1 Coherent states and path integral quantization.

1.1 Coherent States

Let q and p be the coordinate and momentum operators. They satisfy the Heisenberg algebra, $[q,p]=i\hbar$. Let us introduce the creation and annihilation operators $a^\dagger$ and a, by their standard relations

$$q = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) \tag{1}$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^\dagger - a\right) \tag{2}$$

Let us consider a Hilbert space spanned by a complete set of harmonic oscillator states $\{|n\rangle\}$, with $n=0,\ldots,\infty$. Let $\hat{a}^\dagger$ and $\hat{a}$ be a pair of creation and annihilation operators acting on that Hilbert space, and satisfying the commutation relations

$$\left[\hat{a},\hat{a}^\dagger\right]=1\ , \quad \left[\hat{a}^\dagger,\hat{a}^\dagger\right]=0\ , \quad [\hat{a},\hat{a}]=0 \tag{3}$$

These operators generate the harmonic oscillators states $\{|n\rangle\}$ in the usual way,

$$|n\rangle = \frac{1}{\sqrt{n!}}\ \left(\hat{a}^\dagger\right)^n\ |0\rangle \tag{4}$$

$$\hat{a}|0\rangle = 0 \tag{5}$$

where $|0\rangle$ is the vacuum state of the oscillator.

Let us denote by $|z\rangle$ the *coherent state*

$$|z\rangle = \ e^{z\hat{a}^\dagger}\ |0\rangle \tag{6}$$

$$\langle z| = \ \langle 0|\ e^{\bar{z}\hat{a}} \tag{7}$$

where z is an arbitrary complex number and $\bar{z}$ is the complex conjugate. The coherent state $|z\rangle$ has the defining property of being a wave packet with optimal spread, *i.e.,* the Heisenberg uncertainty inequality is an equality for these coherent states.

How does $\hat{a}$ act on the coherent state $|z\rangle$?

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty}\frac{z^n}{n!}\ \hat{a}\left(\hat{a}^\dagger\right)^n\ |0\rangle \tag{8}$$

Since

$$\left[\hat{a},\left(\hat{a}^\dagger\right)^n\right] = n\ \left(\hat{a}^\dagger\right)^{n-1} \tag{9}$$

we get

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty}\frac{z^n}{n!}\ \left(\left[\hat{a},\left(\hat{a}^\dagger\right)^n\right] + \left(\hat{a}^\dagger\right)^n\ \hat{a}\right)\ |0\rangle \tag{10}$$

Thus, we find

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, n \left(\hat{a}^\dagger\right)^{n-1} \; |0\rangle \equiv z \, |z\rangle \tag{11}$$

Therefore $|z\rangle$ is a right eigenvector of $\hat{a}$ and z is the (right) eigenvalue.

Likewise we get

$$\begin{aligned}
\hat{a}^\dagger|z\rangle &= \; \hat{a}^\dagger \; \sum_{n=0}^{\infty} \frac{z^n}{n!} \; \left(\hat{a}^\dagger\right)^n \; |0\rangle \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \; \left(\hat{a}^\dagger\right)^{n+1} \; |0\rangle \\
&= \sum_{n=0}^{\infty} (n+1) \; \frac{z^n}{(n+1)!} \; \left(\hat{a}^\dagger\right)^{n+1} \; |0\rangle \\
&= \sum_{n=1}^{\infty} n \; \frac{z^{n-1}}{n!} \; \left(\hat{a}^\dagger\right)^n \; |0\rangle
\end{aligned} \tag{12}$$

Thus,

$$\hat{a}^\dagger|z\rangle = \frac{\partial}{\partial z} \; |z\rangle \tag{13}$$

Another quantity of interest is the overlap of two coherent states, $\langle z|z'\rangle$,

$$\langle z|z'\rangle = \langle 0|e^{\bar{z}\hat{a}} \;\; e^{z'\hat{a}^\dagger} \; |0\rangle \tag{14}$$

We will calculate this matrix element using the Baker-Hausdorff formulas

$$e^{\hat{A}} \; e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}\left[\hat{A}, \hat{B}\right]} = e^{\left[\hat{A}, \hat{B}\right]} \; e^{\hat{B}} \; e^{\hat{A}} \tag{15}$$

which holds provided the commutator $\left[\hat{A}, \hat{B}\right]$ is a c-number, *i.e.,* it is proportional to the identity operator. Since $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, we find

$$\langle z|z'\rangle = e^{\bar{z}z'} \; \langle 0|e^{z'\hat{a}^\dagger} \; e^{\bar{z}\hat{a}}|0\rangle \tag{16}$$

But

$$e^{\bar{z}\hat{a}} \; |0\rangle = |0\rangle \tag{17}$$

and

$$\langle 0| \; e^{z'\hat{a}^\dagger} = \langle 0| \tag{18}$$

Hence we get

$$\langle z|z'\rangle = e^{\bar{z}z'} \tag{19}$$

An arbitrary state $|\psi\rangle$ of this Hilbert space can be expanded in the harmonic oscillator basis states $\{|n\rangle\}$,

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}} \; |n\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!} \; \left(\hat{a}^\dagger\right)^n \; |0\rangle \tag{20}$$

The projection of the state $|\psi\rangle$ onto the coherent state $|z\rangle$ is

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!} \; \langle z| \left(\hat{a}^\dagger\right)^n |0\rangle \tag{21}$$

Since

$$\langle z| \; \hat{a}^\dagger = \bar{z} \; \langle z| \tag{22}$$

we find

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!} \; \bar{z}^n \equiv \psi(\bar{z}) \tag{23}$$

Therefore the projection of $|\psi\rangle$ onto $|z\rangle$ is the *anti-holomorphic* (*i.e.*, *anti-analytic*) function $\psi(\bar{z})$. In other words, in this representation, the space of states $\{|\psi\rangle\}$ are in one-to-one correspondence with the space of anti-analytic functions.

In summary, the coherent states $\{|z\rangle\}$ satisfy

$$\begin{array}{ll} \hat{a}|z\rangle = z|z\rangle & \langle z|\hat{a} = \partial_{\bar{z}}\langle z| \\ \hat{a}^\dagger|z\rangle = \partial_z|z\rangle & \langle z|\hat{a}^\dagger = \bar{z}\langle z| \\ \langle z|\psi\rangle = \psi(\bar{z}) & \langle\psi|z\rangle = \bar{\psi}(z) \end{array} \tag{24}$$

Next we will prove the *resolution of identity*

$$\hat{I} = \int \frac{dz d\bar{z}}{2\pi i} \; e^{-z\bar{z}}|z\rangle\langle z| \tag{25}$$

Let $|\psi\rangle$ and $|\phi\rangle$ be two arbitrary states

$$\begin{aligned} |\psi\rangle &= \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}} \; |n\rangle \\ |\psi\rangle &= \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}} \; |n\rangle \\ \langle\phi|\psi\rangle &= \sum_{n=0}^{\infty} \frac{\phi_n\psi_n}{n!} \end{aligned} \tag{26}$$

Let us compute the matrix element

$$\langle\phi|\hat{I}|\psi\rangle = \sum_{m.n} \frac{\bar{\phi}_n\psi_n}{n!}\langle n|\hat{I}|m\rangle \tag{27}$$

Thus we need to find

$$\langle n|\hat{I}|m\rangle = \int \frac{dzd\bar{z}}{2\pi i}\; e^{-|z|^2}\langle n|z\rangle\langle z|m\rangle \tag{28}$$

Recall that the integration measure is defined to be given by

$$\frac{dzd\bar{z}}{2\pi i} = \frac{d\text{Re}z d\text{Im}z}{\pi} \tag{29}$$

where

$$\langle n|z\rangle = \frac{1}{\sqrt{n!}}\langle 0|\left(\hat{a}\right)^n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle \tag{30}$$

and

$$\langle z|m\rangle = \frac{1}{\sqrt{m!}}\langle z|\left(\hat{a}^\dagger\right)^m|0\rangle = \frac{\bar{z}^m}{\sqrt{m!}}\langle z|0\rangle \tag{31}$$

Now, since $|\langle 0|z\rangle|^2 = 1$, we get

$$\langle n|\hat{I}|m\rangle = \int \frac{dzd\bar{z}}{2\pi i}\,\frac{e^{-|z|^2}}{\sqrt{n!m!}} z^n\bar{z}^m = \int_0^\infty \rho d\rho \int_0^{2\pi}\frac{d\varphi}{2\pi}\,\frac{e^{-\rho^2}}{\sqrt{n!m!}}\rho^{n+m}e^{i(n-m)\varphi} \tag{32}$$

Thus,

$$\langle n|\hat{I}|m\rangle = \frac{\delta_{n,m}}{n!}\int_0^\infty dx\; x^n e^{-x} = \langle n|m\rangle \tag{33}$$

Hence, we have found that

$$\langle\phi|\hat{I}|\psi\rangle = \langle\phi|\psi\rangle \tag{34}$$

for any pair of states $|\psi\rangle$ and $|\phi\rangle$. Therefore $\hat{I}$ is the identity operator in that space. We conclude that the set of coherent states $\{|z\rangle\}$ is an *over-complete* set of states.

Furthermore, since

$$\langle z|\left(\hat{a}^\dagger\right)^n\left(\hat{a}\right)^m|z'\rangle = \bar{z}^n z'^m\langle z|z'\rangle = \bar{z}^n z'^m e^{\bar{z}z'} \tag{35}$$

we conclude that the matrix elements of any arbitrary normal ordered operator of the form

$$\hat{A} = \sum_{n,m} A_{n,m}\left(\hat{a}^\dagger\right)^n\left(\hat{a}\right)^m \tag{36}$$

are equal to

$$\langle z|\hat{A}|z'\rangle = \left(\sum_{n,m} A_{n,m}\bar{z}^n z'^m\right) e^{\bar{z}z'} \tag{37}$$

Therefore, if $\hat{A}(\hat{a},\hat{a}^\dagger)$ is an arbitrary *normal ordered* operator (relative to the state $|0\rangle$), its matrix elements are given by

$$\langle z|\hat{A}(\hat{a},\hat{a}^\dagger)|z'\rangle = A(\bar{z},z')e^{\bar{z}z'} \tag{38}$$

where $A(\bar{z}, z')$ is a function of two complex variables $\bar{z}$ and z', obtained from $\hat{A}$ by the formal replacement

$$\hat{a} \leftrightarrow z' \, , \qquad \hat{a}^\dagger \leftrightarrow \bar{z} \tag{39}$$

For example, the matrix elements of the the operator $\hat{N} = \hat{a}^\dagger \hat{a}$, which measures the number of excitations, is

$$\langle z|\hat{N}|z'\rangle = \langle z|\hat{a}^\dagger \hat{a}|z'\rangle = \bar{z}z' \, e^{\bar{z}z'} \tag{40}$$

1.2 Path Integrals and Coherent States

As usual we will want to compute the matrix elements of the evolution operator $\mathcal{U}$,

$$\mathcal{U} = e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})} \tag{41}$$

where $\hat{H}(\hat{a}^\dagger, \hat{a})$ is a normal ordered operator. Thus, if $|i\rangle$ and $|f\rangle$ denote two arbitrary initial and final states, we can write the matrix element of $\mathcal{U}$ as

$$\langle f|e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})}|i\rangle = \lim_{\epsilon\to 0, N\to\infty} \langle f| \left(1 - -i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)^N |i\rangle \tag{42}$$

However now, instead of inserting a complete set of states at each intermediate time t_j (with $j = 1, \ldots, N$), we will insert an over-complete set $\{|z_j\rangle\}$ at each time t_j through the insertion of the resolution of the identity,

$$\begin{aligned}
&\langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)^N |i\rangle = \\
&= \int \left(\prod_{j=1}^N \frac{dz_j d\bar{z}_j}{2\pi i}\right) e^{-\sum_{j=1}^N |z_j|^2} \langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|z_N\rangle \\
&\times \langle z_N| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|z_{N-1}\rangle \ldots \langle z_1| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|i\rangle = \\
&\equiv \int \left(\prod_{j=1}^N \frac{dz_j d\bar{z}_j}{2\pi i}\right) e^{-\sum_{j=1}^N |z_j|^2} \left[\prod_{k=1}^{N-1} \langle z_{k+1}| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|z_k\rangle\right] \\
&\times \langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|z_N\rangle \langle z_1| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|z_i\rangle
\end{aligned} \tag{43}$$

In the limit $\epsilon \to 0$ these matrix elements are

$$\begin{aligned}
\langle z_{k+1}| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\right)|z_k\rangle =& \langle z_{k+1}|z_k\rangle - i\frac{\epsilon}{\hbar}\langle z_{k+1}|\hat{H}(\hat{a}^\dagger, \hat{a})|z_k\rangle \\
=& \langle z_{k+1}|z_k\rangle \left[1 - i\frac{\epsilon}{\hbar}H(\bar{z}_{k+1}, z_k)\right]
\end{aligned} \tag{44}$$

where $H(\bar{z}_{k+1}, z_k)$ is a *function* which is obtained from the *normal ordered* Hamiltonian by the substitutions $\hat{a}^\dagger \to \bar{z}_{k+1}$ and $\hat{a} \to z_k$. Hence, we can write the following expression for the matrix element

$$
\begin{aligned}
&\langle f| e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})} |i\rangle = \\
&= \lim_{\epsilon\to 0, N\to\infty} \int \left(\prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i} \right) e^{-\sum_{j=1}^{N} |z_j|^2} e^{\sum_{j=1}^{N-1} \bar{z}_{j+1} z_j} \prod_{j=1}^{N-1} \left[1 - i\frac{\epsilon}{\hbar} H(\bar{z}_{k+1}, z_k) \right] \\
&\times \langle f|z_N\rangle\langle z_1|i\rangle \left[1 - i\frac{\epsilon}{\hbar} \frac{\langle f|\hat{H}|z_N\rangle}{\langle f|z_N\rangle} \right] \left[1 - i\frac{\epsilon}{\hbar} \frac{\langle z_1|\hat{H}|i\rangle}{\langle z_1|i\rangle} \right]
\end{aligned}
\tag{45}
$$

By further expanding the initial and final states in coherent states

$$
\begin{aligned}
\langle f| =& \int \frac{dz_f d\bar{z}_f}{2\pi i} e^{-|z_f|^2} \bar{\psi}_f(z_f) \langle z_f| \\
|i\rangle =& \int \frac{dz_i d\bar{z}_i}{2\pi i} e^{-|z_i|^2} \psi_i(\bar{z}_i) |z_i\rangle
\end{aligned}
\tag{46}
$$

we find

$$
\begin{aligned}
&\langle f| e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})} |i\rangle = \\
&= \int \mathcal{D}z\mathcal{D}\bar{z}\; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{\hbar}{2i} \left(z\partial_t \bar{z} - \bar{z}\partial_t z \right) - H(z, \bar{z}) \right]} e^{\frac{1}{2}(|z_i|^2 + |z_f|^2)} \bar{\psi}_f(z_f)\psi_i(\bar{z}_i)
\end{aligned}
\tag{47}
$$

This is the coherent-state form of the path integral. We can identify in this expression the Lagrangian L as the quantity

$$
L = \frac{\hbar}{2i} \left(z\partial_t \bar{z} - \bar{z}\partial_t z \right) - H(z, \bar{z}) \tag{48}
$$

$$
= \frac{1}{2}(p\partial_t q - q\partial_t p) - H(q, p) \tag{49}
$$

Therefore the coherent state path integral is, in this case, equivalent, to the path integral over phase space.

Notice that the Lagrangian in the coherent-state representation is first order in time derivatives. because of this feature we are not guaranteed that the paths are necessarily differentiable. This property leads to all kinds of subtleties that for the most part we will ignore in what follows.

2 Path integral for spin.

We will now discuss the use of path integral methods to describe a quantum mechanical spin. Consider a quantum mechanical system which consists of a spin in the spin-S representation of the group $SU(2)$. The space of states of the spin-S representation is $2S+1$-dimensional, and it is spanned by the basis $\{|S,M\rangle\}$ which are the eigenstates of the operators $\vec{S}^2$ and S_3, *i.e.,*

$$\begin{aligned} \vec{S}^2 \, |S,M\rangle = & \quad S(S+1) \, |S,M\rangle \\ S_3 \, |S,M\rangle = & \quad M \, |S,M\rangle \end{aligned} \tag{50}$$

with $|M| \leq S$ (in integer-spaced intervals). This set of states is complete ad it forms a basis of this Hilbert space. The operators S_1, S_2 and S_3 obey the $SU(2)$ algebra,

$$[S_a, S_b] = i\epsilon_{abc} S_c \tag{51}$$

where $a, b, c = 1, 2, 3$.

The simplest physical problem involving spin is the coupling to an external magnetic field $\vec{B}$ through the Zeeman interaction

$$H_{\text{Zeeman}} = \mu \, \vec{B} \cdot \vec{S} \tag{52}$$

where μ is the Zeeman coupling constant (*i.e.,* the product of the Bohr magneton and the gyromagnetic factor).

Let us denote by $|0\rangle$ the *highest weight state* $|S,S\rangle$. Let us define the spin raising and lowering operators $S^{\pm}$,

$$S^{\pm} = S_1 \pm iS_2 \tag{53}$$

The highest weight state $|0\rangle$ is annihilated by S^+,

$$S^+|0\rangle = S^+|S,S\rangle = 0 \tag{54}$$

Clearly, we also have

$$\begin{aligned} \vec{S}^2|0\rangle = & \quad S(S+1)|0\rangle \\ S_3|0\rangle = & \quad S|0\rangle \end{aligned} \tag{55}$$

Let us consider now the state $|\vec{n}\rangle$,

$$|\vec{n}\rangle = e^{i\theta(\vec{n}_0 \times \vec{n} \cdot \vec{S}} \; |0\rangle \tag{56}$$

where $\vec{n}$ is a three-dimensional unit vector ($\vec{n}^2 = 1$), $\vec{n}_0$ is a unit vector pointing along the direction of the quantization axis (*i.e.,* the "North Pole" of the unit sphere) and θ is the *colatitude*, (see Fig. 2)

$$\vec{n} \cdot \vec{n}_0 = \cos\theta \tag{57}$$

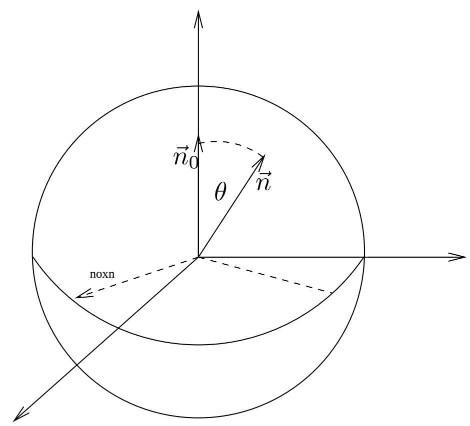

Figure 1:

As we will see the state $|\vec{n}\rangle$ is a coherent spin state which represents a spin polarized along the $\vec{n}$ axis. The state $|\vec{n}\rangle$ can be expanded in the basis $|S, M\rangle$,

$$|\vec{n}\rangle = \sum_{M=-S}^{S} D_{MS}^{(S)}(\vec{n})\; |S, M\rangle \tag{58}$$

Here $D_{MS}^{(S)}(\vec{n})$ are the representation matrices in the spin-S representation.

It is important to note that there are many rotations that lead to the same state $|\vec{n}\rangle$ from the highest weight $|0\rangle$. For example any rotation along the direction $\vec{n}$ results only in a change in the phase of the state $|\vec{n}\rangle$. These rotations are equivalent to a multiplication on the right by a rotation about the z axis. However, in Quantum Mechanics this phase has no physically observable consequence. Hence we will regard all of these states as being physically equivalent. In other terms, the states for *equivalence classes* (or *rays*) and we must pick one and only one state from each class. These rotations are generated by S_3, the (only) diagonal generator of $SU(2)$. Hence, the physical states are not in one-to-one correspondence with the elements of $SU(2)$ but instead with the elements of the right *coset* $SU(2)/U(1)$, with the $U(1)$ generated by S_3. (In the case of a more general group we must divide out the Maximal Torus generated by all the diagonal generators of the group.) In mathematical language, if we consider all the rotations at once , the spin coherent states are said to form a Hermitian line bundle.

A consequence of these observations is that the D matrices do not form a group under matrix multiplication. Instead they satisfy

$$D^{(S)}(\vec{n}_1)D^{(S)}(\vec{n}_2) = D^{(S)}(\vec{n}_3)\; e^{i\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_3) S_3} \tag{59}$$

where the phase factor is usually called a *cocycle*. Here $\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ is the (oriented) area of the spherical triangle with vertices at $\vec{n}_1, \vec{n}_2, \vec{n}_3$. However, since the sphere is a closed surface, which area do we actually mean? "Inside" or "outsider"? Thus, the phase factor is ambiguous by an amount determined by 4π, the total area of the sphere,

$$e^{i4\pi M} \tag{60}$$

However, since M is either an integer or a half-integer this ambiguity in Φ has no consequence whatsoever,

$$e^{i4\pi M} = 1 \tag{61}$$

(we can also regard this result as a requirement that M be quantized).

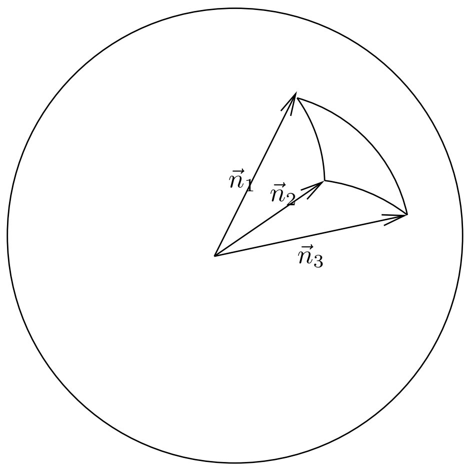

Figure 2:

The states $|\vec{n}\rangle$ are coherent states which satisfy the following properties (see Perelomov's book *Coherent States*). The overlap of two coherent states $|\vec{n}_1\rangle$ and $|\vec{n}_2\rangle$ is

$$\begin{aligned} \langle\vec{n}_1|\vec{n}_2\rangle &= \langle 0|D^{(S)}(\vec{n}_1)^\dagger D^{(S)}(\vec{n}_2)|0\rangle \\ &= \langle 0|D^{(S)}(\vec{n}_0)e^{i\Phi(\vec{n}_1,\vec{n}_2,\vec{n}_0)S_3}|0\rangle \\ &= \left(\frac{1+\vec{n}_1\cdot\vec{n}_2}{2}\right)^S e^{i\Phi(\vec{n}_1,\vec{n}_2,\vec{n}_0)S} \end{aligned} \tag{62}$$

The (diagonal) matrix element of the spin operator is

$$\langle\vec{n}|\vec{S}|\vec{n}\rangle = S\,\vec{n} \tag{63}$$

Finally, the (over-complete) set of coherent states $\{|\vec{n}\rangle\}$ have a resolution of the identity of the form

$$\hat{I} = \int d\mu(\vec{n})\;|\vec{n}\rangle\langle\vec{n}| \tag{64}$$

where the integration measure $d\mu(\vec{n})$ is

$$d\mu(\vec{n}) = \left(\frac{2S+1}{4\pi}\right)\delta(\vec{n}^2-1)d^3n \tag{65}$$

Let us now use the coherent states $\{|\vec{n}\rangle\}$ to find the path integral for a spin. In imaginary time τ (and with periodic boundary conditions) the path integral is simply the partition function

$$Z = \text{tr}e^{-\beta H} \tag{66}$$

where $\beta = 1/T$ (T is the temperature) and H is the Hamiltonian. As usual the path integral form of the partition function is found by splitting up the imaginary time interval $0 \le \tau \le \beta$ in N_τ steps each of length $\delta\tau$ such that $N_\tau \delta\tau = \beta$. Hence we have

$$Z = \lim_{N_\tau\to\infty,\delta\tau\to 0} \text{tr}\left(e^{-\delta\tau H}\right)^{N_\tau} \tag{67}$$

and insert the resolution of the identity at every intermediate time step,

$$\begin{aligned} Z &= \lim_{N_\tau\to\infty,\delta\tau\to 0}\left(\prod_{j=1}^{N_\tau}\int d\mu(\vec{n}_j)\right)\left(\prod_{j=1}^{N_\tau}\langle\vec{n}(\tau_j)|e^{-\delta\tau H}|\vec{n}(\tau_{j+1})\rangle\right) \\ &\simeq \lim_{N_\tau\to\infty,\delta\tau\to 0}\left(\prod_{j=1}^{N_\tau}\int d\mu(\vec{n}_j)\right)\left(\prod_{j=1}^{N_\tau}\left[\langle\vec{n}(\tau_j)|\vec{n}(\tau_{j+1})\rangle - \delta\tau\langle\vec{n}(\tau_j)|H|\vec{n}(\tau_{j+1})\rangle\right]\right) \end{aligned} \tag{68}$$

However, since

$$\frac{\langle\vec{n}(\tau_j)|H|\vec{n}(\tau_{j+1})\rangle}{\langle\vec{n}(\tau_j)|\vec{n}(\tau_{j+1})\rangle} \simeq \langle\vec{n}(\tau_j)|H|\vec{n}(\tau_j)\rangle = \mu S\vec{B}\cdot\vec{n}(\tau_j) \tag{69}$$

and

$$\langle\vec{n}(\tau_j)|\vec{n}(\tau_{j+1})\rangle = \left(\frac{1+\vec{n}(\tau_j)\cdot\vec{n}(\tau_{j+1})}{2}\right)^S e^{i\Phi(\vec{n}(\tau_j),\vec{n}(\tau_{j+1}),\vec{n}_0)S} \tag{70}$$

we can write the partition function in the form

$$Z = \lim_{N_\tau\to\infty,\delta\tau\to 0}\int \mathcal{D}\vec{n}\; e^{-S_E[\vec{n}]} \tag{71}$$

where $S_E[\vec{n}]$ is given by

$$\begin{aligned} -S_E[\vec{n}] &= iS\sum_{j=1}^{N_\tau}\Phi(\vec{n}(\tau_j),\vec{n}(\tau_{j+1}),\vec{n}_0) \\ &+S\sum_{j=1}^{N_\tau}\ln\left(\frac{1+\vec{n}(\tau_j)\cdot\vec{n}(\tau_{j+1})}{2}\right) - \sum_{j=1}^{N_\tau}(\delta\tau)\mu S\;\vec{n}(\tau_j)\cdot\vec{B} \end{aligned} \tag{72}$$

The first term of the r. h. s. of Eq. 82 contains the expression $\Phi(\vec{n}(\tau_j), \vec{n}(\tau_{j+1}), \vec{n}_0)$ which has a simple geometric interpretation: it is the sum of the areas of the N_τ contiguous spherical triangles. These triangles have the pole $\vec{n}_0$ as a common vertex, and their other pairs of vertices trace a spherical polygon with vertices at $\{\vec{n}(\tau_j)\}$. In the time continuum limit this spherical polygon becomes the *history* of the spin, which traces a closed oriented curve $\Gamma = \{\vec{n}(\tau)\}$ (with $0 \leq \tau \leq \beta$). Let us denote by Ω^+ the region of the sphere whose boundary is Γ and which contains the pole $\vec{n}_0$. The complement of this region is Ω^- and it contains the opposite pole $-\vec{n}_0$. Hence we find that

$$\lim_{N_\tau \to \infty, \delta\tau \to 0} \Phi(\vec{n}(\tau_j), \vec{n}(\tau_{j+1}), \vec{n}_0) = \mathcal{A}[\Omega^+] = 4\pi - \mathcal{A}[\Omega^-] \tag{73}$$

where $\mathcal{A}[\Omega]$ is the area of the region Ω. Once again, the ambiguity of the area leads to the requirement that S should be an integer or a half-integer.

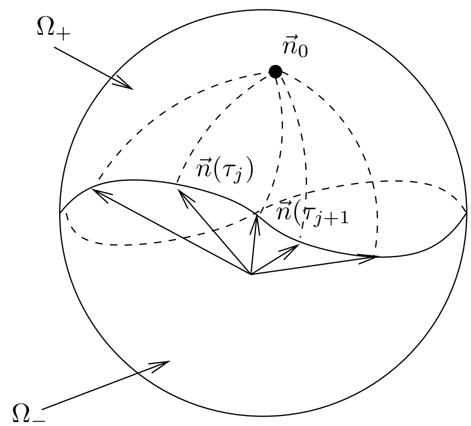

Figure 3:

There is a simple an elegant way to write the area enclosed by Γ. Let $\vec{n}(\tau)$ be a history and Γ be the set of points o the 2-sphere traced by $\vec{n}(\tau)$ for $0 \leq \tau \leq \beta$. Let us define $\vec{n}(\tau, s)$ (with $0 \leq s \leq 1$) to be an *arbitrary* extension of $\vec{n}(\tau)$ from the curve Γ to the interior of the upper cap Ω^+, such that

$$\begin{aligned} \vec{n}(\tau, 0) &= \vec{n}(\tau) \\ \vec{n}(\tau, 1) &= \vec{n}_0 \\ \vec{n}(\tau, 0) &= \vec{n}(\tau + \beta, 0) \end{aligned} \tag{74}$$

Then the area can be written in the compact form

$$\mathcal{A}[\Omega^+] = \int_0^1 ds \int_0^\beta d\tau \; \vec{n}(\tau, s) \cdot \partial_\tau \vec{n}(\tau, s) \times \partial_s \vec{n}(\tau, s) \equiv S_{\rm WZ}[\vec{n}] \tag{75}$$

In Mathematics this expression for the area is called the (simplectic) 2-form, and in the Physics literature is usually called a Wess-Zumino action, $S_{\rm WZ}$, or *Berry's Phase*.

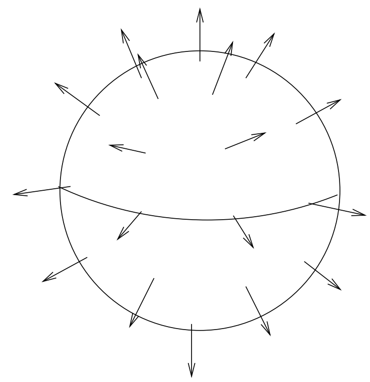

Figure 4: A hairy ball or monopole

Thus, in the (formal) time continuum limit, the action S_E becomes

$$\mathcal{S}_E = -iS\;\mathcal{S}_{\rm WZ}[\vec n] + \frac{S\delta\tau}{2}\int_0^\beta d\tau\,\left(\partial_\tau \vec n(\tau)\right)^2 + \int_0^\beta d\tau\;\mu S\;\vec B\cdot \vec n(\tau) \tag{76}$$

Notice that we have kept (temporarily) a term of order $\delta\tau$, which we will drop shortly.

How do we interpret Eq. 76 ? Since $\vec n(\tau)$ is constrained to be a point on the surface of the unit sphere, *i.e.,* $\vec n^2 = 1$, the action $S_E[\vec n]$ can be interpreted as the action of a particle of mass $M = S\delta\tau \to 0$ and $\vec n(\tau)$ is the position vector of the particle at (imaginary) time τ. Thus, the second term is a (vanishingly small) kinetic energy term, and the last term of Eq. (76) is a potential energy term.

What is the meaning of the first term? In Eq. (75) we saw that $\mathcal{S}_{\rm WZ}[\vec n]$, the the so-called Wess-Zumino or Berry phase term in the action, is the *area* of the (positively oriented) region $\mathcal{A}[\Omega_+]$ "enclosed" by the "path" $\vec n(\tau)$. In fact,

$$\mathcal{S}_{\rm WZ}[\vec n] = \int_0^1 ds \int_0^\beta d\tau \vec n\cdot \partial_\tau \vec n \times \partial_s \vec n \tag{77}$$

is the area of the oriented surface Ω^+ whose boundary is the oriented path $\Gamma = \partial\Omega^+$ (see Fig. 3). Using Stokes Theorem we can write the the expression $S\mathcal{A}[\vec n]$ as the circulation of a vector field $\vec A[\vec n]$,

$$\oint_{\partial\Omega} d\vec n\cdot \vec A[\vec n(\tau)] = \iint_{\Omega^+} d\vec S\cdot \vec\nabla_{\vec n}\times \vec A[\vec n(\tau)] \tag{78}$$

provided the "magnetic field" $\vec{\nabla}_{\vec{n}} \times \vec{A}$ is "constant", namely

$$\vec{B} = \vec{\nabla}_{\vec{n}} \times \vec{A}[\vec{(}\tau)] = S\, \vec{n}(\tau) \tag{79}$$

What is the total flux Φ of this magnetic field?

$$\begin{aligned} \Phi &= \int_{\text{sphere}} d\vec{S} \cdot \vec{\nabla}_{\vec{n}} \times \vec{A}[\vec{n}] \\ &= S \int d\vec{S} \cdot \vec{n} \equiv 4\pi S \end{aligned} \tag{80}$$

Thus, the total number of flux quanta N_ϕ piercing the unit sphere is

$$N_\phi = \frac{\Phi}{2\pi} = 2S = \text{magnetic charge} \tag{81}$$

We reach the condition that the magnetic charge is *quantized*, a result known as the *Dirac quantization condition*.

Is this result consistent with what we know about charged particles in magnetic fields? In particular, how is this result related to the physics of spin? To answer these questions we will go back to real time and write the action

$$\mathcal{S}[\vec{n}] = \int_0^T dt \left[\frac{M}{2} \left(\frac{d\vec{n}}{dt} \right)^2 + \vec{A}[\vec{n}(t)] \cdot \frac{d\vec{n}}{dt} - \mu S \vec{n}(t) \cdot \vec{B} \right] \tag{82}$$

with the constraint $\vec{n}^{\,2} = 1$ and where the limit $M \to 0$ is implied.

The classical hamiltonian associated to the action of Eq. (82) is

$$H = \frac{1}{2M} \left[\vec{n} \times \left(\vec{p} - \vec{A}[\vec{n}] \right) \right]^2 + \mu S \vec{n} \cdot \vec{B} \equiv H_0 + \mu S \vec{n} \cdot \vec{B} \tag{83}$$

It is easy to check that the vector $\vec{\Lambda}$,

$$\vec{\Lambda} = \vec{n} \times \left(\vec{p} - \vec{A} \right) \tag{84}$$

satisfies the algebra

$$[\Lambda_a, \Lambda_b] = i\hbar \epsilon_{abc} \left(\Lambda_c - \hbar S n_c \right) \tag{85}$$

where $a, b, c = 1, 2, 3$, ϵ_{abc} is the (third rank) Levi-Civita tensor, and with

$$\vec{\Lambda} \cdot \vec{n} = \vec{n} \cdot \vec{\Lambda} = 0 \tag{86}$$

the generators of rotations for this system are

$$\vec{L} = \vec{\Lambda} + \hbar S \vec{n} \tag{87}$$

The operators $\vec{L}$ and $\vec{\Lambda}$ satisfy the (joint) algebra

$$\begin{aligned} &[L_a, L_b] = -i\hbar \epsilon_{abc} L_s \qquad \left[L_a, \vec{L}^2 \right] = 0 \\ &[L_a, n_b] = i\hbar \epsilon_{abc} n_c \qquad [L_a, \Lambda_b] = i\hbar \epsilon_{abc} \Lambda_c \end{aligned} \tag{88}$$

Hence

$$\left[L_a, \vec{\Lambda}^2\right] = 0 \;\Rightarrow\; [L_a, H] = 0 \tag{89}$$

since the operators L_a satisfy the angular momentum algebra, we can diagonalize $\vec{L}^2$ and L_3 simultaneously. Let $|m, \ell\rangle$ be the simultaneous eigenstates of $\vec{L}^2$ and L_3,

$$\vec{L}^2|m,\ell\rangle = \hbar^2 \ell(\ell+1)|m,\ell\rangle \tag{90}$$

$$L_3|m,\ell\rangle = \hbar m|m,\ell\rangle \tag{91}$$

$$H_0|m,\ell\rangle = \frac{\hbar^2}{2MR^2}\left(\frac{\ell(\ell+1)-S}{2S}\right)|m,\ell\rangle \tag{92}$$

where $R = 1$ is the radius of the sphere. The eigenvalues ℓ are of the form $\ell = S + n$, $|m| \leq \ell$, with $n \in \mathbb{Z}^+ \cup \{0\}$ and $2S \in \mathbb{Z}^+ \cup \{0\}$. Hence each level is $2\ell+1$-fold degenerate, or what is equivalent, $2n+1+2S$-fold degenerate. Then, we get

$$\vec{\Lambda}^{\,2} = \vec{L}^{\,2} - \vec{n}^{\,2}\hbar^2 S^2 = \vec{L}^{\,2} - \hbar^2 S^2 \tag{93}$$

Since $M = S\delta t \to 0$, the *lowest* energy in the spectrum of H_0 are those with the *smallest* value of ℓ, *i. e.* states with $n = 0$ and $\ell = S$. The degeneracy of this "Landau" level is $2S + 1$, and the gap to the next excited states diverges as $M \to 0$. Thus, in the $M \to 0$ limit, the *lowest* energy states have the same degeneracy as the spin-S representation. Moreover, the operators $\vec{L}^{\,2}$ and L_3 become the corresponding spin operators. thus, the equivalency found is indeed correct.

Thus, we have shown that the quantum states of a scalar (non-relativistic) particle bound to a magnetic monopole of magnetic charge $2S$, obeying the Dirac quantization condition, are identical to those of those of a spinning particle!

We close this section with some observations on the semi-classical motion. From the (real time) action (already in the $M \to 0$ limit)

$$\mathcal{S} = -\int_0^T dt\; \mu S\; \vec{n}\cdot\vec{B} + S\int_0^T dt \int_0^1 ds\; \vec{n}\cdot\partial_t\vec{n}\times\partial_s\vec{n} \tag{94}$$

we can derive a Classical Equation of Motion by looking at the stationary configurations. The variation of the second term in Eq. (94) is

$$\delta\mathcal{S} = S\;\delta\int_0^T dt\int_0^1 ds\; \vec{n}\cdot\partial_t\vec{n}\times\partial_s\vec{n} = S\int_0^T dt\delta\vec{n}(t)\cdot\vec{n}(t)\times\partial_t\vec{n}(t) \tag{95}$$

the variation of the first term in Eq. (94) is

$$\delta\int_0^T dt\;\mu S\vec{n}(t)\cdot\vec{B} = \int_0^T dt\;\delta n(t)\cdot\mu S\vec{B} \tag{96}$$

Hence,

$$\delta\mathcal{S} = \int_0^T dt\;\delta\vec{n}(t)\cdot\left(-\mu S\vec{B} + S\vec{n}(t)\times\partial_t\vec{n}(t)\right) \tag{97}$$

which implies that the classical trajectories must satisfy the equation of motion

$$\mu\vec{B} = \vec{n} \times \partial_t \vec{n} \tag{98}$$

If we now use the vector identity

$$\vec{n} \times \vec{n} \times \partial_t \vec{n} = (\vec{n} \cdot \partial_t \vec{n})\,\vec{n} - \vec{n}^{\,2} \partial_t \vec{n} \tag{99}$$

and

$$\vec{n} \cdot \partial_t \vec{n} = 0, \qquad \text{and} \qquad \vec{n}^{\,2} = 1 \tag{100}$$

we get the classical equation of motion

$$\partial_t \vec{n} = \mu\vec{B} \times \vec{n} \tag{101}$$

Therefore, the classical motion is *precessional* with an angular velocity $\vec{\Omega}_{\rm pr} = \mu\vec{B}$.