1. Spin Waves in a Quantum Heisenberg Antiferromagnet

Consider the 1d Heisenberg chain

\[ H = J \sum_{x=-N/2+1}^{N/2} S(x) \cdot S(x+1), \]

where the spins \( S(x) = (S_1(x), S_2(x), S_3(x)) \) are the angular momentum operators in the spin-\( S \) representation of \( SU(2) \),

\[ [S_i(x), S_j(x')] = \delta_{xx'} i\varepsilon_{ijk} S_k(x). \]

In this problem, we will consider the antiferromagnetic case, \( J > 0 \), with periodic boundary conditions \( S(x+N) = S(x) \). We can thus write down a basis for our Hilbert space \( \{ |S, M(x)\rangle \} \), where

\[ S^2(x)|S, M(x)\rangle = S(S+1)|S, M(x)\rangle, \quad S_3(x)|S, M(x)\rangle = M(x)|S, M(x)\rangle, \]

and \( M(x) = -S, -S+1, \ldots, S-1, S \).

1. Let us define the raising and lowering operators

\[ S^\pm(x) = S_1(x) \pm iS_2(x). \]

Then clearly

\[ [S^+(x), S^-(x')] = 2\delta_{xx'} S_3(x), \quad [S_3(x), S^\pm(x')] = \delta_{xx'} S^\pm(x). \]
In terms of these operators, we can write the Hamiltonian as

\[ H = J \sum_{x=-N/2+1}^{N/2} \left[ \frac{1}{2} (S^+(x)S^-(x+1) + S^-(x)S^+(x+1)) + S_3(x)S_3(x+1) \right] \]. (1.6)

The Heisenberg equations of motion for \( S^\pm, S_3 \) are

\[-i\partial_t S^\pm (x, t) = [H, S^\pm (x)]\]

\[-i\partial_t S_3 (x, t) = [H, S_3 (x)]\]

\[-i\partial_t S_3 (x, t) = \frac{J}{2} \left[ (S^- (x+1) + S^- (x-1)) S^+ (x) + (S^+ (x-1) + S^+ (x+1)) S^- (x) \right]. \] (1.7)

These equations are nonlinear.

2. Let us define the spin deviation operator \( n(x) = S + (-1)^{1+x} S_3(x) \). Then the basis we chose for our Hilbert space above are eigenstates of \( n(x) \), so we can rename them \( \{ |n(x)\rangle \} \). For even sites, \( S^\pm \) act on this basis as

\[ S^+ |n\rangle = \left[ 2S \left( 1 - \frac{n-1}{2S} \right) \right]^{1/2} |n-1\rangle, \]

\[ S^- |n\rangle = \left[ 2S(n+1) \left( 1 - \frac{n}{2S} \right) \right]^{1/2} |n+1\rangle, \] (1.9)

where we have left the \( x \)-dependence implicit. For odd sites, the action of these operators is interchanged. These expressions have the flavor of the analogous relations for harmonic oscillator creation and annihilation operators

\[ a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle. \] (1.11)

Indeed, we can define operators \( a^\dagger, a \) (on even sites), where \([a(x), a^\dagger (x')] = \delta_{xx'}\) so that the above equations are satisfied. These operators are related to \( S^\pm, S_3 \) as follows

\[ S^+ = \sqrt{2S} \left[ 1 - \frac{n}{2S} \right]^{1/2} a, \]

\[ S^- = \sqrt{2S} a^\dagger \left[ 1 - \frac{n}{2S} \right]^{1/2}, \]

\[ n = a^\dagger a. \] (1.12)

For odd sites, we can define operators \( b, b^\dagger \) which satisfy the above equations with \( S^+ \) and \( S^- \) exchanged. Applying these equations to a state |\( n \rangle\), we can immediately
see that they are consistent with (1.9)-(1.10) (note that you still need to show this explicitly in your problem set to receive full credit).

3. We can now rewrite the Heisenberg Hamiltonian in terms of \( a \) and \( b \) operators (and shifting the sum to start at \( x = 0 \))

\[
H = JS \sum_{\langle x_A, y_B \rangle} \left[ a^\dagger(x_A) b^\dagger(y_B) \sqrt{\left( 1 - \frac{n(x_A)}{2S} \right) \left( 1 - \frac{n(y_B)}{2S} \right)} + \text{h.c.} \right] - S \left( 1 - \frac{n(x_A)}{S} \right) \left( 1 - \frac{n(y_B)}{S} \right),
\]

(1.15)

where we will use \( \langle x_A, y_B \rangle \) to denote nearest neighbor pairs with \( x \) even (A sublattice) and \( y \) odd (B sublattice).

4. In the limit \( S \to \infty \),

\[
H = -NJ S^2 + JS \sum_{\langle x_A, y_B \rangle} \left[ a^\dagger(x_A) b^\dagger(y_B) + a(x_A) b(y_B) + a^\dagger a(x_A) + b^\dagger b(y_B) + \mathcal{O}(S^{-1}) \right].
\]

(1.16)

This Hamiltonian is indeed quadratic in \( a \) and \( b \)!

5. The equations of motion are now linear,

\[
-i \partial_t a(x,t) = [H, a] = JS \left[ b^\dagger(x+1) + b^\dagger(x-1) + 2a(x) + \mathcal{O}(S^{-1}) \right],
\]

(1.17)

\[
-i \partial_t b(x,t) = [H, b] = JS \left[ a^\dagger(x+1) + a^\dagger(x-1) + 2b(x) + \mathcal{O}(S^{-1}) \right],
\]

(1.18)

where the equations for \( a^\dagger \) and \( b^\dagger \) are simply the Hermitian conjugates of those above. The funny factors of 2 come from the fact that a particular site \( x \) is a member of 2 nearest neighbor bonds.

6. We can expand \( a, b \) in Fourier components

\[
a(x_A) = \sqrt{\frac{2}{N}} \sum_q e^{iqx_A} a(q), \quad b(y) = \sqrt{\frac{2}{N}} \sum_p e^{-ipy} b(p),
\]

(1.19)

then the Hamiltonian becomes

\[
H = -NJ S^2 + 2JS \sum_q \left[ a^\dagger(q) a(q) + b^\dagger(q) b(q) + \cos(q) (a(q) b(q) + a^\dagger(q) b^\dagger(q)) \right],
\]

(1.20)
where the \( \cos(q) \) factor comes from the fact
\[
\frac{2}{N} \sum_p \sum_{(x_A,y_B)} e^{i(q x_A - p y_B)} = \frac{2}{N} \sum_p \sum_{x_A} (e^{i(q x_A - p(x_A + 1))} + e^{i(q x_A - p(x_A - 1))})
\]
\[
= 2 \sum_p \delta_{pq} \cos(p)
\]
\[
= 2 \cos(q).
\]

We can rewrite the Hamiltonian in matrix form as
\[
H = -N J S^2 + 2 J S \sum_q \left( \begin{array}{c} a^\dagger(q) \\ b(q) \end{array} \right) h(q) \left( \begin{array}{c} a(q) \\ b^\dagger(q) \end{array} \right),
\]
where
\[
h(q) = \left( \begin{array}{cc} 1 & \cos(q) \\ \cos(q) & 1 \end{array} \right).
\]

The matrix \( h(q) \) can be made diagonal via the Bogoliubov transformation
\[
\left( \begin{array}{c} c(q) \\ d^\dagger(q) \end{array} \right) = \left( \begin{array}{cc} \cosh \theta_q & \sinh \theta_q \\ \sinh \theta_q & \cosh \theta_q \end{array} \right) \left( \begin{array}{c} a(q) \\ b^\dagger(q) \end{array} \right),
\]
meaning that \( h \) transforms as
\[
h(q) \mapsto \left( \begin{array}{cc} \cosh(2\theta_q) - \cos(q) \sinh(2\theta_q) & \cos(q) \cosh(2\theta_q) - \sinh(2\theta_q) \\ \cos(q) \cosh(2\theta_q) - \sinh(2\theta_q) & \cosh(2\theta_q) - \cos(q) \sinh(2\theta_q) \end{array} \right),
\]
where we have used the identities \( \cosh^2 \theta + \sinh^2 \theta = \cosh(2\theta) \) and \( 2 \sinh \theta \cosh \theta = \sinh(2\theta) \). We wish to choose \( \theta_q \) so that the off-diagonal entries of this matrix vanish, i.e.
\[
\tanh(2\theta_q) = \cos(q).
\]
Now, using the fact that
\[
\text{sech}^2(2\theta_q) = 1 - \tanh^2(2\theta_q) \Rightarrow \cosh(2\theta_q) = \frac{1}{\sqrt{1 - \cos^2(q)}},
\]
we finally obtain
\[
H = -N J S^2 + \int_{-\pi/2}^{\pi/2} dq \frac{\omega(q)}{2\pi} (c^\dagger(q)c(q) + d^\dagger(q)d(q)),
\]
where
\[
\omega(q) = 2 J S \sqrt{1 - \cos^2(q)} = 2 J S |\sin(q)|
\]
and have replaced the sum over $q$ with an integral over the Brillouin zone $[-\pi/2, \pi/2]$ (due to the fact that the unit cell of the antiferromagnet has size 2). We now have a quantum theory of spin waves with linear dispersion!

7. The ground state for this system corresponds to the set of states for which the spin wave terms contribute zero energy, i.e.

$$E_0 = -NJS^2.$$  

(1.32)

8. We can organize our Hilbert space into eigenstates of the number operators $c^\dagger c(q) \equiv n_c(q)$ and $d^\dagger d(q) \equiv n_d(q)$. The single particle eigenstates are the states $|1(q), 0\rangle = c^\dagger(q)|0, 0\rangle$ and $|0, 1(q)\rangle = d^\dagger(q)|0, 0\rangle$. These states have energy

$$\epsilon(q) = 2JS|\sin(q)|,$$  

(1.33)

above the ground state, meaning that states with momentum 0 have energy equal to that of the vacuum state. Near this point, the dispersion is linear in $q$

$$\epsilon(q) \approx 2JS|q|,$$  

(1.34)

so the spin wave velocity near this point is

$$v_s = \left. \frac{\partial \epsilon(q)}{\partial |q|} \right|_{q=0} = 2JS.$$  

(1.35)

2. THE TWO-COMPONENT COMPLEX SCALAR FIELD

Consider a theory of two species of complex scalar field$^1$,

$$\mathcal{L} = |\partial \phi_I|^2 - m^2_0|\phi_I|^2, \ I = 1, 2.$$  

(2.1)

This Lagrangian is invariant under global $SU(2)$ rotations

$$\phi_I \rightarrow U_{IJ}(\theta)\phi_J = \exp(i\theta^a\sigma_{IJ}^a)\phi_J, \ \phi_I^\dagger \rightarrow \phi_J^\dagger U_{IJ}^\dagger(\theta) = \phi_J^\dagger \exp(-i\theta^a\sigma_{IJ}^a), \ a = x, y, z.$$  

(2.2)

1. The canonical momentum $\Pi_I$ conjugate to $\phi_I$ is what you would expect,

$$\Pi_I = \frac{\delta \mathcal{L}}{\delta (\partial_t \phi_I)} = \partial^t \phi_I^\dagger,$$  

(2.3)

$^1$ Note that the problem set states the Lagrangian with an extra overall factor of $1/2$. This is not the usual convention for a complex scalar field, and so we drop it here.
and so the Hamiltonian (density) is

\[ \mathcal{H} = \Pi_I \partial_t \phi_I + \pi^I \partial_t \phi^I - \mathcal{L} = |\Pi_I|^2 - \left( \partial_i \phi_I^\dagger \right) \left( \partial^i \phi_I \right) + m_0^2 |\phi_I|^2 \]  \quad (2.4)

and the momentum density is

\[ \mathcal{P}^i = \Pi_I \partial_i \phi_I + \text{h.c.} \]  \quad (2.5)

This can be derived from the stress tensor as in the lecture notes – the only difference is the presence of the index \( I \).

2. Infinitesimal \( SU(2) \) rotations map

\[ \phi_I \mapsto \phi_I + i \delta \theta^a \sigma^a_{IJ} \phi_J, \quad \phi_I^\dagger \mapsto \phi_I^\dagger - i \delta \theta^a \sigma^a_{IJ} \phi_J^\dagger, \]  \quad (2.6)

and so the conserved current associated with these rotations is

\[ J^\mu_{,a} = - \left( \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_I)} \frac{\delta \phi_I}{\delta \theta^a} + \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_I^\dagger)} \frac{\delta \phi_I^\dagger}{\delta \theta^a} \right) = i \left( \phi_I^\dagger \sigma^a_{IJ} \partial^\mu \phi_J - \partial^\mu \phi_I^\dagger \sigma^a_{IJ} \phi_J \right). \]  \quad (2.7)

The minus sign is a convention that will be useful below. Notice how the indices are ordered. Messing up this ordering would lead to problems below.

Thus, we find that there are three conserved charges, one for each generator of \( SU(2) \) (i.e. Pauli matrix)

\[ Q^a = \int d^3x \, J^0_{,a}. \]  \quad (2.8)

Notice that the current \( J^\mu_{,a} \) looks a lot like the global \( U(1) \) current we saw in class. The only difference is that it involves the generators \( \sigma^a \).

3. We can quantize the theory by promoting \( \phi \) and \( \pi \) to operators on Hilbert space, \( \hat{\phi} \) and \( \hat{\Pi} \) respectively, and imposing equal-time canonical commutation relations

\[ [\hat{\phi}_I(x, t), \hat{\Pi}_J(x', t)] = i \delta_{IJ} \delta(x - x'). \]  \quad (2.9)

The quantum mechanical Hamiltonian and total momentum are thus obtained by plugging these hatted fields into the expressions we obtained classically in part (1)

\[ \hat{\mathcal{H}} = \int d^3x \, \mathcal{H}(\hat{\phi}_I, \hat{\Pi}_I), \]  \quad (2.10)

\[ \hat{\mathcal{P}}^i = \int d^3x \, \mathcal{P}^i(\hat{\phi}_I, \hat{\Pi}_I). \]  \quad (2.11)
4. The quantum mechanical generators of the global $SU(2)$ symmetry are obtained by plugging hatted fields into the charges $Q^a$ obtained classically,

$$\dot{Q}^a = Q^a(\hat{\phi}_I, \hat{\Pi}_I) = \int d^3x \left[ i(\hat{\phi}_I^\dagger \sigma^a_{IJ} \hat{\Pi}_J^\dagger - \hat{\Pi}_I \sigma^a_{IJ} \hat{\phi}_J) \right].$$

These operators clearly generate a $SU(2)$ algebra,

$$\left[ \dot{Q}^a, \dot{Q}^b \right] = 2i\varepsilon^{abc} \dot{Q}^c$$

which follows from the fact (suppressing the $t$ dependence of the $\hat{\phi}$’s and $\hat{\Pi}$’s)

$$[\hat{\Pi}_I \hat{\phi}_J(x), \hat{\Pi}_K \hat{\phi}_L(x')] = i(\hat{\Pi}_I \hat{\phi}_L(x) \delta_{JK} - \hat{\Pi}_K \hat{\phi}_J(x) \delta_{IL}) \delta(x - x'),$$

meaning

$$-[\hat{\Pi}_I \sigma^a_{IJ} \hat{\phi}_J(x), \hat{\Pi}_K \sigma^b_{KL} \hat{\phi}_L(x')] = -i(\hat{\Pi}_I \hat{\phi}_L(x) \sigma^a_{IJ} \sigma^b_{KL} - \hat{\Pi}_K \hat{\phi}_J(x) \sigma^a_{IJ} \sigma^b_{KL}) \delta(x - x')$$

$$= -i\phi_I [\sigma^a, \sigma^b]_{IJ} \hat{\Pi}_J(x) \delta(x - x')$$

$$= +2i\varepsilon^{abc}[\hat{\Pi}_I \sigma^c_{IJ} \hat{\phi}_J(x)] \delta(x - x').$$

Thus,

$$\left[ \dot{Q}^a, \dot{Q}^b \right] = \int d^3x d^3x' \delta(x - x') 2i\varepsilon^{abc} \left[ i(\hat{\phi}_I^\dagger \sigma^c_{IJ} \hat{\Pi}_J^\dagger(x) - \hat{\Pi}_I \sigma^c_{IJ} \hat{\phi}_J(x)) \right]$$

$$= 2i\varepsilon^{abc} \dot{Q}^c. \quad (2.17)$$

This is why the minus sign we introduced above is useful: we would have otherwise obtained a $SU(2)$ algebra with a different sign on the right hand side from the one satisfied by the Pauli matrices. We can further convince ourselves that the $\dot{Q}^a$’s generate the global $SU(2)$ symmetry by noting that under infinitesimal $SU(2)$ rotations, $\hat{\phi}$ transforms as

$$\hat{\phi}_I \mapsto (\delta_{IJ} + i\delta \sigma^a_{IJ}) \hat{\phi}_J = \hat{\phi}_I - i\delta \sigma^a_{IJ} \left[ Q^a, \hat{\phi}_I \right] = e^{-i\delta \sigma^a_{IJ} Q^a} \hat{\phi}_I e^{i\delta \sigma^a_{IJ} Q^a}. \quad (2.18)$$

It is also straightforward to check that the $\dot{Q}^a$’s commute with the Hamiltonian.

5. The Heisenberg equations of motion are (continuing to suppress time dependence)

$$\partial_t \hat{\phi}_I = -i \left[ \hat{\phi}_I(x), \hat{H} \right] = -i \int d^3x' \hat{\Pi}_J^\dagger [\hat{\Pi}_J(x'), \hat{\phi}_I(x)] + 0 = \hat{\Pi}_I(x), \quad (2.19)$$

$$\partial_t \hat{\Pi}_I = -i \left[ \hat{\Pi}_I(x), \hat{H} \right] = -i \int d^3x' \left( -\partial_t \hat{\phi}_J^\dagger [\hat{\Pi}_J(x), \partial_i \hat{\phi}_I(x')] + m_0^2 \hat{\phi}_J \left[ \hat{\Pi}_I(x), \hat{\phi}_J(x') \right] \right)$$

$$= (-\partial_i \partial_t - m_0^2) \hat{\phi}_I(x), \quad (2.20)$$
where we have integrated by parts to obtain the last equality, and we note that any funny looking signs come from the fact that we are keeping track of raised and lowered indices (i.e. $\partial_i \partial^i = -\nabla^2$ since we are using the $(+, -, -, -)$ metric signature). The first equation is just the classical equation for the canonical momentum. Plugging it into the second, we recover the Klein-Gordon equation,

$$ (\partial_\mu \partial^\mu + m_0^2) \hat{\phi}_I = 0. \quad (2.21) $$

6. We now pass to a representation of the $\hat{\phi}_I$ operators in terms of creation and annihilation operators via the mode expansion

$$ \hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(k)} \left[ \hat{a}_I(k) e^{-ik_\mu x^\mu} + \hat{b}_I^\dagger(k) e^{ik_\mu x^\mu} \right], \quad (2.22) $$

where $k^\mu = (\omega(k), k)$ and

$$ \omega(k) = \sqrt{|k|^2 + m_0^2} \quad (2.23) $$

is the Klein-Gordon dispersion.

We have introduced four sets of creation and annihilation operators, two for each species of complex scalar field. These operators satisfy the canonical commutation relations

$$ [\hat{a}_I(k), \hat{a}_J^\dagger(k')] = [\hat{b}_I(k), \hat{b}_J^\dagger(k')] = (2\pi)^3 2\omega(k) \delta_{IJ} \delta(k-k'), \quad (2.24) $$

where all other combinations of $\hat{a}$’s and $\hat{b}$’s commute. It is easy to check that this ansatz reproduces the canonical commutation relations for the $\hat{\phi}$ and $\hat{\Pi}$ fields.

7. We now write the $SU(2)$ charges in terms of creation and annihilation operators. We first note

$$ \hat{\Pi}_I = \int \frac{d^3k}{2(2\pi)^3} i \left[ \hat{a}_I^\dagger(k) e^{ik_\mu x^\mu} - \hat{b}_I(k) e^{-ik_\mu x^\mu} \right], \quad (2.25) $$

and so, suppressing species indices and hats and adopting the notation $\int_x \equiv \int d^3x, \int_k \equiv \int \frac{d^3k}{(2\pi)^3},$

$$ -i \int_x \hat{\Pi}_I \sigma^a \hat{\phi} = \int_{k,k',x} \frac{1}{2\omega(k)} \left[ \right. \left( a^\dagger(k') \sigma^a a(k) - b(k') \sigma^a b^\dagger(k) \right) e^{i(k_\mu - k'_\mu) x^\mu} \\
\left. + (a^\dagger(k') \sigma^a b^\dagger(k) e^{i(k_\mu + k'_\mu) x^\mu} - b(k') \sigma^a a(k) e^{-i(k_\mu + k'_\mu) x^\mu} \right]. \quad (2.26) $$

Now we use the fact

$$ \int_x e^{iq_\mu x^\mu} = (2\pi)^3 e^{i\omega(q)t} \delta(q), \quad (2.27) $$
\[-i \int_x \hat{\Pi} \sigma^\alpha \phi = \int_k \frac{1}{2\omega(k)} \left[ a^\dagger(k)\sigma^\alpha a(k) - b(k)\sigma^\alpha b^\dagger(k) \right. \\
\left. + (a^\dagger(k)\sigma^\alpha b^\dagger(-k)e^{2\omega(k)t} - b(k)\sigma^\alpha a(-k)e^{-2\omega(k)t}) \right]. \quad (2.28)\]

Similarly,
\[+i \int_x \hat{\Pi}^\dagger \sigma^\alpha \phi^\dagger = \int_k \frac{1}{2\omega(k)} \left[ a^\dagger(k)\sigma^\alpha a(k) - b(k)\sigma^\alpha b^\dagger(k) \right. \\
\left. - (a^\dagger(-k)\sigma^\alpha b^\dagger(k)e^{2\omega(k)t} - b(-k)\sigma^\alpha a(k)e^{-2\omega(k)t}) \right]. \quad (2.29)\]

The terms with explicit time dependence cancel, leading to
\[\hat{Q}^a = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(k)} \left[ a^\dagger(k)\sigma^\alpha a(k) - b(k)\sigma^\alpha b^\dagger(k) \right]. \quad (2.30)\]

8. The ground state of the system is the vacuum $|0\rangle$, which is annihilated by $\hat{a}_I$ and $\hat{b}_I$. We can obtain the normal ordered Hamiltonian and momentum operators relative to this state in exactly the same way as for a single component complex scalar field. The only difference is the presence of the index $I$,
\[\hat{H} = \int \frac{d^3k}{(2\pi)^3} \omega(k)(\hat{a}_I^\dagger \hat{a}_I + \hat{b}_I^\dagger \hat{b}_I), \quad (2.31)\]
\[\hat{P}_i = \int \frac{d^3k}{(2\pi)^3} \omega(k) k_i (\hat{a}_I^\dagger \hat{a}_I + \hat{b}_I^\dagger \hat{b}_I), \quad (2.32)\]

and the normal ordered charge operator is
\[\hat{Q}^a = \int d^3k \frac{1}{\omega(k)} [a^\dagger(k)\sigma^\alpha a(k) - b^\dagger(k)\sigma^\alpha b(k)]. \quad (2.33)\]

9. The single-particle states are
\[\hat{a}_1^\dagger(k)|0\rangle, \hat{a}_2^\dagger(k)|0\rangle, \hat{b}_1^\dagger(k)|0\rangle, \hat{b}_2^\dagger(k)|0\rangle, \quad (2.34)\]
each of these states has energy
\[\langle : \hat{H} : \rangle = \omega(k) = \sqrt{|k|^2 + m_0^2}. \quad (2.35)\]

This can be seen by invoking the canonical commutation relations for the creation and annihilation operators. The single particle excited states thus form the $j = 1/2$ representation of $SU(2)$, leading to a twofold species degeneracy for the particle and antiparticle states.