Here you will look again at problem 2 of problem set No. 1 in which you studied some of the properties of the dynamics of a *charged* (complex) scalar field \( \phi(x) \) coupled to the electromagnetic field \( A_\mu(x) \). Recall that the Lagrangian density \( \mathcal{L} \) for this system is

\[
\mathcal{L} = (D_\mu \phi(x))^* (D_\mu \phi(x)) - m_0^2 |\phi(x)|^2 - \frac{\lambda}{4!} (|\phi(x)|^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\] (1)

where \( D_\mu \) is the covariant derivative

\[
D_\mu \equiv \partial_\mu + i e A_\mu
\] (2)

e is the electric charge and \( * \) denotes complex conjugation. In this problem set you will determine several important properties of this field theory at the classical level.

1. Derive an expression for the *locally conserved current* \( j_\mu(x) \), associated with the *global* symmetry

\[
\phi(x) \rightarrow \phi'(x) = e^{i\theta(x)} \phi(x)
\]

\[
\phi^*(x) \rightarrow \phi'^*(x) = e^{-i\theta(x)} \phi^*(x)
\]

\[
A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x)
\]

in terms of the fields of the theory.

2. Show that the conservation of the current \( j_\mu \) implies the existence of a *constant of motion*. Find an explicit form for this constant of motion.

3. Consider now the case of the *local* (or *gauge*) transformation

\[
\phi(x) \rightarrow \phi'(x) = e^{i\theta(x)} \phi(x)
\]

\[
\phi^*(x) \rightarrow \phi'^*(x) = e^{-i\theta(x)} \phi^*(x)
\]

\[
A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)
\]

where \( \theta(x) \) and \( \Lambda(x) \) are two functions. What should be the relation between \( \theta(x) \) and \( \Lambda(x) \) for this transformation to be a symmetry of the Lagrangian of the system?
4. Show that, if the system has the local symmetry of the previous section, there is a locally conserved gauge current \( J_\mu(x) \). Find an explicit expression for \( J_\mu \) and discuss in which way it is different from the current \( j_\mu \) of Section 1. Find an explicit expression for the associated constant of motion and discuss its physical meaning.

5. Find the Energy-Momentum tensor for this system. Show that it can be written as the sum of two terms

\[
T^{\mu\nu} = T^{\mu\nu}(A) + T^{\mu\nu}(\phi, A)
\]

where \( T^{\mu\nu}(A) \) is the energy-momentum tensor for the free electromagnetic field and \( T^{\mu\nu}(\phi, A) \) is the tensor which results by modifying the energy-momentum tensor for the decoupled complex scalar field \( \phi \) by the minimal coupling procedure.

6. Find explicit expressions for the Hamiltonian \( \mathcal{H}(x) \) and the linear momentum \( \vec{P}(x) \) densities for this system. Give a physical interpretation for all of the terms that you found for each quantity.

7. Consider now the case of an infinitesimal Lorentz transformation

\[
x_\mu \rightarrow x'_\mu + \omega_{\mu\nu}x^\nu
\]

where \( \omega_{\mu\nu} \) infinitesimal and antisymmetric. Show that the invariance of the Lagrangian of this system under these Lorentz transformation leads to the existence of a conserved tensor \( M_{\mu\nu\lambda} \). Find the explicit form of this tensor. Give a physical interpretation for its spatial components. Does the conservation of this tensor impose any restriction on the properties of the energy-momentum tensor \( T^{\mu\nu} \)? Explain. **Warning:** Be very careful in how you treat the fields. Recall that not all of the fields are scalars!

8. In this section you will consider again the same system but in a polar representation for the scalar field \( \phi \), i.e.,

\[
\phi(x) = \rho(x) e^{i\omega(x)}
\]

In problem set 1, problem II, you showed that for \( m_0^2 < 0 \) the lowest energy states of the system can be well approximated by freezing the amplitude mode \( \rho \) to a constant value \( \rho_0 \) which you obtained by an energy minimization argument. In this section you are asked to find the form of (A) the conserved gauge current \( J_\mu \), (B) the total energy \( E \) and (C) the total linear momentum \( \vec{P} \) in this limit.

9. Consider now the analytic continuation to imaginary time of this theory. Find the energy functional of the equivalent system in classical statistical mechanics. Give a physical interpretation for each of the terms of this
energy functional. If $D$ is the dimensionality of space-time for the original system, what is the dimensionality of space for the equivalent classical problem?

**Warning**: Be very careful in the way you continue the components of the vector potential.