Physics 582, Problem Set 2 Solutions

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Symmetries and Conservation Laws

In this problem set we return to a study of scalar electrodynamics which has the Lagrangian

\[ L = |D_\mu \phi|^2 - m_0^2 |\phi|^2 - \frac{\lambda}{4!} |\phi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]  

where \( \phi \) is a complex scalar field, \( A_\mu \) is a \( U(1) \) gauge field, and \( D_\mu \) is the covariant derivative

\[ D_\mu = \partial_\mu + ie A_\mu. \]

We will perform an analysis of this theory at the classical level, similar to what was done in the lecture notes for a Dirac fermion coupled to a gauge field.

1. Under an infinitesimal global \( U(1) \) rotation the fields transform as

\[ \phi(x) \rightarrow e^{i\theta} \phi(x) \approx (1 + i\theta) \phi(x) \implies \delta\phi(x) = i\theta \phi(x) \]

\[ \delta\phi^*(x) = -i\theta \phi^*(x) \]

\[ A_\mu(x) \rightarrow A_\mu(x) \implies \delta A_\mu(x) = 0. \]

The corresponding change in the Lagrangian is

\[ \delta L = \frac{\delta L}{\delta \phi} \delta\phi + \frac{\delta L}{\delta \phi^*} \delta\phi^* + \frac{\delta L}{\delta (\partial_\mu \phi)} \delta \partial_\mu \phi + \frac{\delta L}{\delta (\partial_\mu \phi^*)} \delta \partial_\mu \phi^* \]

\[ = \partial_\mu \left[ \frac{\delta L}{\delta (\partial_\mu \phi)} \delta\phi + \frac{\delta L}{\delta (\partial_\mu \phi^*)} \delta\phi^* \right] \]

\[ = \theta \partial_\mu [i\phi D_\mu \phi^* - i\phi^* D_\mu \phi]. \]

where we used the Euler-Lagrange equations in passing to the second line. Since \( \delta L = 0 \) under a \( U(1) \) rotation, we conclude that the current

\[ j^\mu = i\phi (D_\mu \phi)^* - i\phi^* D_\mu \phi \]

is conserved.
2. We have that
\[ \partial_\mu j^\mu = 0 \implies \partial_0 j^0 = -\partial_i j^i. \] (0.11)

Integrating the above expression over all space and using Gauss’ Law, we find
\[ \partial_0 \int d^3x j^0 = -\int d^3x \partial_i j^i = 0, \] (0.12)
where we have assumed that the currents vanish at spatial infinity. So,
\[ Q \equiv \int d^3x j^0 = \int d^3x \left[ i\phi(D^0\phi)^* - i\phi^*D^0\phi \right] \] (0.13)
is a constant of motion.

3. Under a gauge transformation,
\[ F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda = F_{\mu\nu}, \] (0.14)
while
\[ D_\mu \phi \to (\partial_\mu + ieA_\mu + i\epsilon \partial_\mu \Lambda)(e^{i\theta} \phi) = e^{i\theta} D_\mu \phi + e^{i\theta}(i\partial_\mu \theta + i\epsilon \partial_\mu \Lambda)\phi. \] (0.15)

So, in order for this transformation to be a symmetry of the Lagrangian, we require
\[ \theta(x) = -e\Lambda(x). \] (0.16)

4. Under an infinitesimal gauge transformation, we have
\[ \phi(x) \to e^{-ie\Lambda} \phi(x) \approx (1 - ie\Lambda)\phi(x) \implies \delta \phi(x) = -ie\Lambda \phi(x) \] (0.17)
\[ \implies \delta \phi^*(x) = ie\Lambda \phi^*(x) \] (0.18)

\[ A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \implies \delta A_\mu(x) = \partial_\mu \Lambda(x). \] (0.19)

The variation of the Lagrangian under such a transformation is
\[ \delta \mathcal{L} = \delta \mathcal{L} + \delta \mathcal{L} \] (0.20)
\[ = \partial_\mu \left[ \frac{\delta \mathcal{L}}{\delta A_\mu} \delta \phi + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \delta (\partial_\mu \phi) + (\phi \leftrightarrow \phi^*) \right] + \frac{\delta \mathcal{L}}{\delta A_\mu} \delta A_\mu + \frac{\delta \mathcal{L}}{\delta (\partial_\mu A_\nu)} \delta (\partial_\mu A_\nu) \] (0.21)
\[ = \partial_\mu (\delta \mathcal{L} j^\mu) + \frac{\delta \mathcal{L}}{\delta A_\mu} j^\mu - F^{\mu\nu} \partial_\mu \Lambda \] (0.22)
\[ = -e\Lambda \partial_\mu j^\mu + \partial_\mu \Lambda \left[ -e j^\mu + \frac{\delta \mathcal{L}}{\delta A_\mu} \right] \] (0.23)
where \( j^\mu(x) \) is the conserved current defined in Eq. (0.10) and the third term in the penultimate equation vanished as the anti-symmetric tensor \( F^{\mu\nu} \) is contracted with a symmetric tensor. Since \( \Lambda \) can be chosen arbitrarily, both \( \partial_\mu j^\mu \) and the quantity in square brackets must vanish. The vanishing of the first term just tells us, as before, that \( j^\mu \) is conserved. We can define the gauge current as

\[
J^\mu(x) \equiv \frac{\delta L}{\delta A^\mu} = e j^\mu(x) = e \left[ i\phi(D^\mu \phi)^* - i\phi^* D^\mu \phi \right].
\]

(0.24)

This tells us that the current \( j^\mu \) couples to the gauge field \( A^\mu \) and carries gauge (electric) charge \( e \). The corresponding constant of motion is simply

\[
Q_{\text{gauge}} = e Q = e \int d^3 x \left[ i\phi(D^0 \phi)^* - i\phi^* D^0 \phi \right]
\]

(0.25)

which has the interpretation as the total electric charge.

5. The energy momentum tensor resulting from invariance of the Lagrangian due to space-time translations is given by

\[
\tilde{T}^{\mu\nu} = -g^{\mu\nu} L + \frac{\delta L}{\delta (\partial_\mu \phi)} \delta \partial_\mu \phi + \frac{\delta L}{\delta (\partial_\mu \phi^*)} \delta \partial_\mu \phi^* + \frac{\delta L}{\delta (\partial_\mu A_\nu)} \delta \partial_\mu A_\nu.
\]

(0.26)

Using the results from the previous sections we find

\[
\tilde{T}^{\mu\nu} = \tilde{T}^{\mu\nu}_{\text{EM}} + \tilde{T}^{\mu\nu}_M
\]

(0.27)

where

\[
\tilde{T}^{\mu\nu}_{\text{EM}} = \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\lambda} \partial^\nu A_\lambda
\]

(0.28)

\[
\tilde{T}^{\mu\nu}_M = g^{\mu\nu} \left[ -|D_\alpha \phi|^2 + m_0^2 |\phi|^2 + \frac{\lambda}{4!} |\phi|^4 \right] + (D^\mu \phi)^* \partial^\nu \phi + (D^\mu \phi) \partial^\nu \phi^*. \quad (0.29)
\]

Note that \( \tilde{T}^{\mu\nu} \) is neither symmetric nor gauge invariant. In order to remedy this, we recall that we can define an improved energy-momentum tensor (known as the Belifante energy-momentum tensor) \( T^{\mu\nu} = \tilde{T}^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \) where \( K^{\lambda\mu\nu} \) is anti-symmetric in \( \lambda \) and \( \mu \). If we choose \( K^{\lambda\mu\nu} = F^{\mu\lambda} A_\nu \) and use the equations of motion

\[
\partial_\lambda F^{\mu\lambda} = e \left[ i\phi(D^\mu \phi)^* - i\phi^*(D^\mu \phi) \right],
\]

(0.30)

we find

\[
\partial_\lambda K^{\lambda\mu\nu} = F^{\mu\lambda} \partial_\lambda A^\nu + A^\nu \left( ie\phi(D^\mu \phi)^* - ie\phi^*(D^\mu \phi) \right). \quad (0.31)
\]
Some rearrangement yields

\[ T^{\mu\nu} = T_{EM}^{\mu\nu} + T_{M}^{\mu\nu} \]

\[ T_{EM}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} + F^{\mu\lambda} F_{\lambda}^{\nu} \]

\[ T_{M}^{\mu\nu} = g^{\mu\nu} \left[ -(D_\alpha \phi)^* (D^\alpha \phi) + m_0^2 |\phi|^2 + \frac{\lambda}{4!} |\phi|^4 \right] + (D^\mu \phi)^* (D^\nu \phi) + (D^\mu \phi) (D^\nu \phi)^*. \]

(0.32)

Hence \( T^{\mu\nu} \) is manifestly symmetric and gauge-invariant. In the remainder of the problem set we will make use of the Belifante energy-momentum tensor, \( T^{\mu\nu} \). We will refer to \( \tilde{T}^{\mu\nu} \) as the canonical energy-momentum tensor to distinguish between the two when necessary. Note that because we used the equations of motion to derive the Belifante energy-momentum tensor, strictly speaking these expressions only hold at the classical level.

6. We first recall that the conjugate momenta are given by

\[ \Pi^* = \frac{\delta L}{\delta (\partial_t \phi)} = D^t \phi, \quad \Pi = \frac{\delta L}{\delta (\partial_t \phi)} = (D^t \phi)^*, \quad \Pi_{A\mu} = \frac{\delta L}{\delta (\partial_t A_\mu)} = F^{\mu t} = (0, E^i). \]

(0.33)

The Hamiltonian density\(^1\) is given by

\[ \mathcal{H} = T^{00} \]

\[ = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + F^{0\lambda} F_{\lambda}^0 + (D^0 \phi)^* (D^0 \phi) - (D_i \phi)^* (D^i \phi) + m_0^2 |\phi|^2 + \frac{\lambda}{4!} |\phi|^4 \]

\[ = \frac{1}{2} (E^2 + B^2) + \Pi \Pi^* - (D_i \phi)^* (D^i \phi) + m_0^2 |\phi|^2 + \frac{\lambda}{4!} |\phi|^4. \]

(0.34)

The first term is the electromagnetic energy density, the second and third terms correspond to energy costs associated with fluctuations of \( \phi \) in time and space (like a kinetic energy term), and the remaining terms are potential energy terms for \( \phi \). Due to our use of a modified energy-momentum tensor, the Gauss’ Law term which appeared in the previous problem set does not appear in this calculation. The momentum density

\(^1\) Strictly speaking, this is the energy density, not the Hamiltonian density (although often they are given by the same formal expression).
The first term – the Poynting vector – is the momentum carried by the electromagnetic field while the remaining terms represent the momentum carried by the scalar field.

7. Following the derivation in the notes, we have that under a general, infinitesimal coordinate transformation,

$$x_\mu \mapsto x_\mu' = x_\mu + \delta x_\mu \implies \frac{\partial x'_\mu}{\partial x_\nu} = g_{\mu \nu} + \partial_\nu \delta x_\mu. \quad (0.40)$$

So, to first order the Jacobian is given by

$$J = 1 + \partial_\mu \delta x_\mu + O(\delta x^2).$$

Using this and the equations of motion, we can write the variation of the action as

$$0 = \delta S = \int d^4x \left\{ \partial_\mu \left[ g^{\mu \nu} \mathcal{L} - \frac{\delta L}{\delta (\partial_\mu \phi)} \partial_\nu \phi - \frac{\delta L}{\delta (\partial_\mu \phi^*)} \partial_\nu \phi^* - \frac{\delta L}{\delta (\partial_\mu A_\nu)} \partial_\mu A_\nu \right] \right\} \quad (0.41)$$

$$+ \partial_\mu \left[ \frac{\delta L}{\delta (\partial_\mu \phi)} \delta T \phi + \frac{\delta L}{\delta (\partial_\mu \phi^*)} \delta T \phi^* + \frac{\delta L}{\delta (\partial_\mu A_\nu)} \delta T A_\nu \right], \quad (0.42)$$

where we have defined the total changes in the fields in terms of their functional changes and changes due to the coordinate transformation:

$$\delta T \phi = \delta \phi + \partial_\mu \delta \phi x^\mu, \quad \delta T \phi^* = \delta \phi^* + \partial_\mu \phi^* \delta x^\mu, \quad \delta T A_\mu = \delta A_\mu + \partial_\nu A_\mu \delta x^\nu. \quad (0.44)$$

Now, under an infinitesimal Lorentz transformation, we have that

$$x_\mu \mapsto x_\mu' = x_\mu + \omega_{\mu \nu} x^\nu$$

and so

$$\delta x_\mu = \omega_{\mu \nu}, \quad \delta \phi = \delta \phi^* = 0, \quad \delta A_\mu = \omega_{\mu \nu} A^\nu \quad (0.45)$$

since $\phi$ and $\phi^*$ are scalars while $A^\mu$ transforms as a four-vector. Plugging these expressions into the variation of the action and following the steps in the notes, we obtain

$$0 = \delta S = \int d^4x \left\{ \partial_\mu \left[ -\tilde{T}^\mu_\nu x_\rho - F^\mu_\nu A_\rho \right] \omega^{\nu \rho} \right\}. \quad (0.46)$$
Since $\omega^{\mu\nu}$ is anti-symmetric, the anti-symmetric part of the quantity in square brackets, namely,
\[
\tilde{M}^{\mu\nu\rho} \equiv \tilde{T}^{\mu\nu} x^\rho - \tilde{T}^{\mu\rho} x^\nu + F^{\mu\nu} A^\rho - F^{\mu\rho} A^\nu
\] (0.47)
must satisfy
\[
\partial_\mu \tilde{M}^{\mu\nu\rho} = 0. \tag{0.48}
\]
This conservation law implies, using $\partial_\mu \tilde{T}^{\mu\nu} = 0$
\[
0 = \partial_\mu \tilde{M}^{\mu\nu\rho} = \tilde{T}^{\nu\rho} - \tilde{T}^{\rho\nu} + \partial_\mu (F^{\mu\nu} A^\rho) - \partial_\mu (F^{\mu\rho} A^\nu) = T^{\mu\nu} - T^{\nu\mu}. \tag{0.49}
\]
It is easy to see that both the canonical and Belinfante energy-momentum tensors satisfy these equalities.

Now, focusing on the spatial components, we can define
\[
\tilde{L}_j \equiv \frac{1}{2} \int d^3 x \epsilon_{jkl} \tilde{M}^{0jk} = \int d^3 x \left[ \epsilon_{jkl} x^k \tilde{T}^{0l} + (E \times A)^j \right] \tag{0.50}
\]
which almost looks like an angular momentum, were it not for the fact that this expression is not manifestly gauge invariant. We can instead define
\[
M^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu
\] (0.51)
which is manifestly gauge invariant and also clearly satisfies $\partial_\mu M^{\mu\nu\rho} = 0$. We can then assign
\[
L^j = \frac{1}{2} \int d^3 x \epsilon_{jkl} M^{0jk} = \int d^3 x \epsilon_{jkl} x^k P_l \tag{0.52}
\]
the meaning of a physical angular momentum.

8. As in the previous problem set, we write $\phi(x) = \rho(x) e^{i\omega(x)}$, gauge fix $\omega(x) = 0$, and take $m_0^2 < 0$ so that $\rho$ acquires a non-zero vacuum expectation value $\rho_0$. We then expand $\rho$ as $\rho = \rho_0 + \delta \rho$ where $\delta \rho$ represents the fluctuations of $\rho$ about the ground state. The effective Lagrangian (as obtained in the previous problem set) is then
\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} A_\mu A_\nu - m_0^2 \rho^2 - \frac{\lambda}{4!} \rho^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \nonumber
\]
\[
= \frac{1}{2} (\partial_\mu \delta \rho)^2 + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\rho_0 + \delta \rho) A_\mu A_\nu - m_0^2 (\rho_0 + \delta \rho)^2 - \frac{\lambda}{4!} (\rho_0 + \delta \rho)^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{0.53}
\]
The conjugate momentum of $\delta \rho$ is therefore $\Pi_\rho = \delta \mathcal{L}/\delta (\partial_0 \delta \rho) = \partial_0 \rho$. 
(a) Recall that we found the gauge current to be
\[ J^\mu(x) \equiv \frac{\delta \mathcal{L}}{\delta A_\mu} = e j^\mu(x) = e \left[ i \phi (D^\mu \phi)^* - i \phi^* D^\mu \phi \right]. \tag{0.54} \]

Plugging in \( \rho = \rho_0 + \delta \rho \), we find
\[ J^\mu = 2 e^2 A^\mu (\rho_0 + \delta \rho)^2 \tag{0.55} \]

(b) Likewise, the Hamiltonian density\(^2\) is
\[
\mathcal{H} = \frac{1}{2} (E^2 + B^2) + m_0^2 (\rho_0 + \delta \rho)^2 + \frac{\lambda}{4!} (\rho_0 + \delta \rho)^4 \\
+ (\partial_0 \delta \rho)^2 + e^2 (A_0)^2 (\rho_0 + \delta \rho)^2 - \partial^i \delta \rho \partial_i \delta \rho + e^2 A_i A_i (\rho_0 + \delta \rho)^2 \\
= \frac{1}{2} (E^2 + B^2) + m_0^2 (\rho_0 + \delta \rho)^2 + \frac{\lambda}{4!} (\rho_0 + \delta \rho)^4 \\
+ \Pi_0^2 + e^2 (A_0)^2 (\rho_0 + \delta \rho)^2 - \partial^i \delta \rho \partial_i \delta \rho + e^2 A_i A_i (\rho_0 + \delta \rho)^2. \tag{0.56} \]

(c) Finally, the momentum density is
\[
P^k = (E \times B)^k + 2 (\partial^0 \rho)^2 (\partial^k \rho) + 2 e^2 A^0 A^k (\rho_0 + \delta \rho)^2 \\
= (E \times B)^k + 2 \Pi_0 (\partial^0 \rho) + 2 e^2 A^0 A^k (\rho_0 + \delta \rho)^2. \tag{0.57} \]

9. Under a Wick rotation,
\[ x_0 \to -i x_D, \quad \partial_0 = \frac{\partial}{\partial x_0} \to i \partial_D. \tag{0.59} \]

As for the gauge potential, \( A_\mu \), since it is a co-vector, it must transform in the same way as \( \partial_\mu \). Thus we have under a Wick rotation that
\[ A_0 \to i A_D, \quad D_0 \to i D_0. \tag{0.60} \]

Using these relations we find that
\[ iS = i \int d x_0 d^d x \mathcal{L}_M \to - \int d^D x \mathcal{L}_E \tag{0.61} \]

where
\[
\int d^D x \mathcal{L}_E = \int d^D x \left[ |\partial \phi|^2 + m_0^2 |\phi|^2 + \frac{\lambda}{4!} |\phi|^4 + \frac{1}{2} (E^2 + B^2) \right. \\
+ e^2 A_\alpha A_\alpha |\phi|^2 + e A_\alpha (i \phi \partial_\alpha \phi^* - i \phi^* \partial_\alpha \phi) \right] \\
= \int d^D x \left[ |\partial \phi|^2 + m_0^2 |\phi|^2 + \frac{\lambda}{4!} |\phi|^4 + \frac{1}{2} (E^2 + B^2) \right. \\
+ A_\alpha (i \phi (D_\alpha \phi)^* - i \phi^* D_\alpha \phi). \tag{0.64} \]

\(^2\) Again, strictly speaking, this is the energy density.
Here we have defined $|\partial \phi|^2 \equiv \partial_D \phi^* \partial_D \phi + |\nabla \phi|^2$. The first three terms correspond to the energy density of a massive scalar field, the fourth term the energy density of the electromagnetic field, and the remaining terms correspond to the usual $J \cdot A$ coupling of a conserved current to the electromagnetic field. Note that since we started with $D = d + 1$-dimensional Minkowski spacetime, our Wick rotated Euclidean theory has $D$ spatial dimensions.