Quantum Field Theory: Problem Set 5
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Problem 1  Fermions in One Dimension

In the continuum limit, the Hamiltonian becomes

\[
H = \sum_{\sigma = \uparrow, \downarrow} \int dx \psi_\sigma^\dagger (-v_f \frac{\partial}{\partial x}) \sigma_3 \psi_\sigma(x) + \int dx \left[ \frac{\Pi^2(x)}{8M a_0^2} + \frac{1}{2} \Delta^2(x) \right] + \sum_{\sigma = \uparrow, \downarrow} \int dx \sqrt{2g} \Delta(x) \bar{\psi}_\sigma(x) \psi_\sigma(x)
\]

Using the adiabatic where the phonon is not excited this is equivalent to taking \( M \to \infty \). In this limit the Hamiltonian takes the form

\[
H = \sum_{\sigma = \uparrow, \downarrow} \int dx \psi_\sigma^\dagger(x) \left[ -v_f \frac{\partial}{\partial x} \sigma_3 + \sqrt{2g} \Delta(x) \sigma_2 \right] \psi_\sigma(x) + \frac{1}{2} \int dx \Delta^2(x)
\]

where in the chiral basis, \( \gamma_0 = \sigma_2, \gamma_1 = i \sigma_1, \) and \( \gamma_5 = \gamma_0 \gamma_1 = \sigma_3 \). This linear differential equation can be solved by making the plane wave ansatz

\[
\psi_\sigma(x) = \int \frac{dk}{2\pi} e^{ikx} \psi_{k,\sigma}; \quad \psi_\sigma^\dagger(x) = \int \frac{dk}{2\pi} e^{-ikx} \psi_{k,\sigma}^\dagger
\]

Inserting this ansatz in the Hamiltonian gives

\[
H = \sum_{\sigma = \uparrow, \downarrow} v_F \int dx \int \frac{dp}{(2\pi)^2} \bar{\psi}_{q,\sigma} \left[ -\gamma_3 p + \sqrt{2g} \Delta(x) \right] \psi_{p,\sigma} e^{i(p-q)x} + \frac{1}{2} \int dx \Delta^2(x)
\]

In the chiral basis one can write \( \psi_{p,\sigma} = \begin{pmatrix} R_{p,\sigma} \\ L_{p,\sigma} \end{pmatrix} \). By defining \( m = \frac{\sqrt{2g}}{v_F} \Delta \), one can write an eigenvalue equation as

\[
\begin{pmatrix} p & -im \\ im & -p \end{pmatrix} \begin{pmatrix} R_{p,\sigma} \\ L_{p,\sigma} \end{pmatrix} = \lambda \begin{pmatrix} R_{p,\sigma} \\ L_{p,\sigma} \end{pmatrix}
\]
This has a non-trivial solution if \( \lambda \) satisfies the characteristic equation,

\[-p^2 + \lambda^2 - m^2 = 0 \quad \Rightarrow \quad \lambda_\pm = \pm \sqrt{p^2 + m^2} = \pm E\]

From here we know that the Hamiltonian is diagonal in the basis of the eigenvectors which are given by,

\[
a_{p,\sigma} = (p - E)R_p - imL_{p,\sigma} \\
b_{p,\sigma} = -(p + E)L_{p,\sigma} + imR_{p,\sigma}
\]

At the end of the day the Hamiltonian is

\[
H = \sum_{\sigma=\uparrow,\downarrow} v_F \int \frac{dp}{2\pi} \begin{pmatrix} a_{p,\sigma}^\dagger b_{p,\sigma}^\dagger \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} a_{p,\sigma} \\ b_{p,\sigma} \end{pmatrix} + \frac{1}{2} \Delta^2
\]

which is diagonal. Notice that the particles, \( b_{p,\sigma}^\dagger \) has negative energy therefore it makes sense to perform the particle-hole transformation, \( b_{p,\sigma}^\dagger = c_{p,\sigma} \).

: \( H := \frac{1}{2} \Delta^2 + 2v_F \int \frac{dp}{2\pi} E(p) \begin{pmatrix} a_{p,\sigma}^\dagger a_{p,\sigma} + c_{p,\sigma}^\dagger c_{p,\sigma} - 1 \end{pmatrix} \]

The ground state is the state which is annihilated by

\[
a_{p,\sigma} |0\rangle = 0 \quad ; \quad c_{p,\sigma} |0\rangle = 0
\]

The ground state energy is equal to

\[
E_{\text{gnd}} = \langle \text{gnd}|H|\text{gnd}\rangle = \frac{1}{2} \Delta^2 - 2 \int \frac{dp}{2\pi} \sqrt{2g\Delta^2 + (v_F p)^2}
\]

Note that the momentum integral has a natural cutoff according to the natural lattice constant 1/\( a \). Since consider any momentum higher than this would be unphysical. Hence one only needs to consider the states within the momentum shell \( p \in (-E_c/p_f, E_c/p_f) \). One finds

\[
\frac{d}{d\Delta} E_{\text{gnd}} = \Delta - \frac{1}{\pi} \int_{0}^{E_c/p_f} dp \frac{\sqrt{2g\Delta}}{2g\Delta^2 + (v_F p)^2} = 0
\]

One can make the following substitutions: \( v_F p = \sqrt{2g\Delta} \sinh(y) \) and \( dp = \frac{\sqrt{2g}}{v_F} \Delta \cosh(y) dy \). This gives

\[
\Delta = \frac{v_F (E_c/p_f)}{\sqrt{2g} \sinh(v_F \pi/\sqrt{2g})}
\]

2. The energy spectrum can be found by acting on the ground state with the creation operators and computing their commutation relations with the Hamiltonian. Both happen to be the same and equal to

\[
E(p) = v_F \sqrt{p^2 + 2g\Delta^2}
\]
The single particle states can be created as
\[ a_{p,\sigma}^\dagger |gnd\rangle = |0, p_\sigma\rangle; \quad c_{p,\sigma}^\dagger |gnd\rangle = |0, p_\sigma\rangle \]

3. We would like to study the Lagrangian under the discrete transformation: \( \psi \rightarrow \gamma_5 \psi, \Delta \rightarrow -\Delta \). From the definition, \((\gamma_5^\dagger)^2 = (\gamma_5)^2 = 1\) and \( \{\gamma_\mu, \gamma_5\} = 0 \). The Hamiltonian is then
\[
H = \sum_{\sigma = \uparrow, \downarrow} \int dx \; \psi_\sigma^\dagger(x) \gamma^0 \left( -v_f \gamma^1 \frac{\partial}{\partial x} + \sqrt{2g\Delta} \right) \psi_\sigma(x) + \int dx \; \left[ \frac{\Pi^2(x)}{8Ma_0^2} + \frac{1}{2}\Delta^2 \right]
\]

Under the transformations, the middle term is invariant. The terms involving the fermions change as
\[
\sum_{\sigma = \uparrow, \downarrow} \int dx \; \psi_\sigma^\dagger(x)(\gamma_5^\dagger)\gamma^0 \left( -v_f \gamma^1 \frac{\partial}{\partial x} - \sqrt{2g\Delta} \right) \gamma_5 \psi_\sigma(x)
\]
\[
\sum_{\sigma = \uparrow, \downarrow} \int dx \; \psi_\sigma^\dagger(x)(-1)\gamma^0 \gamma_5 \left( -v_f \gamma^1 \frac{\partial}{\partial x} - \sqrt{2g\Delta} \right) \gamma_5 \psi_\sigma(x)
\]

It’s obvious from this that anticommuting \( \gamma_5 \) past \( \gamma^1 \) adds another minus sign which makes the whole Hamiltonian invariant under the transformations. Furthermore, by using the anticommutation relation it is easy to see that \( \bar{\psi} \psi \rightarrow -\bar{\psi} \psi \) under these discrete transformations.

4. To compute the two point function \( \langle gnd|\bar{\psi}\psi|gnd\rangle \), we need to rewrite the ground state in terms of \( a_{p,\sigma} \) and \( c_{p,\sigma} \). Recall the transformations:
\[
a_{p,\sigma} = (p - E)R_p - imL_{p,\sigma} \\
b_{p,\sigma} = -(p + E)L_{p,\sigma} + imR_{p,\sigma}
\]

From this one can construct a transformation matrix that connects the right-left basis to the eigenvectors basis. This is
\[
U = \begin{pmatrix}
\frac{p - E}{\sqrt{2E^2 - 2Ep}} & \frac{-im}{\sqrt{2E^2 - 2Ep}} \\
\frac{-im}{\sqrt{2E^2 + 2Ep}} & \frac{p + E}{\sqrt{2E^2 + 2Ep}}
\end{pmatrix}
\]

By simply doing matrix multiplications one finds that
\[
\langle gnd|\bar{\psi}\psi|gnd\rangle = \int \frac{dp}{2\pi} \frac{\sqrt{2g\Delta}}{\sqrt{2g\Delta^2 + (v_fp)^2}}
\]

This integral can be computed by making the substitution, \( v_fp = \sqrt{2g\Delta} \sinh(y) \). This yields
\[
\langle gnd|\bar{\psi}\psi|gnd\rangle = \frac{\sqrt{2g\Delta}}{2\pi v_f} \sinh^{-1} \left( \frac{E_cv_f}{\sqrt{2g\Delta p}} \right)
\]

From this relation one sees that the correlation vanishes when \( \Delta = 0 \) and has a non-zero when \( \Delta \neq 0 \).
Problem 2 Grassmann Variables

1. Recall the anticommutation relation of the Grassmann variables, \( \{a, b\} = 0 \). From here we can derive that a function of Grassmann variables can only be up to linear order. Hence, \( g(a^*) \sim g_0 + g_1 a^* \). Defining the inner product between the two functions to be

\[
\langle f | g \rangle = \int da^* da e^{a^* a} f(a^* g(a))
\]

One can Taylor expand this expansion to be

\[
\langle f | g \rangle = \int da^* da (1 + a^* a) (\bar{f}_0 g_0 + \bar{f}_1 g_1)
\]

Since the quadratic terms are zero and \( \int da = 0, \int daa = 1 \), one obtains

\[
\langle f | g \rangle = \bar{f}_0 g_0 + \bar{f}_1 g_1
\]

2. A Taylor expansion of an arbitrary function, \( A(a^*, a) \) is

\[
A_{00} + A_{10} a^* + A_{01} a + A_{11} a^* a
\]

Taylor expand the integrand gives

\[
(Af)b^* = \int da^* da (A_{00} + A_{10} b^* + A_{01} a + A_{11} b^* a)(f_0 + f_1 a^*)(1 - a^* a)
\]

Since the quadratic terms are zero and \( \int da = 0, \int daa = 1 \), one obtains

\[
\langle f | g \rangle = \bar{f}_0 g_0 + \bar{f}_1 g_1
\]

This is also a matrix multiplication.
3. To find the operator identity we have to evaluate it with a test function.

\[ \hat{a}^* \hat{a}^* f(a^*) = \hat{a}^* (a^* f(a^*)) = a^* (a^* f(a^*)) = 0 \]

\[ \hat{a} \hat{a} f(a^*) = \hat{a} \left( \frac{d}{da^*} f(a^*) \right) = \frac{d}{da^*} \left( \frac{d}{da^*} f(a^*) \right) = 0 \]

\[ \{ \hat{a}^*, \hat{a} \} f(a^*) = [\hat{a}^* \hat{a} + \hat{a} \hat{a}^*] f(a^*) = \left[ \hat{a}^* \left( \frac{d}{da^*} f(a^*) \right) + \hat{a} (a^* f(a^*)) \right] = f_0 + f_1 a^* = f(a^*) \]

This implies that \( \{ \hat{a}^*, \hat{a} \} = 1 \)

4. If \( \{ \eta_j \}_{j=1}^N \) is a set of \( N \) Grassmann variables, then

\[ Z = \int \prod_{j=1}^N d\eta_j^* d\eta_j e^{-\sum_k \eta_k^* M_{kk} \eta_k} \]

One can transform to the basis where the matrix, \( M \) is diagonal. The new Grassmann variables are \( \eta' = U \eta \). where the matrix \( U \) is unitary. The integral then becomes

\[ Z = \int \prod_{j=1}^N d(\eta_j')^* d\eta_j' \det(U^\dagger) \det(U) e^{-\sum_k (\eta_k')^* M_{kk} \eta_k} = \int \left( \prod_{j=1}^N d(\eta_j')^* d\eta_j \right) \prod_k (1 - \eta_k^* M_{kk} \eta_k) = \det(M) \]

\[ \cdots \cdots \]

**Problem 3  Dirac Fermions**

1. Adding the current coupling terms to the action. The generating function is given by

\[ Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi \ e^{i \int dx \bar{\psi} (i\partial - m) \psi + \bar{\eta} \psi + \bar{\psi} \eta} = \int D\bar{\psi} D\psi \ e^{i \int dx \bar{\psi} M_{ab} \psi_a + \bar{\eta} \psi_a + \bar{\psi} \eta_a} \]

where the indices are the spinor indices. We can complete the squares by making the substitution \( \eta_a = \psi_a(x) + \int S_F^{ab}(x - y) \eta_b(y) \) where \( M_{ab}(x) S_F^{bc} = \delta_{ac}(x - y) \). The generating function can be written as

\[ Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi e^{i \int \bar{\eta} \psi M_{ab} \psi_a - \int d^2 x \bar{\eta}_a(x) M^{-1}_{ab} (x - y) \eta_b(y) \]

We can integrate out the Fermions to give the generating function that depends on the determinant,

\[ Z = \det(i\partial - m) e^{i \int \bar{\eta} \psi M_{ab} \psi_a - \int d^2 x \bar{\eta}_a(x) M^{-1}_{ab} (x - y) \eta_b(y) \]

where \( M^{-1} = i\partial - m \).

2. The Feynman propagator can be written as a functional derivative of the generating function.

\[ S_F^{ab}(x - y) = \langle x, \alpha | \frac{1}{i\partial - m} | x, \beta \rangle = \frac{\delta^2}{\delta \bar{\eta}_a(x) \delta \eta_b(y)} Z[\bar{\eta}, \eta] \bigg|_{\eta = \bar{\eta} = 0} \]
3. To find the four point function, we take the respective derivatives of the generating function four times. For example

\[ S_F^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{Z[0]} \frac{\delta^4 Z[\bar{\eta}, \eta]}{\delta \bar{\eta}_a(x_1) \delta \bar{\eta}_b(x_2) \delta \eta_c(x_3) \delta \eta_d(x_4)} \]

\[ = \frac{\delta^3 Z[\bar{\eta}, \eta]}{\delta \bar{\eta}_a(x_1) \delta \bar{\eta}_b(x_2) \delta \eta_c(x_3)} e^{-\frac{i}{2} \int d^4x \bar{\eta}_a \eta^{\beta}(x-y) \eta_\beta} \int dy \bar{\eta}_a(y) S_F^{ad}(y-x_4) \]

\[ = \frac{\delta^2 Z[\bar{\eta}, \eta]}{\delta \bar{\eta}_a(x_1) \delta \bar{\eta}_b(x_2)} e^{-\frac{i}{2} \int d^4x \bar{\eta}_a S^{\alpha\beta}(x-y) \eta_\beta} \int dy \bar{\eta}_a(y) S_F^{ad}(y-x_4) \]

\[ = \frac{\delta Z[\bar{\eta}, \eta]}{\delta \eta_a(x_1)} e^{-\frac{i}{2} \int d^4x \eta^{\alpha}(x-y) \eta_\alpha} S_F^{bd} S_F^{ad} S_F^{ac} \int dy \bar{\eta}_a(y) S_F^{ad}(y-x_4) \]

\[ = \int dy \bar{\eta}_a(y) S_F^{ad}(y-x_4) S_F^{be}(x_2 - x_3) \]

The minus sign comes from commuting the derivative past the Grassmann variable. The final result is the following

\[ S_F^{(4)}(x_1, x_2, x_3, x_4) = S_F^{bd}(x_2 - x_4) S_F^{ad}(x_1 - x_3) - S_F^{ad}(x_1 - x_4) S_F^{bc}(x_2 - x_3) \]

\[ \cdots \cdots \]

Problem 4 Functional Determinant

1. The energy of the system can be computed from the Hamiltonian

\[ H = \frac{1}{2} \int dx \Pi^2 + (\nabla \phi)^2 + m^2 \phi^2 \]

The classical solution of this has to satisfy the Euler-Lagrange equation.

\[ -\partial_\mu \partial^\mu \phi - m^2 \phi = (-\partial_0 \partial^0 + \nabla^2 - m^2) \phi = 0 \]

The solution, \( \phi_c \) that satisfies both the boundary condition, \( \phi(x, t) = \phi(x + L, t) \) and \( m^2(\text{const}) = 0 \) is \( \phi_c = 0 \). To get the classical energy we plug this back into the Hamiltonian, yields \( E_c = H(\phi_c) = 0 \).

2. We can compute the path integral by expanding around the classical solution, \( \phi \sim \phi_c + \phi \). The path integral becomes

\[ Z = \int D\phi e^{iS(\phi)} = \text{det}[-\Box + m^2]^{-(1/2)} e^{iS(\phi_c)} + ... \]

If we analytically continue this to imaginary time, we know that the generating function is equivalent to statistical partition function in one higher spatial dimension. If the
$T = 0$ limit is taken, the ground state contribution dominates the partition function, ie. $Z = e^{-\beta E_G}$. Comparing to the expression of the generating function we can write the ground state energy as

$$E_G = \lim_{\beta \to \infty} \left( \frac{1}{\beta} S[\phi_c] + \frac{1}{2\beta} \log(\text{det}[-\partial_\tau \partial^\tau + \nabla^2 + m^2]) \right)$$

(1)

3. From part 1 we immediately know that that the classical contribution is equal to 0. The term involving the determinant can be solved by first consider that the determinant is the product of the eigenvalues

$$\log(\text{det}[-\partial_\tau \partial^\tau + \nabla^2 + m^2]) = \sum \log(\prod \lambda) = -\sum \frac{\partial}{\partial s} \lambda^s \bigg|_{s=0^+}$$

One can solve for this determinent term using the heat-kernel method. The operator we would like to find the kernel of is simply the Klein-Gordon operator.

$$(-\nabla^2 + m^2)G(x, y; \tau) = \partial_\tau G(x, y; \tau)$$

with the boundary condition that

$$\lim_{\tau \to 0^+} G(x, y; \tau) = \delta(x_0 - y_0) \sum_{n=-\infty}^{\infty} \delta(x_1 - (y_1 + nL))$$

$$G(x, y; \tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-(\omega^2 + m^2)\tau + i\omega(x_0 - y_0)} \frac{1}{L} \sum_{-\infty}^{\infty} e^{-k^2\tau + ik(x_1 - y_1)}$$

By completing the square, we can integrate over $\omega$. The result is

$$G(x, y; \tau) = \frac{1}{L(4\pi \tau)^{1/2}} e^{-m^2\tau - \frac{(x_0 - y_0)^2}{4\tau}} \sum_{-\infty}^{\infty} e^{-k^2\tau + ik(x_1 - y_1)}$$

I can sum the terms and write it in the form that converges.

$$\sum_{-\infty}^{\infty} e^{-k^2\tau + ik(x_1 - y_1)} = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} dn \ e^{-(\frac{2\pi n}{L})^2 + i\frac{2\pi n}{L} (x_1 - y_1)} e^{i2\pi \nu \tau}$$

$$= \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} dn \ e^{-(\frac{2\pi \nu}{L})^2 + i\frac{2\pi \nu}{L} ((x_1 - y_1) + \nu L)}$$

We can complete the square then integrate over $n$. The sum becomes

$$\sqrt{L^2/4\pi \tau} \sum_{\nu=-\infty}^{\infty} e^{-\frac{1}{4\tau}((x_1 - y_1) + \nu L)^2}$$

The final answer for the kernel is:

$$G(x, y; \tau) = \frac{1}{4\pi \tau} \sum_{\nu=-\infty}^{\infty} e^{-m^2\tau - \frac{(x_0 - y_0)^2}{4\tau} + ((x_1 - y_1) + \nu L)^2}$$
This sum can be splitted into the cases when $\nu = 0$ and when it is non-zero. The answer can be looked up from the lecture notes. At the end of the day we can plug this back into eq.1 we got

$$E_G = \frac{m^2}{2\pi a^2} \left( \log \left( \frac{m}{a} \right) - \frac{1}{2} \right) L - \frac{1}{6} \frac{\pi}{L}$$

From this we can read off $A = \frac{\pi}{6}$ and $\eta = 1$

4. The Casimir force is given by $-\frac{\partial F}{\partial L}$, which is $\sim -\frac{1}{6} \frac{\pi}{L}$, Hence this is an attractive force.

Problem 5  Weakly Interacting Bose Gase

1. Recall that the propagator in quantum mechanics is written as $\langle f | e^{-i\hat{H}(\phi,\phi^*)} | i \rangle$, where the Hamiltonian is

$$\hat{H} := \int d^3x \left[ \hat{\phi}^\dagger(x) \left( \frac{\hat{P}^2}{2m} - \mu \right) \hat{\phi}(x) + \hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)V(x-y)\hat{\phi}(x)\hat{\phi}(y) \right]$$

We can write this in a path integral representation be using the coherent states basis. This is

$$|\{\phi\}⟩ = e^{\int d^d x φ(x)\hat{φ}(x) |0\rangle}; \hat{φ}|\{\phi\}⟩ = φ(x)|\{\phi\}⟩$$

In this basis one can write

$$\langle f | e^{-i\hat{H}(\phi,\phi^*)} | i \rangle = \int DφDφ^* \exp \left[ \int_{t_i}^{t_f} dt \left( \int d^d x (φ_\partial φ^* - φ^*_\partial φ) - iH(φ, φ^*) \right) \right] \times \langle f |\{φ\}⟩\langle\{φ\}|i⟩ e^{i\int d^d x(φ(x,t_i))^2 + |φ(x,t_f)|^2}$$

We can relate this to the partition function by making several mappings: $β = T = t_f - t_i$, the initial and final state is the same since it is a trace. Therefore the partition function becomes

$$Z = \text{tr} e^{-\beta\hat{H}} = \int DφDφ^* \exp \left[ \int_0^\beta dt \left( \int d^d x (φ_\partial φ^* - φ^*_\partial φ) - iH(φ, φ^*) \right) \right]$$

Wick rotate gives the Euclidean action as

$$S_E(φ, φ^*) = \int_0^\beta dτ \int d^d x (-φ^*_\partial_τ φ + H[φ^*, φ])$$

2. The classical solutions can be found by solving the Euler-Lagrange equations:

$$\dot{φ} - \frac{1}{2m} \nabla^2 φ + (λ - μ) φ + 2λ φ^* φ^2 = 0$$

$$-\dot{φ}^* - \frac{1}{2m} \nabla^2 φ^* + (λ - μ) φ^* + 2λ φ(φ^*)^2 = 0$$
Notice that $\phi = \rho e^{i\alpha}$ is a solution regardless of the value of $\alpha$, hence the solution is not unique.

3. The scalar complex field can be written as two real components, $a, b$. Expanding the Lagrangian for small fluctuations into the fields, one finds that the two-point function is $A^{-1}$, which is

$$A = \begin{pmatrix} \mu & i\omega + \frac{p^2}{2m} - \mu \\ i\omega + \frac{p^2}{2m} - \mu & \mu \end{pmatrix}$$

4. The Euclidean action can be written as

$$S_{\text{eff}} = \int d^2x \left( -\frac{1}{2} (\phi^* \partial_0 \phi - \phi \partial_0 \phi^*) - \phi^* \left[ -\frac{\hbar^2}{2m} \nabla^2 + \mu \right] \phi - |\phi|^2 \lambda \right)$$

Now we can look at the small fluctuation around the classical solution. This is done by considering $\phi = (\rho + \delta \rho)^{1/2} e^{i\theta}$. We Taylor expand this in the action and keep up to the second order. We will look at each term separately. First

$$\partial_0 \phi = e^{i\theta} \left( \frac{1}{2(\rho + \delta \rho)^{1/2}} \delta \rho + \frac{i}{2} (\rho + \delta \rho)^{1/2} \dot{\theta} \right) ; \quad \partial_0 \phi^* = e^{-i\theta} \left( \frac{1}{2(\rho + \delta \rho)^{1/2}} \delta \rho - \frac{i}{2} (\rho + \delta \rho)^{1/2} \dot{\theta} \right)$$

Hence

$$\phi^* \partial_0 \phi - \phi \partial_0 \phi^* = i(\rho + \delta \rho) \dot{\theta}$$

For the momentum term, the derivatives can be evaluated similarly. First note that by integrating by parts $\phi^* \nabla^2 \phi = -|\nabla \phi|^2$. This term is easier to evaluate,

$$-|\nabla \phi|^2 = \frac{(\nabla \delta \rho)^2}{2(\rho + \delta \rho)} - \left( \rho + \frac{1}{2} \delta \rho \right) (\nabla \theta)^2$$

The other terms are

$$\mu |\phi| = \rho \mu + \mu \delta \rho$$

$$\lambda |\rho|^2 = \lambda (\rho^2 + 2\rho \delta \rho + \delta \rho^2)$$

Collecting all the terms and dropping the constant pieces and the total derivative terms gives

$$S_{\text{eff}} = \int d^2x \left[ -\delta \rho \left( -\frac{\hbar^2 \nabla^2}{2m\rho} + \lambda \right) \delta \rho - \delta \rho (i \dot{\theta} + 2 \rho \lambda) - \frac{\hbar^2 \rho}{4m} (\nabla \theta)^2 \right]$$

Next step is to integrate out $\delta \rho$. This is just a Gaussian integral. In the slowly varying limit, one obtains

$$S_{\text{eff}} = \int d^2x \frac{1}{4\lambda} (i \dot{\theta}(x) + 2 \rho \lambda)^2 + \frac{\hbar^2}{4m} \rho (\nabla \theta)^2$$
Dropping out the constant and total derivatives terms,

\[ S_{\text{eff}} = \int d^2x \frac{1}{2} \left[ \frac{1}{2\lambda} \dot{\theta}^2(x) + \frac{\hbar^2}{2m} \rho(\nabla \theta)^2 \right] \]

5. From part 4, \( a = \frac{1}{2\lambda} \) and \( b = \frac{\hbar^2 \rho}{2m} \). The action in real time is

\[ S_{\text{eff}} = \int dt \int d^3x \frac{1}{2} \left[ a(\partial_x \theta)^2(x) + b(\nabla \theta)^2 \right] \]

We can then rescale \( \theta \) and redefine \( v_{\text{eff}} = \frac{\rho \hbar^2}{m} \). The effective action then has the same form as of the wave propagating with effective velocity, \( v_{\text{eff}} \). 

\[ \cdots \cdots \cdots \]