Identical Particles
(for the most part)

Until now we have discussed the QM of single particle systems. We will now look at more realistic situation of multi-particle states.

Consider a system of $N=2$ particles (spin included).

Suppose for the moment that we know the basis for each particle in isolation. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert space for each particle. The combined two-particle system lives in (a subspace of) the direct (or tensor) product $\mathcal{H}_1 \otimes \mathcal{H}_2$. If the two particles are identical, these two Hilbert spaces are actually the same space, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. A set of basis states of this two-particle system of identical particles is $\{ |\alpha_1, \alpha_2 \rangle \}$ where $|\alpha_i \rangle$ is a separate set of states in $\mathcal{H}$. Here, $|\alpha_i \rangle$ are the states for particle 1 and 2 respectively.

Q: Is the state $|\alpha_1, \alpha_2 \rangle$ different in any essential way from $|\alpha_1, \alpha_2 \rangle$?

A: No. The reason is that all possible operators (i.e., observables) are symmetric under the exchange
of the labels, e.g., the Hamiltonian is

\[ H = H_1 + H_2 + H_{12} \]

and it is invariant under the permutation or exchange (i.e., two particle exchanges) \((1 \leftrightarrow 2)\) of the particle labels, i.e.,

\[ \langle \alpha_1, \alpha_2 | H | \beta_1, \beta_2 \rangle = \langle \alpha_2, \alpha_1 | P_{12}^+ H P_{12} | \beta_2, \beta_1 \rangle \]

where \( P_{12} | \beta \beta_1 \rangle = | \beta_1 \beta \rangle \)

\[ P_{12} | \alpha_2 \alpha_1 \rangle = | \alpha_1 \alpha_2 \rangle \]

Note: \(( P_{12}^+ = P_{12}^{-1} \text{ see below})\)

\[ \implies P_{12}^{-1} H P_{12} = H \quad \text{(by symmetry)} \implies \]

\[ \langle \alpha_1, \alpha_2 | H | \beta_1, \beta_2 \rangle = \langle \alpha_2, \alpha_1 | H | \beta_2, \beta_1 \rangle \]

The same applies to all observables.

Since \( P_{12}^{-1} H P_{12} = H \implies H P_{12} = P_{12} H \implies [H, P] = 0 \)

\[ \implies \text{we can choose the eigenstates of } H \text{ to be also eigenstates of } P. \]

But \( P_{12} | \alpha_1 \alpha_2 \rangle = | \alpha_2 \alpha_1 \rangle \quad \text{and} \quad P_{12}^2 | \alpha_1 \alpha_2 \rangle = P_{12} | \alpha_1 \alpha_2 \rangle = | \alpha_1 \alpha_2 \rangle \)

\[ \implies P_{12}^2 = I \implies P_{12}^+ = P_{12}^{-1} \]

If \( P_{12} | \alpha_1 \alpha_2 \rangle = \lambda | \alpha_1 \alpha_2 \rangle \implies P_{12}^2 | \alpha_1 \alpha_2 \rangle = | \alpha_1 \alpha_2 \rangle \)

(i.e., \( | \alpha_1 \alpha_2 \rangle \) is an eigenvector)

\[ \implies \lambda^2 = 1 \implies \lambda = \pm 1 \]

(Note: this rule may be violated in two dimensions).
The eigenstates of $P_{12}$ are either symmetric or antisymmetric under exchange. (Or pairwise permutation).

Postulate: Any state of a set of $N$ identical particles should be physically equivalent to a state obtained by (any) number of pairwise permutations.

The states of a general QM system are either symmetric or antisymmetric under pairwise permutations. Particles whose states are symmetric under exchange are called bosons and particles whose states are antisymmetric under exchange are called fermions. Note: Antisymmetry $\Rightarrow$ Pauli Principle.

There is a fundamental principle of QM (which actually is inherent to relativistic Quantum Field Theory) called the Spin - Statistics Theorem which states that particles with half-integer spin are fermions, integer spin are bosons.

In reality this means (though we have two types of particles, the term statistics derives from the origin in counting states in statistical physics).
Fermions: electrons, protons, neutrons, quarks, He, nuclei...

Bosons: photons, gluons, pions, phonons, spin-waves, etc.

**N-particle States**

From the above considerations, it follows that we need to construct N-particle states in such a way that they are either symmetric or antisymmetric under pair-wise permutations or exchanges. If we have a group of N objects, we can define a permutation by constructing an algorithm that generates any sequence of N objects from a standard one (say 1, 2, 3, ..., N).

Thus, a permutation of N elements is denoted by $(1, 2, 3, ..., N)$, which tells us what is the label assigned to each object. It is easy to show that all permutations can be constructed by a sequence of pair-wise permutations.

Permutations form a group: $S_N$
Thus, the states can be arranged (or classified) according to their transformation properties under permutations. As we are constructing representations of $S_N$, In practice, this is achieved by noting that the $N$ objects can be partitioned into several sets with $N_1, \ldots, N_k$ elements each, \( N = N_1 + N_2 + \cdots + N_k \) (for some $k$).

We consider two operations: total symmetrization and total antisymmetrization. But there are more steps.

Suppose we pick $N=22$. Two partitions are

\[ 6 + 4 + 4 + 3 + 3 + 1 + 1 \quad \text{and} \quad 7 + 5 + 5 + 3 + 1 + 1 \]

\[ \text{Young Tableaus} \]

We now pick the state and symmetrize it w.r.t. the variables in each row separately. Then we antisymmetrize w.r.t. variables in different rows (i.e., along each column). Then we further antisymmetrize what's left after fixing the variables in the first column and continue until all are done.
We now pick the state, i.e., its wave function, and symmetrize it w.r.t. the variables in each row separately. Next we antisymmetrize w.r.t. the variables on the first column. We continue by antisymmetrizing w.r.t. the variables on the second column and so on until we are done.

The states produced by this method are neither symmetric nor antisymmetric. Also the wave functions thus generated are not all linearly independent, but there are methods to construct an appropriate set of basis states (see Messiah).

3) Each Young diagram determines a symmetry of a type of wave function w.r.t. permutations.

4) To each energy level we associate a Young diagram which determines the symmetry of the state under permutations. This is possible since 
\[ [\hat{P}_{ij}, \hat{H}] = 0 \] (however, \([\hat{P}_{ij}, \hat{D}_{kl}] \neq 0 \) in general)
Notice that

\[ \Psi_{\text{sym}} \quad \text{is fully symmetric} \]

\[ N = 22 \]

and

\[ \Psi_{\text{antisym}} \quad \text{is fully antisymmetric} \]

The main point is that the representations of \( S_N \) are in one-to-one correspondence with the Young Tableaux. Imagine now that we are looking at a problem with several particles with spin in a system without spin-orbit interaction. So \( \mathbf{H} \) does not depend explicitly on \( \mathbf{r} \). We can then write the wave function in a factorized form

\[ \Psi(r_1, \sigma_1, \ldots, r_N, \sigma_N) = \Psi_0(r_1, \ldots, r_N) X_s(\sigma_1, \ldots, \sigma_N) \]

If the particles are fermions (e.g., electrons)

\[ P_{ij} \Psi(r_1, \sigma_1, \ldots, r_i, \sigma_i, \ldots, r_j, \sigma_j, \ldots) = -\Psi(r_1, \sigma_1, \ldots, r_i, \sigma_i, \ldots, r_j, \sigma_j, \ldots) \]

\[ \Rightarrow \] either \( \Psi_0(r_1, \ldots, r_N) \) is symmetric and \( X_s(\sigma_1, \ldots, \sigma_N) \) is antisymmetric

or \( \Psi_0 \) is antisymmetric and \( X_s \) is symmetric.
For the orbital wave function, quantization reduces to wave since the center of mass factorizes (and it is symmetric) and the relative coordinate
$
\mathbf{r}_{ij} \to + \mathbf{r}_{ji} = -\mathbf{r}_{ij}
$
under permutation.

Example: two fermions interacting through a pair potential:

\[
\psi_0 (r_1, r_2) = \Psi_{\text{CM}} (R_{\text{CM}}) \Psi (\mathbf{R})
\]

\[
\Psi_{\text{even}} (-\mathbf{R}) = (-1)^I \Psi_{\text{even}} (\mathbf{R})
\]

The spin wave function have either $S=0$ or $S=1$

Since $X_{\text{singlet}}$ is antisymmetric

$X_{\text{triplet}}$ is symmetric

Thus the allowed states have

(a) even \( \psi_0 \) symmetric and \( S=0 \) \( X_s \) odd

(b) odd \( \psi_0 \) odd and \( S=1 \) \( X_s \) even

Not all states are allowed.
Another example: \( N = 4 \) particles

\[
\begin{align*}
\bar{\psi} \\
\psi
\end{align*}
\]

\[
\begin{array}{cccc}
\chi_0 & \bar{\chi}_0 & \chi_1 & \bar{\chi}_1 \\
\chi_2 & \bar{\chi}_2 & \chi_3 & \bar{\chi}_3
\end{array}
\]

\[
\begin{array}{cccc}
S = 2 & S = 1 & S = 0
\end{array}
\]

Exchange Interactions and Spin

Consider first a problem with two quasi-\( \frac{1}{2} \) \( \bar{S}_1 \) and \( \bar{S}_2 \).

The operator

\[
P_{12} = \frac{1}{12} \left( 1 + \frac{6}{\hbar^2} \bar{S}_1 \cdot \bar{S}_2 \right)
\]

exchanges the quas.

Indeed, since \( \bar{S} = \bar{S}_1 + \bar{S}_2 \) and \( 2 \bar{S}_1 \bar{S}_2 = \bar{S}^2 - \bar{S}_1^2 - \bar{S}_2^2 \)

\[
\Rightarrow \frac{6}{\hbar^2} \bar{S}_1 \bar{S}_2 = \frac{2}{\hbar^2} (\bar{S}^2 - \frac{3}{2})
\]

acting on states \( |S, M\rangle \)

\[
\Rightarrow P_{12} = \frac{1}{2} + S(S+1) - \frac{3}{2} = S(S+1) - 1
\]

\[
\Rightarrow P_{12} |10, 0\rangle = -1 \quad |10, 0\rangle \quad \text{antisymmetric}
\]

\[
\Rightarrow P_{12} |11, 1\rangle = +1 \quad |11, 1\rangle \quad \text{symmetric}
\]

\[
\Rightarrow P_{12} |\uparrow \downarrow \rangle = 1 \quad |\uparrow \downarrow \rangle \quad P_{12} |\downarrow \uparrow \rangle = 1 \quad |\downarrow \uparrow \rangle
\]

\[
\Rightarrow P_{12} |\uparrow \uparrow \rangle = 1 \quad |\uparrow \uparrow \rangle \quad P_{12} |\downarrow \downarrow \rangle = 1 \quad |\downarrow \downarrow \rangle
\]
Consider now the Hydrogen molecule. Let \( \vec{R}_a \) and \( \vec{R}_b \) be the (fixed) nuclei (protons) with \( |\vec{R}_{ab}| > 2a_0 \).

Let \( H \) be the interaction Hamiltonian (Heitler and London):

\[
H = \left( \frac{p_{1}^{2}}{2m} - \frac{e^{2}}{r_{1a}} + \frac{p_{2}^{2}}{2m} - \frac{e^{2}}{r_{2b}} \right) + \left( \frac{e^{2}}{\vec{R}_{ab}} + \frac{e^{2}}{r_{12}} - \frac{e^{2}}{r_{1b}} - \frac{e^{2}}{r_{2a}} \right)
\]

Let \( \psi_{I}(\vec{r}_1, \vec{r}_2) \) be the wave function of the electrons, and I will choose at first:

\[
\psi_{I}(r_1, r_2) = \psi_a(r_1) \psi_b(r_2)
\]

where

\[
\left( \frac{p_{1}^{2}}{2m} - \frac{e^{2}}{r_{1a}} \right) \psi_a(r_1) = E_0 \psi_a(r_1)
\]

\[
\left( \frac{p_{2}^{2}}{2m} - \frac{e^{2}}{r_{2b}} \right) \psi_b(r_2) = E_0 \psi_b(r_2)
\]

i.e. each electron in the same state of its hydrogen atom. (\( \Rightarrow \) even though the energies are the same, the wave functions are different, these are \( \neq \) states!)
But
\[ \psi_{\Pi}(r_1, r_2) = \psi_a(r_2) \psi_b(r_1) = \psi^*_{\Pi}(r_2, r_1) \]
is just as good a wave function.

Let us construct the wave functions
\[ \psi_s(r_1, r_2) = \frac{1}{\sqrt{2}} \left( \psi_\Pi(r_1, r_2) + \psi_{\Pi}(r_2, r_1) \right) = \frac{1}{\sqrt{2}} \left( \psi_a(r_1) \psi_b(r_2) + \psi_b(r_1) \psi_a(r_2) \right) \]
\[ \psi_t(r_1, r_2) = \frac{1}{\sqrt{2}} \left( \psi_a(r_1) \psi_b(r_2) - \psi_b(r_1) \psi_a(r_2) \right) \]

We will diagonalize \( H \) in this subspace.

First we define
\[ 1 = \int d^3r |\psi_a(r)|^2 = \int d^3r |\psi_b(r)|^2 \]
(overlapping)
\[ l = \int d^3r \psi_a^*(r) \psi_b(r) \]
(overlap) \( l \in \mathbb{R} \)

\[ V = \int d^3r_1 \int d^3r_2 \left| \psi_{\Pi} \right|^2 \left( \frac{e^2}{r_{ab}} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1b}} - \frac{e^2}{r_{2a}} \right) = \langle \Psi | H_{\text{intra}} | \Psi \rangle \]
\[ = \int d^3r_1 \int d^3r_2 \left| \psi_{\Pi} \right|^2 \left( \frac{e^2}{r_{ab}} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1a}} - \frac{e^2}{r_{2b}} \right) \]
(overlapping)
\[ U = \int d^3r_1 \int d^3r_2 \psi_{\Pi}^* \psi_{\Pi} \left( \frac{e^2}{r_{ab}} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1a}} - \frac{e^2}{r_{2b}} \right) = \langle \Psi | H_{\text{exchange}} | \Psi \rangle \]
(exchange) \( U \in \mathbb{C}^{\ast} \)

(I will also take \( U = U^{\ast} \))

The normalization of the wave functions are:
\[ \int d^3r_1 \int d^3r_2 |\psi_{s, a}|^2 = 1 \pm |l|^2 \]
\[ \int d^3r_1 \int d^3r_2 |\psi_{s, a}|^2 = 0 \]
\[
\tilde{\Psi}_{5,A} = \frac{1}{\sqrt{1 \pm |\ell|^2}} \left[ \frac{1}{\sqrt{2}} \right] \Psi_{\alpha}(r_1) \Psi_{\beta}(r_2) \pm \Psi_{\beta}(r_1) \Psi_{\alpha}(r_2)
\]

\[
\int d^3r_1 \int d^3r_2 \tilde{\Psi}_{5,A}^* H_0 \tilde{\Psi}_{5,A} = 2E_0
\]

\[
\int d^3r_1 \int d^3r_2 \tilde{\Psi}_{5,A}^* H_{\text{int}} \tilde{\Psi}_{5,A} = \frac{V \pm \frac{U + U^*}{2}}{1 \pm |\ell|^2}
\]

\[
\int d^3r_1 \int d^3r_2 \tilde{\Psi}_{5,A}^* H_{\text{int}} \tilde{\Psi}_{A,5} = \pm \frac{1}{2} \frac{(U^* - U)}{\sqrt{1 - |\ell|^2}}
\]

and we have the effective 2x2 matrix

\[
H_{\text{eff}} = \begin{bmatrix}
2E_0 + \frac{V + U}{1 + |\ell|^2} & 0 \\
0 & 2E_0 + \frac{V - U}{1 - |\ell|^2}
\end{bmatrix}
\]

\(\Rightarrow\) The new eigenvalues are

\[E_{\pm} = 2E_0 \pm \frac{V \pm U}{1 \pm |\ell|^2}\]

with 

\[ (+) \leftrightarrow \tilde{\Psi}_{5,A} \]

\[ (-) \leftrightarrow \tilde{\Psi}_{5,A}^* \]

Note: If \( |\ell_{3ab}| \gg a_0 \Rightarrow E_+ - E_- \rightarrow 0 \text{ min } U \rightarrow 0 \text{ and } \ell \rightarrow 0.\]

According to our general principle, the actual wave function is a product of the orbital and spin wave functions.
Since the $e^-$ are fermions the total wave function must be antisymmetric under exchange.

$\psi_S(r_1, r_2) \chi_S=0 (\sigma_1 \sigma_2) \leftrightarrow E^+$

$\psi_A(r_1, r_2) \chi_S=1 (\sigma_1 \sigma_2) \leftrightarrow E^-$

Since $E_\pm = 2E_0 \pm \frac{V \pm U}{1 \pm \ell^2}$

$= 2E_0 \pm \frac{(V \pm U)(1 \pm \ell^2)}{1 - \ell^4}$

$\Rightarrow E_\pm = \left(2E_0 + U - \ell^2 \frac{V^2 + U^2 - U \ell^2}{1 - \ell^4}\right) \pm \frac{U - V \ell^2}{1 - \ell^4}$

and $\Theta$ corresponds to the state with $S = 0$

$\Theta \sim \uparrow \uparrow \sim \downarrow \downarrow \sim S = 1$.

The operator $P_{12} \equiv \frac{1}{2} \left(1 + \gamma \frac{S_1 \cdot S_2}{\hbar^2}\right)$ is diagonal in this basis.

$P_{12} \chi_{S=0} = - \chi_{S=0}$

$P_{12} \chi_{S=1} = + \chi_{S=1}$

$\Rightarrow H_{\text{eff}} = \left(2E_0 + \frac{V - U \ell^2}{1 - \ell^4}\right) I + \frac{U \ell^2 - U}{1 - \ell^4} P_{12}$

$H_{\text{eff}} = 2E_0 + \frac{1}{1 - \ell^4} \left(\frac{V^2 + U^2}{2} - \frac{U}{2} + \frac{U^2}{2} \ell^2\right) I - \frac{1}{2} \frac{S_1 \cdot S_2}{\hbar^2}$
When

$$\bar{J}_{12} = - \frac{2}{\hbar^2} \left( \frac{\sqrt{1-u}}{1-u^2} \right)$$

for the 1s state of Hydrogen (Heitler and London) found that \( \bar{J}_{12} < 0 \Rightarrow \) ground state is a \underline{spin bright}

H-L found that the triplet state has an energy greater than two unbound atoms \( \Rightarrow \) no triplet bound state ("autobinding")

\( \Rightarrow \) The bound state is a \underline{spin triplet}

[For a detailed discussion, see D. Mathis']
Scattering of Identical Particles

Let us turn now to the problem of scattering of two identical particles.

(A) Bosons

Consider two distinguishable bosons (e.g., $\alpha$ particles). The wave function for two $\alpha$-particles look like

$$\psi(\vec{r}_1, \vec{r}_2) = e^{i \frac{\vec{p} \cdot (\vec{r}_1 + \vec{r}_2)}{2}} \psi(\vec{r})$$

where $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ is the CM and $\vec{r}$ is the relative coordinate. Clearly,

$$\Psi_{\text{CM}}(\vec{R}) = e^{i \frac{\vec{P} \cdot \vec{R}}{2}}$$

is symmetric under exchange.

For $\psi(\vec{r})$ to be symmetric, it is necessary that

$$\psi(\vec{r}) = \psi(-\vec{r}) \implies l \text{ must be even.}$$

Neglecting symmetry, the scattered wave function are (asymptotically) of the form

$$e^{i \vec{k} \cdot \vec{r}} + f(\Theta) \frac{e^{ikr}}{r}$$

Then symmetrizing,

$$\psi(\vec{r}) = (e^{i \vec{k} \cdot \vec{r}} + e^{-i \vec{k} \cdot \vec{r}}) + (f(\Theta) + f(\pi - \Theta)) \frac{e^{ikr}}{r}$$

For under $\vec{r} \rightarrow -\vec{r}$, $\Theta \rightarrow \pi - \Theta$
$f(\theta)$ is the scattering amplitude. The fact that we get two terms simply means that the following processes have the same amplitude.

\[ \frac{d\sigma}{d\Omega} = \left| f(\theta) + f(\pi - \theta) \right|^2 \]

\[ = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 + 2 \text{Re}(f^*(\theta)f(\pi - \theta)) \]

\[ \uparrow \quad \uparrow \quad \uparrow \]

\[ \text{(a)} \quad \text{(b)} \quad \text{(c)} \]

For distinguishable particles, exchange contribution.

If \[ \theta = \frac{\pi}{2} \Rightarrow f(\theta) = f(\pi - \theta) = f(\frac{\pi}{2}) \]

\[ \Rightarrow \quad \left. \frac{d\sigma}{d\Omega} \right|_{\theta = \frac{\pi}{2}} = 4 \left| f\left(\frac{\pi}{2}\right) \right|^2 \quad \text{--- indistinguishable} \]

\[ \text{and} \quad \left. \frac{d\sigma}{d\Omega} \right|_{\theta = \frac{\pi}{2}} = 2 \left| f\left(\frac{\pi}{2}\right) \right|^2 \quad \text{--- distinguishable} \]
Partial waves:

\[ f(\theta) = \sum \left( \alpha \right) P_\ell (\cos \theta) \left( \ell \ell + 1 \right) f_\ell \]

\[ \Rightarrow f(\theta) + f(\pi - \theta) = 2 \sum \left( \alpha \right) P_\ell (\cos \theta) f_\ell \]

since \( P_\ell (\cos \theta) = P_\ell (-\cos \theta) = (-1)^\ell P_\ell (\cos \theta) \)

\[ \begin{align*}
\text{(5) Fermions} & \quad \text{now we have to account for spin. Take two electrons} \\
\ell = 1/2 & \Rightarrow \psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) = e^{i \frac{\vec{p} \cdot \vec{R}}{\hbar}} \psi(\vec{r}_1 - \vec{r}_2) \chi(\delta_1, \delta_2) \\
\vec{R} & = \frac{1}{2} (\vec{r}_1 + \vec{r}_2)
\end{align*} \]

Two cases: ① Spin singlet \( S = 0 \) \( \chi_{0,0}(s_1, s_2) \) is antisymmetric.

\[ \Rightarrow \psi(\vec{r}_1, \vec{r}_2) \text{ must be symmetric} \]

\[ \Rightarrow \psi(\vec{r}) = (e^{i \vec{k} \cdot \vec{r}} + e^{-i \vec{k} \cdot \vec{r}}) + (f(\theta) + f(\pi - \theta)) e^{i k r} \]

\[ \frac{d\sigma}{d\Omega} \text{ singlet} = \left| f(\theta) + f(\pi - \theta) \right|^2 \]

\( S = 0 \) \( \Rightarrow \) same predictions as for bosons.

② Spin triplet: \( S = 1 \) \( \Rightarrow \chi_{1,1}(s_1, s_2) \) is symmetric.

\[ \Rightarrow \psi(\vec{r}) \text{ must be antisymmetric} \]
\[ \rho(\theta) \sim (e^{i k r} - e^{-i k r}) + (f(\theta) - f(\pi - \theta)) e^{i k r} \]

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{triplet}} = \left| f(\theta) - f(\pi - \theta) \right|^2 
= \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 - 2 \Re (f(\theta) f(\pi - \theta)) \]

Now we notice that for \( \theta = \frac{\pi}{2} \)

\[ f(\theta) - f(\pi - \theta) = f(\frac{\pi}{2}) - f(\frac{3\pi}{2}) = 0 \]

We get no scattering of electrons at \( \theta = \frac{\pi}{2} \) of the electrons are in a \( S \geq 1 \) state!

For an unpolarized beam we get the average

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{unpol.}} = \frac{3}{4} \left( \frac{d\sigma}{d\Omega} \right)_{\text{triplet}} + \frac{1}{4} \left( \frac{d\sigma}{d\Omega} \right)_{\text{polarized}} \]

\[ = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 - 2 \Re (f(\theta) f(\pi - \theta)) \]

\[ \rightarrow \text{ as } \theta \rightarrow \frac{\pi}{2} \]

\[ \left| f(\frac{\pi}{2}) \right|^2 \]