Solution to Problem Set No. 3
Angular Momentum

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1 Harmonic Oscillators and Angular Momentum

1.1

\[ [\hat{J}_i, \hat{J}_j] = \frac{\hbar^2}{4} [\hat{a}^{\dagger}_\alpha \sigma_i^{\alpha \beta} \hat{a}_\mu \hat{a}_\nu] \]

\[ = \frac{\hbar^2}{4} (\hat{a}^{\dagger}_\alpha \hat{a}_\beta \hat{a}^{\dagger}_\mu \hat{a}_\nu \sigma_i^{\alpha \beta} \sigma_j^{\mu \nu} - \hat{a}^{\dagger}_\mu \hat{a}_\nu \hat{a}^{\dagger}_\alpha \hat{a}_\beta \sigma_i^{\mu \nu} \sigma_j^{\alpha \beta}) \]

Using

\[ \hat{a}^{\dagger}_\mu \hat{a}_\nu \hat{a}^{\dagger}_\alpha \hat{a}_\beta = \frac{1}{2} (\hat{a}^{\dagger}_\alpha \hat{a}_\beta \hat{a}_\mu \hat{a}_\nu - \delta_{\alpha \nu} \hat{a}^{\dagger}_\beta \hat{a}_\mu - \delta_{\beta \mu} \hat{a}^{\dagger}_\alpha \hat{a}_\nu) \]

After relabeling and grouping the same terms

\[ [\hat{J}_i, \hat{J}_j] = \frac{\hbar^2}{4} (\hat{a}^{\dagger}_\alpha \hat{a}_\beta \hat{a}^{\dagger}_\mu \hat{a}_\nu \sigma_i^{\alpha \beta} \sigma_j^{\mu \nu} - \hat{a}^{\dagger}_\mu \hat{a}_\nu \hat{a}^{\dagger}_\alpha \hat{a}_\beta \sigma_i^{\mu \nu} \sigma_j^{\alpha \beta}) \]

\[ = \frac{\hbar^2}{4} (\delta_{\alpha \nu} \hat{a}^{\dagger}_\beta \hat{a}_\mu + \delta_{\beta \mu} \hat{a}^{\dagger}_\alpha \hat{a}_\nu) \sigma_i^{\mu \nu} \sigma_j^{\alpha \beta} \]

\[ = \frac{\hbar^2}{4} \hat{a}^{\dagger}_\alpha \hat{a}_\beta (\sigma_i^{\beta} \sigma_j^{\alpha} - \delta_{\alpha \beta} \sigma_i^{\mu} \sigma_j^{\mu}) \]

\[ = i \hbar \epsilon_{ijk} \hat{J}_k \]

We recovered the commutation relation of angular momentum operators.

1.2

Note that

\[ \sum_i \hat{J}_i^2 = \frac{\hbar^2}{4} \hat{a}^{\dagger}_\alpha \hat{a}_\beta \hat{a}^{\dagger}_\mu \hat{a}_\nu \sum_i \sigma_i^{\alpha \beta} \sigma_i^{\mu \nu} \]

\[ = \frac{\hbar^2}{4} \hat{a}^{\dagger}_\alpha \hat{a}_\beta \hat{a}^{\dagger}_\mu \hat{a}_\nu (2 \delta_{\alpha \nu} \delta_{\beta \mu} - \delta_{\alpha \beta} \delta_{\mu \nu}) \]

\[ = \frac{\hbar^2}{4} [2(2N_1 N_2 + N_1 + N_2 + N_1^2 + N_2^2) - (N_1 + N_2)^2] \]

\[ = \hbar^2 \left( \frac{N_1 + N_2}{2} \right) \left( \frac{N_1 + N_2}{2} + 1 \right) \]
and
\[ \hat{J}_z = \frac{\hbar}{2} \hat{a}_\alpha^\dagger \sigma_z^{\alpha \beta} \hat{a}_\beta \] (13)
\[ = \frac{\hbar}{2} (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \] (14)
\[ = \hbar \left( \frac{N_1 - N_2}{2} \right) \] (15)

So we have identified the following eigenvalues
\[ j = \frac{N_1 + N_2}{2}, \quad m = \frac{N_1 - N_2}{2} \] (16)

1.3

We simply invert (16) to express \( N_\alpha \) in terms of \( j, m \):
\[ N_1 = j + m, \quad N_2 = j - m \] (17)

Hence
\[ |j, m\rangle = \frac{1}{\sqrt{(j + m)!(j - m)!}} (\hat{a}_1^\dagger)^{j + m} (\hat{a}_2^\dagger)^{j - m} |0\rangle \] (18)

Now we construct the raising/lowering operators \( \hat{J}^\pm \) as
\[ \hat{J}^\pm = \hat{J}_x \pm i \hat{J}_y \] (19)
\[ = \frac{\hbar}{2} \left[ (\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2^\dagger \hat{a}_1^\dagger) \pm i (-\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_2^\dagger \hat{a}_1^\dagger) \right] \] (20)
\[ \Rightarrow \hat{J}^+ = \hbar \hat{a}_1^\dagger \hat{a}_2, \quad \hat{J}^- = \hbar \hat{a}_2^\dagger \hat{a}_1 \] (21)

Acting \( \hat{J}^\pm \) on the state \( |j, m\rangle \) we find
\[ \hat{J}^+ = \hbar \sqrt{\frac{(j - m)}{(j + m)!(j - m - 1)!}} (\hat{a}_1^\dagger)^{j + m + 1} (\hat{a}_2^\dagger)^{j - m - 1} |0\rangle \] (22)
\[ = \hbar \sqrt{(j - m)(j + m + 1)} |j, m + 1\rangle \] (23)
and
\[ \hat{J}^- = \hbar \sqrt{\frac{(j + m)}{(j + m - 1)!(j - m)!}} (\hat{a}_1^\dagger)^{j + m - 1} (\hat{a}_2^\dagger)^{j - m + 1} |0\rangle \] (24)
\[ = \hbar \sqrt{(j + m)(j - m + 1)} |j, m - 1\rangle \] (25)

1.4

We have
\[ \hat{K}^\dagger = \hat{a}_1^\dagger \hat{a}_2^\dagger, \quad \hat{K} = \hat{a}_1 \hat{a}_2 \] (26)
Acting $\hat{K}^\dagger$ on state $|j, m\rangle$ gives

$$\hat{K}^\dagger |j, m\rangle = \frac{1}{\sqrt{(j + m)! (j - m)!}} (\hat{a}^\dagger_1)^{j+m+1}(\hat{a}^\dagger_2)^{j-m+1} |0\rangle$$

$$= \sqrt{(j + m + 1)(j - m + 1)} |j + 1, m\rangle$$

Similarly,

$$\hat{K} |j, m\rangle = \begin{cases} \frac{1}{\sqrt{(j+m)(j-m)}} |j - 1, m\rangle, & j \neq \pm m \\ 0, & j = \pm m \end{cases}$$

$\hat{K}^\dagger$ and $\hat{K}$ act as the raising/lowering operator for total angular momentum $j$.

## 2 Addition of Spin Angular Momenta

### 2.1

The dimension of the product Hilbert space is four. The space can either be spanned by the original $|m_1, m_2\rangle$ basis

$$|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$$

or in the new $|j, m\rangle$ basis

$$|1, 1\rangle, |1, 0\rangle, |1, -1\rangle, |0, 0\rangle$$

### 2.2

The possible eigenvalues of $J^2$ are 0 and 1.

### 2.3

The two recursive relations we will use (refer to Sakurai §3.8) are the $J^\pm$ relations

$$\sqrt{(j \pm m)(j \pm m + 1)} \langle m_1 m_2 | j, m \pm 1 \rangle$$

$$= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \langle m_1 \mp 1, m_2 | j m \rangle + \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \langle m_1, m_2 \mp 1 | j m \rangle$$

We will first start with the representation of $j = 1$, see figure below.

Starting from the rightmost corner of the $(m_1, m_2)$ plane, with the coefficient

$$\langle \frac{1}{2}, \frac{1}{2} | 1, 1 \rangle = 1$$

and apply the $J^-$ relation (lower sign), we find

$$\sqrt{(1 + 1)(1 - 1 + 1)} \langle \frac{1}{2}, \frac{1}{2} | 1, 0 \rangle = \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right) \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)} \langle \frac{1}{2}, \frac{1}{2} | 1, 1 \rangle$$

or

$$\langle \frac{1}{2}, -\frac{1}{2} | 1, 0 \rangle = \frac{1}{\sqrt{2}}$$
Now we apply the $J^+$ relation to find the upper left corner:

$$\sqrt{(1)(1+1)} \langle \frac{1}{2}, \frac{1}{2} | 1, 1 \rangle$$

$$= \sqrt{\left(\frac{1}{2} - \frac{1}{2} + 1\right)\left(\frac{1}{2} + \frac{1}{2}\right)} \langle -\frac{1}{2}, \frac{1}{2} | 1, 0 \rangle + \sqrt{\left(\frac{1}{2} - \frac{1}{2} + 1\right)\left(\frac{1}{2} + \frac{1}{2}\right)} \langle \frac{1}{2}, -\frac{1}{2} | 1, 0 \rangle$$

This gives

$$\langle -\frac{1}{2}, \frac{1}{2} | 1, 0 \rangle = \frac{1}{\sqrt{2}}$$

Finally, for the bottom left corner we apply the $J^-$ relation again:

$$\sqrt{(1)(1+1)} \langle -\frac{1}{2}, -\frac{1}{2} | 1, -1 \rangle$$

$$= \sqrt{\left(\frac{1}{2} - \frac{1}{2} + 1\right)\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)} \langle \frac{1}{2}, -\frac{1}{2} | 1, 0 \rangle + \sqrt{\left(\frac{1}{2} - \frac{1}{2} + 1\right)\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)} \langle -\frac{1}{2}, \frac{1}{2} | 1, 0 \rangle$$

This gives

$$\langle -\frac{1}{2}, -\frac{1}{2} | 1, -1 \rangle = 1$$

For the $j = 0$ singlet, the $(m_1, m_2)$ plane comprises of only two points. Any one of the recursive relations will give

$$0 = \langle \frac{1}{2}, -\frac{1}{2} | 0, 0 \rangle + \langle -\frac{1}{2}, \frac{1}{2} | 0, 0 \rangle$$

which together with the normalization condition determines

$$\langle \frac{1}{2}, -\frac{1}{2} | 0, 0 \rangle = \frac{1}{\sqrt{2}}$$

$$\langle -\frac{1}{2}, \frac{1}{2} | 0, 0 \rangle = -\frac{1}{\sqrt{2}}$$
3 Simple Magnets

Note that

$$\sum_{ij} S_i \cdot S_j = \left( \sum_i S_i \right)^2 - \frac{3N\hbar^2}{4} \tag{46}$$

Defining the total spin operator $S$ to be

$$S = \sum_i S_i \tag{47}$$

The Heisenberg Hamiltonian can be written as

$$H = -J \left( S^2 - \frac{3N\hbar^2}{4} \right) \tag{48}$$

Thus the eigenstates of $H$ is also the eigenstates of the total angular momentum operator $S^2$. The Heisenberg Hamiltonian is invariant under simultaneous $SO(3)$ rotations of all spins. It is convenient for us to pick another operator to differentiate these symmetric states. The original set of angular momentum operators $\{S_{iz}\}$ cannot do this job since $[S^2, S_{iz}] \neq 0$. A proper choice is the total $z$-angular momentum operator

$$S_z = \sum_i S_{iz} \tag{49}$$

The problem of finding the eigenstates of the Hamiltonian thus transform into finding the representations of total spin operator and total $S_z$ operator.

Throughout this problem I will use $|\uparrow\uparrow\uparrow\rangle$ to represent a eigenket in $\{S_{iz}\}$ basis and $|s,m\rangle$ to represent a eigenket in $\{S, S_z\}$ basis.

3.1

Let us investigate how to go from the original basis of three (four) spin-1/2 electrons to higher spin objects. Starting form the most basic action, we know that the merging of two spin-1/2 objects gives a spin-1 triplet and a spin-0 singlet:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \tag{50}$$

For three spin-1/2 systems we merge the first two spins to create a singlet and a triplet. We then merge the last electron to them. This can be symbolically understood as follows:

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \oplus 0) \otimes \frac{1}{2} \tag{51}$$

$$= \left( \frac{3}{2} \oplus \frac{1}{2} \right) \oplus \frac{1}{2} \tag{52}$$

We have obtained one quadruplet and two doublets, coming from two different ways to merge three spin-1/2 systems together. The total degree of freedom is $4 + 2 + 2 = 2^3$, just as expected.
For four spins we can either merge the electrons one after one, or merge 1-2 and 3-4 pair first, then merge two triplet-singlet systems together. Either method should give the same decomposition (although the expansion coefficient may be different). I demonstrate the second approach here:

\[ \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \oplus 0) \otimes (1 \oplus 0) \]
\[ = (2 \oplus 1 \oplus 0) \oplus 1 \oplus 1 \oplus 0 \]  

(53)
(54)

Therefore, we have one quintuplet, three triplets and two singlets in the decomposition. The total dimension of the Hilbert space is \( 5 + 3 \times 3 + 2 \times 1 = 2^4 \).

The detailed relation between these two basis can be obtained by carefully work out each step in the symbolic process using a table of Clebsh-Gordan coefficients. I’ll list the final results here (ignoring normalization):

<table>
<thead>
<tr>
<th>( s )</th>
<th>( m )</th>
<th>( N=3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>(</td>
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<tr>
<td>2</td>
<td>1</td>
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<td>0</td>
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</tr>
</tbody>
</table>

Table 1: Detailed relation of \( \{S_{zi}\} \) and \( \{S^2, S_z\} \) basis for \( N=3 \) and \( N=4 \)

Only results of positive \( m \) are listed. To get the expression of negative \( m \), simply invert the direction of each spinor.

(a) In the ferromagnet case, the ground state is simply states with the highest total spin number \( s \). The degeneracies thus come from the allowed \( m \) in such a representation. In \( N = 3 \) there it is a quadruplet, and in \( N = 4 \) it is a quintuplet.

- For \( N = 3 \) (four-fold degeneracy):

\[ |0\rangle = \left| \begin{array}{c} 3 \\ 2 \\ 2 \end{array} \right> , \left| \begin{array}{c} 3 \\ 1 \\ 2 \end{array} \right> , \left| \begin{array}{c} 3 \\ 1 \\ -2 \end{array} \right> , \left| \begin{array}{c} 3 \\ 2 \\ -2 \end{array} \right> \]  

\[ E_0 = -\frac{3J\hbar^2}{2} \]  

(55)
(56)

- For \( N = 4 \) (five-fold degeneracy):

\[ |0\rangle = |2, 2\rangle , |2, 1\rangle , |2, 0\rangle , |2, -1\rangle , |2, -2\rangle \]  

\[ E_0 = -3J\hbar^2 \]  

(57)
(58)

(b) The first excited states are those states who have the next highest quantum number \( s \).
\( N = 3 \):

\[
|1\rangle = \left| \frac{1}{2}, \frac{1}{2} \right>, \left| \frac{1}{2}, -\frac{1}{2} \right>, \quad E_1 = \frac{3J \hbar^2}{2}
\] (59)

As can be seen from the table, each of the states is actually two-fold degenerate. So we have a four-fold degeneracy in total.

\( N = 4 \):

\[
|1\rangle = |1, 1\rangle, |1, 0\rangle, |1, -1\rangle, \quad E_1 = J \hbar^3
\] (60)

As in the case of \( N = 3 \), each of these \( |s, m\rangle \) states are 3-fold degenerate. Thus the degeneracy of the first excited state is nine-fold.

(c) The ground state ket and energy for general mutually correlated \( N \) spins is

\[
|\psi\rangle = |N, m\rangle, \quad -N \leq m \leq N
\]

\[
E = -\frac{J \hbar^2}{4} N(N-1) + \frac{3N \hbar^2}{4}
\] (62)

The degeneracy of the ground state is \( N + 1 \). We can cast the states in \( \{S_{zi}\} \) bases:

\[
|N, m\rangle = \frac{1}{\sqrt{C^N_m}} \sum_{\text{symmetrize}} \left( |\uparrow\rangle^N \otimes |\downarrow\rangle^{N-m} \right)
\] (63)

where the sum is understood to be over all the symmetric permutations of the spins.

3.2

Now that the system is antiferromagnetic, the energy eigenkets is state with lowest total angular momentum quantum number \( s \).

(a) The ground state for \( N = 3 \) is now the two excited states we found in (59)

\[
|0\rangle = \left| \frac{1}{2}, \pm \frac{1}{2} \right>
\] (64)

The first excited states are the ferromagnetic ground states in (55):

\[
|1\rangle = \left| \frac{3}{2}, \frac{3}{2} \right>, \left| \frac{3}{2}, \frac{3}{2} \right>, \left| \frac{3}{2}, -\frac{3}{2} \right>, \left| \frac{3}{2}, -\frac{3}{2} \right>
\] (65)

(b) For \( N = 4 \) the spin zero state now becomes the lowest energy state when \( J > 0 \). This state is also two-fold degenerate.

\[
|0\rangle = |0, 0\rangle
\] (66)

The energy is

\[
E = \frac{3J \hbar^2}{4}
\] (67)

7
3.3

(a) It is sufficient to calculate the commutator \([\chi_{ABC}, S_A]\) since the other two terms are related by a cyclic permutation.

\[
[\chi_{ABC}, S_A] = [\epsilon_{ijk} S_{Ai} S_{Bj} S_{Ck}, S_{Ae} e_x + S_{Ay} e_y + S_{Az} e_z] = [\epsilon_{ijk} S_{Ai}, S_{Ae} e_x + S_{Ay} e_y + S_{Az} e_z] S_{Bj} S_{Ck}
\]

Note that

\[
[\epsilon_{ijk} S_i, S_l] = \epsilon_{ijk} \epsilon_{ilm} S_m
\]

Putting the results of \(S_B, S_C\) (related by a simple cyclic permutation) in, we get

\[
[\chi_{ABC}, S_A] = (\delta_{jl} S_{Ak} - \delta_{kl} S_{Aj}) S_{Bj} S_{Ck} e_l
\]

(b) We label the two degenerate states who have the same representation \(|\frac{1}{2}, \frac{1}{2}\rangle\) by

\[
|a\rangle = |\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle
\]

\[
|b\rangle = |\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle
\]

Acting \(\chi_{ABC}\) on these states, we find

\[
\chi_{ABC} |a\rangle = \frac{h^3}{4} (2|\downarrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle)
\]

\[
= -\frac{h^3}{4} |b\rangle
\]

and

\[
\chi_{ABC} |b\rangle = \frac{h^3}{4} (|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle)
\]

\[
= \frac{h^3}{4} |a\rangle
\]

Define

\[
|\uparrow, +\rangle = \frac{1}{\sqrt{8}} (|b\rangle - |a\rangle)
\]

\[
|\uparrow, -\rangle = \frac{1}{\sqrt{8}} (|b\rangle + |a\rangle)
\]
The states $|\pm\rangle$ are now properly normalized, orthogonal and
\[
\chi_{ABC} |\uparrow, +\rangle = \frac{h^3}{8\sqrt{2}} |\uparrow, +\rangle, \quad \chi_{ABC} |\uparrow, -\rangle = -\frac{h^3}{8\sqrt{2}} |\uparrow, -\rangle
\] (86)

Hence $\chi_{ABC}$ can be used to label the two degenerate states. The two state kets $|\pm\rangle$, despite having the same total spin and total $z$ angular momentum, have different chirality, and can be picked out by the chiral operator $\chi_{ABC}$.

Similarly, we can also apply the same procedure to the two degenerate $|\frac{1}{2}, -\frac{1}{2}\rangle$ states to find two other eigenkets of the chiral operator
\[
\chi_{ABC} |\downarrow, +\rangle = \frac{h^3}{8\sqrt{2}} |\downarrow, +\rangle, \quad \chi_{ABC} |\downarrow, -\rangle = -\frac{h^3}{8\sqrt{2}} |\downarrow, -\rangle
\] (87)

They have the same (in fact, opposite) eigenvalues as their spin-up partners, as they are the mirror reflection along the $z$ direction. Unfortunately, the chiral operator $\chi_{ABC}$ can only distinguish the chirality but not $S_z$.

It is worthwhile to check the first excited states are also eigenstates of the chiral operator:
\[
\chi_{ABC} |\uparrow\uparrow\uparrow\rangle = \chi_{ABC} \left( \frac{|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle}{\sqrt{3}} \right) = 0
\] (88)

Which is expected since $[H, \chi_{ABC}] = 0$ and they are all symmetric in spins.

(c) Consider now the Hamiltonian
\[
H = -J(S_A \cdot S_B + S_B \cdot S_C + S_C \cdot S_A) + g S_A \cdot S_B \times S_C
\]
\[
= -\frac{J}{2} S^2 + g\chi_{ABC} + \frac{9Jh^2}{8}
\] (89) (90)

This Hamiltonian is completely diagonalized in the basis of eigenkets of the operator $\{S^2, S_z, \chi_{ABC}\}$ we just found. The eigenstates and the spectrum are listed in table 2. Some ground state degeneracies have been lifted.
<table>
<thead>
<tr>
<th>label</th>
<th>s</th>
<th>m</th>
<th>$X_{ABC}$</th>
<th>E</th>
</tr>
</thead>
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<tr>
<td>$\frac{3}{2}, \frac{3}{2}$</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>$\frac{3J\hbar^2}{4}$</td>
</tr>
<tr>
<td>$\frac{3}{2}, \frac{1}{2}$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>$\frac{3J\hbar^2}{4}$</td>
</tr>
<tr>
<td>$\frac{3}{2}, -\frac{1}{2}$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$\frac{3J\hbar^2}{4}$</td>
</tr>
<tr>
<td>$\frac{3}{2}, -\frac{3}{2}$</td>
<td>3</td>
<td>-3</td>
<td>0</td>
<td>$\frac{3J\hbar^2}{4}$</td>
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<tr>
<td>$\uparrow, +$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\hbar^3}{8\sqrt{2}}$</td>
<td>$-\frac{3J\hbar^2}{4} + \frac{\hbar^3}{8\sqrt{2}}$</td>
</tr>
<tr>
<td>$\uparrow, -$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{\hbar^3}{8\sqrt{2}}$</td>
<td>$-\frac{3J\hbar^2}{4} - \frac{\hbar^3}{8\sqrt{2}}$</td>
</tr>
<tr>
<td>$\downarrow, +$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\hbar^3}{8\sqrt{2}}$</td>
<td>$-\frac{3J\hbar^2}{4} + \frac{\hbar^3}{8\sqrt{2}}$</td>
</tr>
<tr>
<td>$\downarrow, -$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{\hbar^3}{8\sqrt{2}}$</td>
<td>$-\frac{3J\hbar^2}{4} - \frac{\hbar^3}{8\sqrt{2}}$</td>
</tr>
</tbody>
</table>

Table 2: The energy spectrum