $|\psi_\omega(t)\rangle$: We start by the Schrödinger equation of the un-rotated ket $|\psi(t)\rangle$

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$  \hspace{1cm} (17)

By inserting $e^{i\omega t \hat{S}_z/\hbar} e^{-i\omega t \hat{S}_z/\hbar}$ we have

$$i\hbar \frac{\partial}{\partial t} (e^{i\omega t \hat{S}_z/\hbar} |\psi_\omega(t)\rangle) = \hat{H} e^{i\omega t \hat{S}_z/\hbar} |\psi_\omega(t)\rangle$$  \hspace{1cm} (18)

Apply the time derivative and after a little simplification, the Schrödinger equation now becomes

$$i\hbar \frac{\partial}{\partial t} |\psi_\omega(t)\rangle = (e^{-i\omega t \hat{S}_z/\hbar} \hat{H} e^{i\omega t \hat{S}_z/\hbar} + \omega \hat{S}_z) |\psi_\omega(t)\rangle$$  \hspace{1cm} (19)
Let us examine the term containing $\hat{H}$ more closely:
\[
e^{-i\omega t\hat{S}_z/h}\hat{H}e^{i\omega t\hat{S}_z/h} = -\frac{ge}{2mc} \mathbf{B} \cdot (e^{-i\omega t\hat{S}_z/h}\hat{S}e^{i\omega t\hat{S}_z/h}) \tag{20}
\]

Now since $\hat{S}_z/h$ is the rotation generator around the $z$ axis acting on the spin operator $\hat{S}$, we can think of this inner product as the original spin operator times a inversely rotated magnetic field $\mathbf{B}'$, which has only $B_z$ and $B_x$ component (recall that $\mathbf{B} = B_0\mathbf{e}_z + B_1\cos \omega t \mathbf{e}_x - B_1 \sin \omega t \mathbf{e}_y$).
\[
e^{-i\omega t\hat{S}_z/h}\hat{H}e^{i\omega t\hat{S}_z/h} = -\frac{ge}{2mc} \mathbf{B}' \cdot \hat{S} \tag{21}
\]

\[
e^{-i\omega t\hat{S}_z/h}\hat{H}e^{i\omega t\hat{S}_z/h} = -\frac{ge}{2mc}(B_0\hat{S}_z + B_1\hat{S}_x) \tag{22}
\]

Our Schrödinger equation in rotated frame now becomes
\[
i\hbar \frac{\partial}{\partial t} |\psi_\omega(t)\rangle = -\left(\frac{ge}{2mc}[B_0\hat{S}_z + B_1\hat{S}_x] - \omega \hat{S}_z\right) |\psi_\omega(t)\rangle \tag{23}
\]

One can directly read out the time translation operator to be
\[
\hat{U}_\omega(t) = \exp \left[i \frac{\hbar}{\omega} \left(\frac{geB_0}{2mc} - \omega\right) \hat{S}_z + i \frac{geB_1}{2mc} \hat{S}_x\right] \tag{24}
\]

1.5

From the form of the time evolution operator $\hat{U}_\omega(t)$ we see that it induces a clockwise rotation around the (unnormalized) axis
\[
\mathbf{n} = \left(\frac{geB_0}{2mc} - \omega\right) \mathbf{e}_z + \frac{geB_1}{2mc} \mathbf{e}_x \tag{25}
\]

Defining
\[
\omega_0 = \frac{geB_0}{2mc}, \tag{26}
\]
\[
\Omega = \sqrt{\left(\frac{geB_1}{2mc}\right)^2 + \left(\frac{geB_0}{2mc} - \omega\right)^2} = \omega_0 \sqrt{\left(\frac{B_1}{B_0}\right)^2 + \left(1 - \frac{\omega}{\omega_0}\right)^2}, \tag{27}
\]
\[
\Phi = \sin^{-1} \left(\frac{\omega_0 B_1}{\Omega B_0}\right) = \cos^{-1} \left(\frac{\omega_0 - \omega}{\Omega}\right) \tag{28}
\]

The exact rotation matrix in the spinor space can we worked out to be
\[
D(\Phi, \Omega) = e^{-i\Phi \hat{S}_y/h}e^{i\Omega \hat{S}_z/h}e^{i\phi \hat{S}_y/h} \tag{29}
\]
\[
= \begin{pmatrix} \cos(\Phi/2) & \sin(\Phi/2) \\ -\sin(\Phi/2) & \cos(\Phi/2) \end{pmatrix} \begin{pmatrix} e^{i\Omega t/2} & 0 \\ 0 & e^{-i\Omega t/2} \end{pmatrix} \begin{pmatrix} \cos(\Phi/2) & -\sin(\Phi/2) \\ \sin(\Phi/2) & \cos(\Phi/2) \end{pmatrix} \tag{30}
\]
\[
= \begin{pmatrix} \cos(\Omega t/2) + i \cos \Phi \sin(\Omega t/2) & i \sin \Phi \sin(\Omega t/2) \\ i \sin \Phi \sin(\Omega t/2) & \cos(\Omega t/2) - i \cos \Phi \sin(\Omega t/2) \end{pmatrix} \tag{31}
\]
\[
= \begin{pmatrix} \frac{\omega_0}{\Omega} B_1 \sin(\Omega t/2) & -\frac{\omega_0}{\Omega} B_0 \sin(\Omega t/2) \\ \frac{\omega_0}{\Omega} B_1 \sin(\Omega t/2) & \cos(\Omega t/2) - i \frac{\omega_0 - \omega}{\Omega} \sin(\Omega t/2) \end{pmatrix} \tag{32}
\]
The final state ket $|\psi_\omega(t)\rangle$ in rotated frame is thus

$$|\psi_\omega(t)\rangle = \hat{U}_\omega(t) |\psi_\omega(0)\rangle$$

$$= \begin{pmatrix} \cos(\Omega t/2) + i\omega_0 - \omega \\
\Omega \\ i\omega_0 B_1 \sin(\Omega t/2) \\
\Omega B_0 \sin(\Omega t/2) \end{pmatrix}$$

(33)

(34)

Rotating the ket back to lab frame using $e^{-i\omega \hat{S}_z / \hbar}$, we obtain

$$|\psi(t)\rangle = e^{-i\omega \hat{S}_z / \hbar} |\psi_\omega(t)\rangle$$

$$= \begin{pmatrix} \cos \left(\frac{\Omega t}{2}\right) + i\omega_0 - \omega \\
\Omega \\ i\omega_0 B_1 \sin \left(\frac{\Omega t}{2}\right) \end{pmatrix} e^{i\omega t/2}$$

$$\cos \left(\frac{\Omega t}{2}\right) e^{i\omega t/2}$$

$$\sin \left(\frac{\Omega t}{2}\right) e^{-i\omega t/2}$$

(35)

(36)

1.6

In the resonant case $\omega = \omega_0$, we have $\omega_0 B_1 / \Omega B_0 = 1$. The lab frame state ket becomes

$$|\psi(t)\rangle = \begin{pmatrix} \cos \left(\frac{\Omega t}{2}\right) e^{i\omega t/2} \\
i \sin \left(\frac{\Omega t}{2}\right) e^{-i\omega t/2} \end{pmatrix}$$

(37)

We expect a flip from $+z$ to $-z$ state at time $t = \pi / \Omega$.

2 Charged Particle on a Ring as a Two Level System

2.1

The zero-electric field Hamiltonian commutes with the azimuthal angular momentum operator $-i\hbar \partial_\phi$:

$$[H_0, p_\phi] = \frac{1}{2MR^2} \left[ \left( p_\phi - i\frac{\Phi}{\phi_0} \right)^2 , p_\phi \right]$$

$$= 0$$

(38)

(39)

Therefore the energy eigenfunction is just the eigenfunctions of $p_\phi$:

$$\langle \phi | m \rangle = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m \in \mathbb{Z}$$

(40)

each with energy

$$E_m = \frac{\hbar^2}{2MR^2} \left( m - \frac{\Phi}{\phi_0} \right)^2$$

(41)
2.2

If $\Phi/\phi_0 = m' - 1/2$, the energy becomes

$$E_m = \frac{\hbar^2}{2MR^2} \left( m - m' + \frac{1}{2} \right)^2$$  \hspace{1cm} (42)

One can easily verify that under this situation the states $m = m'$ and $m = m' - 1$ has the same energy (degenerate states). Meanwhile we also identify these two states as the ground states since they have the lowest energy. Moreover we have two-fold degeneracies for the pairs $|m' + n\rangle$ and $|m' - n - 1\rangle$.

The flux acts as a shift of the symmetry point in the $p_\varphi$ space. Picking a half-integer value of $\Phi/\phi_0$ amounts to set the symmetry point in the middle of state $|m'\rangle$ and $|m' - 1\rangle$, making the ground state energy degenerate.

2.3

It is straightforward to show that $|m\rangle$ are also eigenkets of $j_\varphi$:

$$\hat{j}_\varphi |m\rangle = i\frac{\phi_0}{2\pi MR} \left( \frac{\partial}{\partial \varphi} - i\frac{\Phi}{\phi_0} \right) \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$  \hspace{1cm} (43)

$$= -\frac{\phi_0}{2\pi MR} \left( m - \frac{\Phi}{\phi_0} \right) |m\rangle$$  \hspace{1cm} (44)

The current carried by $|m\rangle$ is simply

$$j_\varphi = \langle m | \hat{j}_\varphi | m \rangle = -\frac{\phi_0}{2\pi MR} \left( m - \frac{\Phi}{\phi_0} \right)$$  \hspace{1cm} (45)

2.4

We now set $\Phi/\phi_0 \sim 1/2$, in where $|0\rangle$ and $|1\rangle$ become nearly degenerate. The effective Hamiltonian of this two-state system is

$$H_{\text{eff}} = \begin{pmatrix} \frac{\hbar^2}{2MR^2} \Phi^2 \phi_0^2 & \frac{eR}{2} (E_x + iE_y) \\ \frac{eR}{2} (E_x - iE_y) & \frac{\hbar^2}{2MR^2} \left( 1 - \frac{\Phi}{\phi_0} \right)^2 \end{pmatrix}$$  \hspace{1cm} (46)

In terms of Pauli sigma matrices:

$$H_{\text{eff}} = \frac{\hbar^2}{8MR^2} (1 + 4\Delta^2) I - \frac{eRE_x}{2} \sigma_1 + \frac{eRE_y}{2} \sigma_2 + \frac{\hbar^2 \Delta}{2MR^2} \sigma_3$$  \hspace{1cm} (47)

where $\Delta = \frac{\Phi}{\phi_0} - \frac{1}{2}$.

2.5

Consider an electron under the influence of a constant magnetic field. Ignoring the spatial degree of freedom, the Hamiltonian reads

$$\hat{H} = -\frac{ge}{2mc} \hat{S} \cdot \mathbf{B}$$  \hspace{1cm} (48)

$$= -\frac{ge\hbar}{4mc} (B_x \hat{\sigma}_1 + B_y \hat{\sigma}_2 + B_z \hat{\sigma}_3)$$  \hspace{1cm} (49)
One immediately see the analogy (the term proportional to identity matrix is just a shift of the
total energy, which can be ignored)

\[
\frac{e R E_x}{2} \leftrightarrow \frac{g e h B_x}{4mc}, \quad -\frac{e R E_y}{2} \leftrightarrow \frac{g e h B_y}{4mc}, \quad -\frac{\hbar^2 \Delta}{2MR^2} \leftrightarrow \frac{g e h B_z}{4mc}
\]

(50)

Moreover we can mimic the definition of spin vector of the electron in magnetic field to decompose
the state ket as

\[
|\psi\rangle = \cos \left( \frac{\theta}{2} \right) e^{-i\phi/2} |0\rangle + \sin \left( \frac{\theta}{2} \right) e^{i\phi/2} |1\rangle
\]

(51)

and define the vector-valued polarization vector operator \( \hat{\sigma} \) to be

\[
\hat{\sigma} = \frac{1}{2}(\hat{\sigma}_1 e_x + \hat{\sigma}_2 e_y + \hat{\sigma}_3 e_z)
\]

(52)

such that the expectation value of \( \sigma \) on ket \( |\psi\rangle \) is

\[
\langle \psi | \hat{\sigma} | \psi \rangle = \sin \theta \cos \phi e_x + \sin \theta \sin \phi e_y + \cos \theta e_z
\]

(53)

What we have done is using the \( SU(2) \to SO(3) \) homomorphism to represent a state ket in two-
dimensional complex vector space as an orientation in three dimensions. The three expectation
values of the polarization operator uniquely (up to a phase) specifies the state ket. Just like in
the electron and spin case, finding the time evolution of the polarization operator is equivalent to
solving the time evolution of the state ket.

Under the Heisenberg picture, the time evolution of \( \hat{\sigma} \) is

\[
\frac{\partial}{\partial t} \hat{\sigma} = \frac{i}{\hbar} [\hat{H}, \hat{\sigma}]
\]

(54)

\[
= \frac{e R}{4\hbar} E \times \hat{\sigma}
\]

(55)

Note that the \( E \) above is the fictitious “electric field” in the \( SO(3) \) space, which by definition (47)
has component \((-E_x, E_y, \hbar^2 \Delta/MeR^3\)). This is just the precession equation in classical mechanics.
We can interpret that \( \sigma \) undergoes a rotation around the axis \( E \), with angular frequency

\[
\omega_0 = \frac{e R}{4\hbar} |E| = \frac{e R}{4\hbar} \sqrt{E_x^2 + E_y^2 + \frac{\hbar^2 \Delta^2}{M^2e^2R^6}}
\]

(56)

2.6

Now we consider the case where \( E_x = E \cos \omega t \) and \( E_y = 0 \). We’ll also assume that \( \hbar^2 \Delta/MeR^3 \gg E \)
in order to draw analogy to NMR.

The Schrödinger equation in the effective Hilbert space is

\[
i \hbar \frac{\partial}{\partial t} |\psi\rangle = \left( \frac{e R E}{2} \cos \omega t \hat{\sigma}_1 - \frac{\hbar^2 \Delta}{2MR^2} \hat{\sigma}_3 \right) |\psi\rangle
\]

(57)

Following the same procedure in problem one, we go to the rotated frame by acting the rotation
operator \( e^{-i\omega t \hat{\sigma}_3/2} \). The Schrödinger equation in the new frame reads

\[
i \hbar \frac{\partial}{\partial t} |\psi_\omega\rangle = \left[ \frac{e R E}{2} \cos \omega t (e^{-i\omega t \hat{\sigma}_1/2} \hat{\sigma}_1 e^{i\omega t \hat{\sigma}_1/2}) - \left( \frac{\hbar^2 \Delta}{2MR^2} - \omega \right) \hat{\sigma}_3 \right] |\psi_\omega\rangle
\]

(58)
Note that
\[
\cos \omega t (e^{-i\omega t \hat{\sigma}_3 / 2} \hat{\sigma}_1 e^{i\omega t \hat{\sigma}_3 / 2}) = \cos \omega t (\cos \omega t \hat{\sigma}_1 + \sin \omega t \hat{\sigma}_2)
\]
\[
= \frac{1}{2} \hat{\sigma}_1 + \frac{1}{2} (\cos 2\omega t \hat{\sigma}_1 + \sin 2\omega t \hat{\sigma}_2)
\]
(59)
(60)

We can see that there is a constant piece and an oscillating piece. Assume that the frequency of oscillation is large enough so we can ignore it. The time dependent Schrödinger equation in rotating frame can be approximated by
\[
i \hbar \frac{\partial}{\partial t} |\psi_\omega \rangle = \left[ \frac{eRE}{4} \hat{\sigma}_1 - \left( \frac{\hbar^2 \Delta}{2MR^2} - \omega \right) \hat{\sigma}_3 \right] |\psi_\omega \rangle
\]
(61)
which has the same form as the NMR Hamiltonian in rotating frame (except for a rapidly oscillating term). Following the same procedure we define
\[
\omega_0 = \frac{\hbar^2 \Delta}{2MR^2}
\]
(62)
\[
\omega_1 = \frac{eRE}{4}
\]
(63)
\[
\Omega = \sqrt{\omega_1^2 + (\omega_0 - \omega)^2}
\]
(64)

And write the solution in the lab frame as
\[
|\psi\rangle = \begin{pmatrix}
\cos(\Omega t / 2) + \frac{i \omega_0 - \omega}{\Omega} \sin(\Omega t / 2) e^{i\omega t / 2} \\
-\frac{i \omega_1}{\Omega} \sin(\Omega t / 2) e^{-i\omega t / 2}
\end{pmatrix}
\]
(65)

This motion is periodic in time with a period
\[
T = 2\pi |\text{lcm}(\omega, \Omega)|^{-1}
\]
(66)
The electron will only make a complete transition $|0\rangle \rightarrow |1\rangle$ in the resonant case $\omega = \omega_0$. The time needed to take for one such period is
\[
T^* = \frac{\pi}{\Omega}
\]
(67)