Problem 1 Spin Waves in a Quantum Heisenberg Antiferromagnet

1. First recall the SU(2) commutators for the spins operators. Note that the commutators are zero when the spins are at the different sites. On the same site the commutators are:

\[ [\hat{S}^+, \hat{S}_-] = 2\hat{S}_3 ; \ [\hat{S}_3, \hat{S}^+] = \hat{S}^+ ; \ [\hat{S}_3, \hat{S}^-] = -\hat{S}^- \]

To compute the time dependence of the Heisenberg operators we can use

\[ \partial_t \hat{O} = -i[\hat{O}, \hat{H}] \]

The calculation becomes a lot simpler if one rewrite the Hamiltonian in terms of \( \hat{S}^+, \hat{S}^-, \) and \( \hat{S}_3 \).

\[ \hat{H} = J \sum_j \hat{S}_k(j) \cdot \hat{S}_k(j+1) = J \sum_j (\hat{S}_1(j) \cdot \hat{S}_1(j+1) + (1 \to 2) + (1 \to 3)) \]

\[ = \frac{J}{2} \sum_j (\hat{S}^+(j) \cdot \hat{S}^-(j+1) + \hat{S}^-(j) \cdot \hat{S}^+(j+1) + 2\hat{S}_3(j) \cdot \hat{S}_3(j+1)) \]

To make it more transparent one can write the Hamiltonian in a form that shows explicitly the operators on the \( j^{th} \) site. Doing so gives

\[ \hat{H} = \frac{J}{4} \sum_j \hat{S}^+(j)(\hat{S}^-(j-1) + \hat{S}^-(j+1)) + \hat{S}^-(j)(\hat{S}^+(j-1) + \hat{S}^+(j+1)) + \hat{S}_3(j)(\hat{S}_3(j-1) + \hat{S}_3(j+1)) \]

From this Hamiltonian, the equation of motion of an operator is easily obtained by computing the commutator with the Hamiltonian. For \( \hat{S}^+ \), it is

\[ [\hat{S}^+(j), \hat{H}] = \frac{J}{4} [\hat{S}^+(j), \hat{S}^+(j)](\hat{S}^-(j-1) + \hat{S}^-(j+1)) + [\hat{S}^+(j), \hat{S}^-(j)](\hat{S}^+(j-1) + \hat{S}^+(j+1)) + [\hat{S}^+(j), \hat{S}_3(j)](\hat{S}_3(j-1) + \hat{S}_3(j+1)) \]

\[ = \frac{J}{4} (2\hat{S}_3(j)(\hat{S}^+(j-1) + \hat{S}^+(j+1)) - 2\hat{S}^+(j)(\hat{S}_3(j-1) + \hat{S}_3(j+1)) \]

Similary, one can compute the equations of motion for the other two operators,

\[ \partial_t \hat{S}^+(j) = -i \frac{J}{2} [\hat{S}_3(j)(\hat{S}^+(j-1) + \hat{S}^+(j+1)) - \hat{S}^+(j)(\hat{S}_3(j-1) + \hat{S}_3(j+1))] \]

\[ \partial_t \hat{S}^-(j) = -i \frac{J}{2} [-\hat{S}_3(j)(\hat{S}^-(j-1) + \hat{S}^-(j+1)) - \hat{S}^-(j)(\hat{S}_3(j-1) + \hat{S}_3(j+1))] \]

\[ \partial_t \hat{S}_3(j) = -i \frac{J}{4} [\hat{S}^+(j)(\hat{S}^-(j-1) + \hat{S}^-(j+1)) + \hat{S}^-(j)(\hat{S}^+(j-1) + \hat{S}^+(j+1))] \]

Clearly these are non-linear equations. It is also obvious that there are three coupled differential equations.
2. Note that the Hilbert space is made up of states with \( n < 2S \). Hence, one can expand the \( (1 - \frac{n}{2S})^{1/2} \) and re-sum to get a similar expression but with c-numbers. For even sites,

\[
\hat{S}^+|n> = \sqrt{2S(1 - \frac{n}{2S})^{1/2}}|n> = [2Sn(1 - \frac{n - 1}{2S})]^{1/2}|n - 1>
\]

\[
\hat{S}^-|n> = \sqrt{2S(1 - \frac{n}{2S})^{1/2}}|n> = [2S(n + 1)(1 - \frac{n - 1}{2S})]^{1/2}|n + 1>
\]

\[
\hat{S}_3|n> = (S - \hat{n})|n> = (S - n)|n> = M(j)|n>
\]

3. The sum in the Hamiltonian can be split into over the odd and and the even sites. This gives

\[
H = \frac{J}{2} \sum_{j \in \text{even}} (\hat{S}^+(j) \cdot \hat{S}^-(j + 1) + \hat{S}^-(j) \cdot \hat{S}^+(j + 1) + 2\hat{S}_3(j) \cdot \hat{S}_3(j + 1))
\]

\[
+ \frac{J}{2} \sum_{j \in \text{odd}} \text{(Identical terms)}
\]

The expressions of \( \hat{S}^+, \hat{S}^-, \) and \( \hat{S}_3 \) in terms of \( \hat{a} \) and \( \hat{b} \) can be used in the Hamiltonian. For even \( j \)

\[
\hat{H} = JS[(1 - \frac{\hat{n}(j)}{2S})^{1/2}(1 - \frac{\hat{n}(j + 1)}{2S})^{1/2}\hat{a}(j)\hat{b}(j + 1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j + 1)(1 - \frac{\hat{n}(j)}{2S})^{1/2}(1 - \frac{\hat{n}(j + 1)}{2S})^{1/2}]
\]

\[
+ J(S)(\hat{S}^\dagger(j)\hat{a}(j) + S\hat{b}^\dagger(j + 1)\hat{b}(j + 1) - S^2 + \hat{a}^\dagger(j)\hat{a}(j)\hat{b}^\dagger(j + 1)\hat{b}(j + 1))
\]

For odd \( j \), the roles of \( \hat{a}, \hat{b} \) are switched.

4. In the classical limit, the spin of the system is taken to be really large, \( S \to \infty \). The leading term in the second part of the Hamiltonian is \( S^2 \). Hence for this term we can take up to the linear order in \( S \). Taking this limit one gets

\[
\hat{H}_{\text{even}} = JS[\hat{a}(j)\hat{b}(j + 1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j + 1) + \hat{a}^\dagger(j)\hat{a}(j) + \hat{b}^\dagger(j + 1)\hat{b}(j + 1) - S]
\]

\[
\hat{H}_{\text{odd}} = JS[\hat{b}(j)\hat{a}(j + 1) + \hat{b}^\dagger(j + 1)\hat{a}(j + 1) + \hat{b}^\dagger(j)\hat{a}(j) + \hat{a}^\dagger(j + 1)\hat{a}(j + 1) - S]
\]

Notice that they are quadratic in \( \hat{a} \) and \( \hat{b} \).

5. Taking the classical limit one gets for the even \( j \)

\[
\partial_0\hat{a}(j) = -i\frac{JS}{2}[\hat{b}^\dagger(j + 1) + \hat{b}(j + 1) - 2\hat{a}(j)]
\]

\[
\partial_0\hat{a}^\dagger(j) = -i\frac{JS}{2}[\hat{b}(j + 1) + \hat{b}^\dagger(j + 1) - 2\hat{a}^\dagger(j)]
\]

One can get similar equations for odd \( j \) by switching \( \hat{a}, \hat{b} \). Also note that to the leading order in \( S, \hat{S}_3 \sim S \) so the derivative vanishes.

6. First I employed the fourier transformation provided in the homework in the Hamiltonian. This yields

\[
\hat{H} = JS\frac{2}{N} \sum_{j \in \text{even,q,r}} e^{iqj}\hat{a}(q)e^{-ir(j+1)}\hat{b}(r) + e^{iqj}\hat{a}^\dagger(q)e^{ir(j+1)}\hat{b}^\dagger(r)
\]

\[
+ e^{-iqj}\hat{a}^\dagger(q)e^{-irj}\hat{a}(q) + e^{-iq(j+1)}\hat{b}(q)e^{-ir(j+1)}\hat{b}(q) - \frac{N}{2}S +
\]

\[
JS\frac{2}{N} \sum_{j \in \text{odd,q,r}} e^{iq(j+1)}\hat{a}(q)e^{-irj}\hat{b}(r) + e^{-iq(j+1)}\hat{a}^\dagger(q)e^{irj}\hat{b}^\dagger(r)
\]

\[
+ e^{-iq(j+1)}\hat{a}^\dagger(q)e^{-ir(j+1)}\hat{a}(q) + e^{-iqj}\hat{b}(q)e^{-irj}\hat{b}(q) - \frac{N}{2}S
\]

Using the orthogonality relations,

\[
\sum_{j \in \text{even}} e^{i(q-r)j} = \sum_{j \in \text{odd}} e^{i(q-r)j} = \frac{N}{2}\delta_{q-r,0}
\]

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Now we can sum over the lattice sites to get rid off one sum over the momentum.

\[ \hat{H} = JS \sum_q [e^{i\theta} \hat{a}(q) \hat{b}(q) + e^{i\theta} \hat{a}^\dagger(q) \hat{b}^\dagger(q) + \hat{a}^\dagger(q) \hat{a}(q) + \hat{b}^\dagger(q) \hat{b}(q)] \]

\[ + [e^{-i\theta} \hat{a}(q) \hat{b}(q) + e^{-i\theta} \hat{a}^\dagger(q) \hat{b}^\dagger(q) + \hat{a}^\dagger(q) \hat{a}(q) + \hat{b}^\dagger(q) \hat{b}(q)] - NS \]

We can use the trigonometry relation with the exponential functions to regroup terms.

\[ \hat{H} = 2SJ \sum_q [\hat{a}(q) \hat{b}(q) \cos(q) + \hat{a}^\dagger(q) \hat{b}^\dagger(q) \cos(q) + \hat{a}^\dagger(q) \hat{a}(q) + \hat{b}^\dagger(q) \hat{b}(q) - \frac{N}{2} S] \]

The Hamiltonian can be diagonalized by applying the Bogoliubov transformations to the basis states:

\[ \hat{a}(q) = \cosh(\theta) \hat{c}(q) - \sinh(\theta) \hat{d}^\dagger(q), \quad \hat{b}(q) = \cosh(\theta) \hat{d}(q) - \sinh(\theta) \hat{c}^\dagger(q) \]

The coefficients \( \cosh(\theta) \) and \( \sinh(\theta) \) can be chosen so that the Hamiltonian is diagonal in this new basis. By plugging in the transformations, the off-diagonal terms of the Hamiltonian are

\[ (\cosh^2(\theta) \hat{c}(q) \hat{d}^\dagger(q) + \sinh^2(\theta) \hat{d}^\dagger(q) \hat{c}^\dagger(q) + \cosh^2(\theta) \hat{c}^\dagger(q) \hat{d}^\dagger(q) + \sinh^2(\theta) \hat{d}(q) \hat{c}(q)) \cos(q) - (\cosh(\theta) \sinh(\theta) \hat{c}^\dagger(q) \hat{d}^\dagger(q) + \sinh(\theta) \cosh(\theta) \hat{d}^\dagger(q) \hat{c}(q) + \sinh(\theta) \cosh(\theta) \hat{c}(q) \hat{d}(q) + \cosh(\theta) \sinh(\theta) \hat{d}(q) \hat{c}^\dagger(q)) \]

For the Hamiltonian to be diagonal these off-diagonal terms must be zero. From this we find that

\[
\cos(q) = \tanh(2\theta)
\]

The Hamiltonian then simplifies to

\[
\hat{H} = 2SJ \sum_q [- \cosh(\theta) \sinh(\theta) (\hat{c}(q) \hat{c}^\dagger(q) + \hat{d}^\dagger(q) \hat{d}(q)) - \cosh(\theta) \sinh(\theta) (\hat{c}^\dagger(q) \hat{c}(q) + \hat{d}(q) \hat{d}^\dagger(q))] \cos(q) + \cosh^2(\theta) \hat{c}(q) \hat{c}^\dagger(q) + \sinh^2(\theta) \hat{d}^\dagger(q) \hat{d}(q) + \cosh^2(\theta) \hat{d}^\dagger(q) \hat{d}(q) + \sinh^2(\theta) \hat{c}^\dagger(q) \hat{c}(q) \]

Normal ordering and use the following identities: \( \sinh(2\theta) = -1 - \sin(q) \). Note that by applying the condition in eq.1, \( \sinh(\theta) = |\sin(q)| \). The sum can also be approximated as \( \frac{1}{\pi} \sum_q \rightarrow \int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} \), which is even as the \( \delta q \rightarrow 0 \). Hence

\[
\hat{H} = -NS(J(S + 1) + 2SNJ \int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} |\sin(q)| (\hat{n}_c(q) + \hat{n}_d(q)) \]

The ground state is the state annihilated by the the number operators \( \hat{n}_c(q) + \hat{n}_d(q) \). It is obvious that \( \hat{d}(q)|0> = \hat{c}(q)|0> = 0 \). They are some collective configuration of the original spins. The single particle states are \( \hat{d}^\dagger(q)|0> \) and \( \hat{c}^\dagger(q)|0> \). These states have the same energy so one can label them with another quantum number(1 and 2) such that one can distinguish between

\[
\hat{c}^\dagger(q)|0> = |q,1>
\hat{d}^\dagger(q)|0> = |q,2>
\]

The dispersion relation is easily read off from the Hamiltonian as

\[
E(q) = 2NSJ|\sin(q)|
\]

This is zero near \( q = n\pi \). Since \( q \in [-\pi/2, \pi/2] \) so one only needs to think of \( q = 0 \). In term of the original lattice, \( q \) is related to the original lattice vector \( k = \pi - q \). One can perform a small angle approximation, \( |\sin(q)| \sim |q| \). Near \( q = 0 \) the energy \( E(q) = 2NSJ|q| \). It is interesting to note that there are two branches of degenerate excitations, \( q = -q \). The wave velocity is given by

\[
v_s = \frac{d\omega(q)}{dq} = 2NSJ.
\]
Problem 2 Two Component Complex Scalar Field

The Lagrangian is given by

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi_a)^* (\partial^\mu \phi_a) - \frac{m_0^2}{2} \phi_a^* \phi_a \]

where \( a = 1, 2 \)

1. Using the definition from the lecture notes the canonical momentum is

\[ \Pi_a = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a} = \frac{1}{2} (\partial_0 \phi_a)^* \]

The Hamiltonian is given by

\[ H = \sum_i \Pi_i \partial_0 q_i - \mathcal{L} = \frac{1}{2} \partial_0 \phi_a^* \partial_0 \phi_a + \frac{1}{2} |\nabla \phi_a|^2 + \frac{m_0^2}{2} |\phi_a|^2 \]

The generalized momentum is given by

\[ P^i = \int d^3x T^0i = \int d^3x \left( \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a} \partial^i \phi_a + \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a} \partial^i \phi_a^* \right) \]

2. The change in the Lagrangian can be written as

\[ \delta \mathcal{L} = i \partial_\mu (\partial^\mu \phi_a^* \sigma^i \phi_b - \partial^\mu \phi_b \sigma^i \phi_a^* \phi_b) \]

Under the transformation \( U_{ab} = [e^{i \theta_k \sigma_k}]_{ab} \), one can read off that infinitesimally

\[ \phi_a = U_{ab} \phi_b = (1 + i \theta_k (\sigma_k)_{ab}) \phi_b \]

Therefore the

\[ \delta \phi_a = i \theta_k (\sigma_k)_{ab} \phi_b \]

\[ \delta \phi_a^* = -i \theta_k (\sigma_k)_{ba} \phi_b^* \]

\( \sigma^k \) are the three Pauli matrices. Hence, \( k = 0, 1, 2, 3 \) and there should be four conserved charges. \( k = 0 \) case is the identity matrix. These are the generators of \( U(2) = U(1) \times SU(2) \). By plugging in the expressions for \( \delta \phi_a \) into eq.2, the change in the Lagrangian is

\[ \delta \mathcal{L} = i \theta_k \partial_\mu (\partial^\mu \phi_a^* \sigma_a^i \phi_b - \partial^\mu \phi_b \sigma^i \phi_a^* \phi_b) \]

If the transformations are a symmetry, the lagrangian should remain the same under the transformations. It is obvious to see that

\[ J^k_\mu = i (\partial_\mu \phi_a^* \sigma^k_a \phi_b - \partial_\mu \phi_b \sigma^k_a \phi_a^* \phi_b) \]

Then the generalized charges are

\[ Q^k = \int d^3x J^k_0 = i \int d^3x i (\Pi_a \sigma_a^k \phi_b - \Pi_b \sigma_a^k \phi_a^*) \]

3. We can quantize the theory by promoting the fields to operators

\[ \hat{H} = \int d^3x \frac{1}{2} \Pi_a \hat{\Pi}_a + \frac{1}{2} \nabla \hat{\phi}_a^* \nabla \hat{\phi}_a + \frac{m^2}{2} \hat{\phi}_a \hat{\phi}_a \]

and impose the canonical equal time commutation relations

\[ [\hat{\Pi}_a(x), \hat{\phi}_b(y)] = i \delta_{ab} \delta(x - y) \]
and with the other commutation relations equal to zero.

4. The charge can also be promoted to an operator

\[ \hat{Q}^k = i \int d^3 x (\hat{\Pi}_a \sigma^k_{ab} \hat{\phi}_b - \hat{\Pi}^k_{ab} \sigma^k_{ab} \hat{\phi}_a) \]

5. The Heisenberg equation of motion can be derived by

\[ \frac{d}{dt} \hat{\phi}_a = -i [\hat{\phi}_a(x), \hat{H}] = -i \frac{1}{2} \int d^3 y (\hat{\phi}(x), \hat{\Pi}_b(y) \hat{\Pi}_b(y) + \nabla \hat{\phi}_b(y) \nabla \hat{\phi}_b(y) + m^2 \hat{\phi}_b(y) \hat{\phi}_b(y)] \]

Only the contribution from the first term is non-zero. Hence

\[ \frac{d}{dt} \hat{\phi}_a = -i \frac{1}{2} \int d^3 y (\hat{\phi}(x), \hat{\Pi}_b(y) \hat{\Pi}_b(y)) = \frac{1}{2} \Pi_a^k(x) \]

The equation of motion for the canonical momentum is found

\[ \frac{d}{dt} \hat{\Pi}_a = -i \frac{1}{2} \int d^3 y \nabla \hat{\phi}_a(y)[\hat{\Pi}_a(x), \nabla \hat{\phi}_b(y)] + m^2 \hat{\phi}_a(y)[\hat{\Pi}_a(x), \hat{\phi}_b(y)] \]

Note that the spatial derivatives is on the y-coordinate. As such it can be pulled through things that depend on x. One can simplify it further to

\[ \frac{d}{dt} \hat{\Pi}_a = -i \frac{1}{2} \int d^3 y \nabla \hat{\phi}_a(y)[\hat{\Pi}_a(x), \hat{\phi}_b(y)] + m^2 \hat{\phi}_a(y)[\hat{\Pi}_a(x), \hat{\phi}_b(y)] \]

The first term will be a derivative of a dirac delta function. By integrating by parts one finds

\[ \frac{d}{dt} \hat{\Pi}_a = \frac{1}{2} (\nabla^2 - m_0^2) \hat{\phi}_a^k \]

by substituting the previous result, this gives the Klein-Gordon equation

\[ \partial^2_t \hat{\phi}_a^k = \nabla^2 \hat{\phi}_a^k \]

6. Since the equations of motion re linear we can solve it by writing the field operator as a Fourier transform

\[ \hat{\phi}_a(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(k)}} (\hat{a}_a(k) e^{i\omega t + ik \cdot x} + \hat{b}_a(k) e^{-i\omega t - ik \cdot x}) \]

the frequency \( \omega(k) \) can be derived from the Klein-Gordon equation and it is equal \( \omega(k) = \sqrt{k^2 + m^2} \).

The expansion for the momentum is equal to \( \partial \phi_a^k \)

\[ \hat{\Pi}_a(x) = \int \frac{d^3 k}{(2\pi)^3} \sqrt{\frac{\omega(k)}{2}} (\hat{a}_a(k) e^{-i\omega t - ik \cdot x} - \hat{b}_a(k) e^{i\omega t + ik \cdot x}) \]

7. Using the expression of \( \hat{Q}^k \) from part 4. The mode expansion of this yields

\[ \hat{Q}^k = \frac{\sigma^k_{ab}}{2} \int d^3 x d^3 k d^3 q \frac{\omega(k)}{\omega(q)} (\hat{a}_a(k) e^{-ik \cdot x} - \hat{b}_a(k) e^{i k \cdot x}) \hat{a}_b(q) e^{iq \cdot x} - \hat{b}_b(q) e^{-iq \cdot x}) \]

Using the fact that \( \int d^3 x e^{i(k-q \cdot x)} = (2\pi)^3 \delta_{k,q} \), the \( \hat{Q} \) becomes

\[ \hat{Q}^k = \frac{\sigma^k_{ab}}{2} \int \frac{d^3 k}{(2\pi)^3} (\hat{a}_a(k) \hat{a}_b(k) - \hat{b}_a(k) \hat{b}_b(k) + \hat{a}_a(k) \hat{b}_b(k) - \hat{b}_a(k) \hat{a}_b(k)) \]

\[ + (\hat{a}_a(k) \hat{a}_b(k) - \hat{b}_a(k) \hat{b}_b(k) + \hat{a}_a(k) \hat{b}_b(k) - \hat{b}_a(k) \hat{a}_b(k)) \]

\[ = \frac{\sigma^k_{ab}}{2} \int \frac{d^3 k}{(2\pi)^3} \{ \hat{a}_a(k) \hat{a}_b(k) - \hat{b}_a(k) \hat{b}_b(k) \} \]
Here we label the state by the momentum and the eigenvalues of $\sigma$. The ground state of this system are the states that’s annihilated by $\hat{a}$, starting from eq. 3

\begin{align*}
E &= \omega(k) = \sqrt{k^2 + m^2}.
\end{align*}

The dispersion relation derived in part 6 given by $E(k) = \omega/2$. This has a four fold degeneracy. The example of the four different states are shown below

| State  | | State  |
|--------|------------------|
| $\hat{a}_1^\dagger(0)$ | $+, \uparrow, k$ | $\hat{b}_1^\dagger(0)$ | $0, \uparrow, k$ |
| $\hat{a}_1^\dagger(0)$ | $+, \downarrow, k$ | $\hat{b}_1^\dagger(0)$ | $0, \downarrow, k$ |

Here we label the state by the momentum and the eigenvalues of $\sigma^0$ and $\sigma^3$.