1. PATH INTEGRAL FOR A PARTICLE IN A DOUBLE POTENTIAL WELL

1. In real time,

\[ \langle q_0, T/2 | q_0, -T/2 \rangle = \langle q_0 | e^{-\frac{i}{\hbar}HT} | q_0 \rangle = \int_{q(t_f) = q_0} Dq e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q, \dot{q})} \]  

(1.1)

where

\[ L = \frac{1}{2} m \dot{q}^2 - \lambda (q^2 - q_0^2)^2. \]  

(1.2)

On Wick rotating \( t \mapsto -i \tau \), we obtain the imaginary time expression

\[ \langle q_0, T/2 | q_0, -T/2 \rangle = \int Dq e^{\frac{i}{\hbar} \int_{t_i}^{t_f} d\tau L_E(q, \dot{q})} \]  

(1.3)

where

\[ L_E = \frac{1}{2} m \dot{q}^2 + \lambda (q^2 - q_0^2)^2. \]  

(1.4)

So, in imaginary time, the potential is \( V(q) = -\lambda (q^2 - q_0^2)^2 \) which is an inverted double potential well.

2. In real time, the equations of motion are readily found to be

\[ m \ddot{q} = -4\lambda q (q^2 - q_0^2) \quad \text{(real time)} \]  

(1.5)

while, in imaginary time,

\[ m \ddot{q} = 4\lambda q (q^2 - q_0^2) \quad \text{(imaginary time).} \]  

(1.6)

Again, we see that the EOM in imaginary time describe a point particle moving in an inverted potential well.

In order to find the trajectory in imaginary time, we make use of the conservation of “energy”,

\[ E = T + V = \frac{1}{2} m \dot{q}^2 - \lambda (q^2 - q_0^2)^2 \]  

(1.7)
with the initial and final conditions \( q(-T/2) = -q_0 \) and \( q(T/2) = q_0 \), respectively. This expression is obtained by multiplying Eq. (1.6) by \( \dot{q} \) and integrating with respect to \( \tau \).

Now, we are interested in the limit \( T \to \infty \). The initial/final conditions \( q(\pm T/2 \to \infty) = \pm q_0 \) can only be enforced if we also require that \( \dot{q}(\pm T/2 \to \infty) = 0 \) (otherwise the particle would travel past \( \pm q_0 \) and not be able to return). So,

\[
E_i = E_f = \frac{1}{2} m(0)^2 - \lambda((\pm q_0)^2 - q_0^2)^2 = 0. 
\]

(1.8)

Thus, in general,

\[
0 = \frac{1}{2} m\dot{q}^2 - \lambda(q^2 - q_0^2)^2 \implies \dot{q} = \pm \sqrt{\frac{2\lambda}{m}(q^2 - q_0^2)^2}. 
\]

(1.9)

This ODE has solutions

\[
q_c(\tau) = \mp q_0 \tanh \left[ q_0 \sqrt{\frac{2\lambda}{m}} (\tau - \tau_0) \right]. 
\]

(1.10)

In order to satisfy \( q(\pm \infty) = \pm q_0 \), we must choose the positive root in the last equation. This solution is unique up to the choice of \( \tau_0 \) as it is the solution to a second order ODE with initial/final conditions for \( q \) and \( \dot{q} \) specified.

In real time, this trajectory would be classically forbidden. In particular, the amplitude (in real time) we calculated corresponds to the amplitude for the particle to tunnel from the well centred at \(-q_0\) to the well at \(q_0\).

3. Without loss of generality, we can set \( \tau_0 = 0 \) in the imaginary time trajectory. We compute,

\[
S[q_c] = \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{2} \dot{q}_c^2 + \lambda(q_c^2 - q_0^2)^2 \right] = \frac{4}{3} q_0^3 \sqrt{2m\lambda}. 
\]

(1.11)

4. We write \( q(\tau) = q_c(\tau) + \xi(\tau) \) where \( \xi(\tau) \) satisfies the initial/final conditions \( \xi(\pm \infty) = 0 \) and \( q_c(\tau) = q_0 \tanh \left[ q_0 \sqrt{\frac{2\lambda}{m}} (\tau - \tau_0) \right] \) is the classical solution to the EOM obtained above. Now, expanding the action to quadratic order in \( \xi(\tau) \), we find

\[
S \approx \int \frac{1}{2} m\ddot{q}_c^2 - V(q_c) d\tau + \int \frac{1}{2} m\dot{\xi}^2 - \frac{1}{2} mV''(q_c)\xi^2 d\tau - \int (m\ddot{q}_c + V'(q_c))\xi d\tau. 
\]

(1.12)

The last integral vanishes as \( q_c \) satisfies the EOM. So, we can write

\[
S[q, \dot{q}] \approx S[q_c, \dot{q}_c] + S_{\text{eff}}[\xi, \dot{\xi}]. 
\]

(1.13)
Since the effective action is quadratic in $\xi$, we can absorb the factor of $1/\hbar$ in the path integral into $\xi$ by defining $\xi = \sqrt{\hbar}\tilde{\xi}$. Dropping the tilde on $\tilde{\xi}$ to lighten notation, we have that

$$
\lim_{T \to \infty} \langle q_0, T/2 | -q_0, -T/2 \rangle = e^{-S_c/\hbar} \int_{\xi(\pm\infty)=0} e^{-S_{\text{eff}}[\xi,\dot{\xi}]}.
$$

(1.14)

On integrating by parts, we can write

$$
S_{\text{eff}}[\xi, \dot{\xi}] = \frac{1}{2} \int \xi \left[ -m \frac{d^2}{d\tau^2} - V''(q_c) \right] \xi d\tau \equiv \frac{1}{2} \int \xi \hat{A} \xi d\tau.
$$

(1.15)

Using the explicit form of $q_c(\tau)$, we can express the differential operator $\hat{A}$ as

$$
\hat{A} = -m \frac{d^2}{d\tau^2} + 12\lambda q_0^2 \tanh \left[ q_0 \sqrt{\frac{2\lambda}{m}} \tau \right] - 4\lambda q_0^2.
$$

(1.16)

In order to evaluate the path integral, we follow the standard procedure and expand $\xi$ in orthonormal eigenstates, $\psi_n$, of $\hat{A}$ with eigenvalues $A_n$: $\xi(\tau) = \sum_n c_n \psi_n(\tau)$. Thus,

$$
\int_{-\infty}^{\infty} \xi \dot{\xi} d\tau = \frac{1}{2} \sum_n c_n^2 A_n.
$$

(1.17)

Defining the integration measure as $D\xi = N \prod_n \frac{dc_n}{\sqrt{2\pi}}$ we find that

$$
\int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}A_n c_n^2} = [A_n]^{-1/2}
$$

(1.18)

and so

$$
\int_{\xi(\pm\infty)=0} e^{-S_{\text{eff}}[\xi,\dot{\xi}]} = \text{Det}[\hat{A}]^{-1/2}
$$

(1.19)

where the explicit form of $\hat{A}$ is given in Eq. (1.16).

### 2. PATH INTEGRAL FOR A CHARGED PARTICLE MOVING ON A PLANE IN THE PRESENCE OF A PERPENDICULAR MAGNETIC FIELD

1. The derivation of the path integral follows that in the free particle case very closely and so we will be a little terse. We have that

$$
\hat{H} = \frac{1}{2m}(p + \frac{e}{c} A)^2 = \frac{p^2}{2m} + \frac{e^2}{2mc^2} A^2 + \frac{e}{2mc} (p \cdot A + A \cdot p).
$$

(2.1)
Now, the transition amplitude is given by
\[ F(q_f, t_f | q_i, t_i) = \int d^2 q \langle q_f, t_f | q', t' \rangle \langle q', t' | q_i, t_i \rangle \] (2.2)
\[ = \int d^2 q_1 \ldots d^2 q_N \langle q_f, t_f | q_N, t_N \rangle \langle q_N, t_N | q_{N-1}, t_{N-1} \rangle \ldots \langle q_1, t_1 | q_i, t_i \rangle \] (2.3)
where we have inserted resolutions of identity at intermediate times. Since the Hamiltonian is time-independent, we can approximate
\[ \langle q_j, t_j | q_{j-1}, t_{j-1} \rangle = \langle q_f | e^{-iH(t_j-t_{j-1})/\hbar} | q_N \rangle \] (2.4)
\[ = \delta(q_j - q_{j-1}) - i\frac{\Delta t}{\hbar} \langle q_j | H | q_{j-1} \rangle + O(\Delta t^2). \] (2.5)

Now, inserting a resolution of identity in terms of momentum eigenstates, we find
\[ \langle q_j | H | q_{j-1} \rangle = \int \frac{d^2 p_j}{(2\pi\hbar)^2} e^{i p_j \cdot (q_j - q_{j-1})/\hbar} \left\{ 1 - i\frac{\Delta t}{2m\hbar} \left[ \frac{p_j^2}{c^2} + \frac{e^2}{c} A(r_j)^2 + \frac{e}{c} p_j \cdot A(r_{j-1}) + \frac{e}{c} p_j \cdot A(r_j) \right] + O(\Delta t^2) \right\}. \] (2.7)

We now adopt the mid-point rule and replace \( r_j, r_{j-1} \mapsto (r_j + r_{j-1})/2 \) (note that in the derivation of the path integral for a free particle, terms depending on \( r_{j-1} \) did not appear). Plugging this back into Eq. (2.5), re-exponentiating, and then plugging the result back into Eq. (2.3), we find
\[ F(q_f, t_f | q_i, t_i) = \lim_{N \to \infty} \int \prod_{j=1}^N d^2 q_j \int \prod_{j=1}^{N+1} d^2 p_j \exp \left\{ \frac{i\Delta t}{\hbar} \sum_{j=1}^{N+1} \left[ p_j \cdot (q_j - q_{j-1})/\Delta t - H \left( p_j, \frac{q_j + q_{j-1}}{2} \right) \right] \right\}. \] (2.9)

In the limit \( N \to \infty \), we find
\[ F(q_f, t_f | q_i, t_i) = \int \mathcal{D}p \mathcal{D}q e^{\int_{t_i}^{t_f} dt [p \cdot \dot{q} - H(p,q)]} \] (2.10)
where
\[ \mathcal{D}p \mathcal{D}q = \lim_{N \to \infty} \prod_{j=1}^N \frac{d^2 q_j d^2 p_j}{(2\pi\hbar)^2}. \] (2.11)
In order to integrate out $p$ (which will amount to effecting a Legendre transform), we must discretize the path integral again. We have that

$$\int d^2p e^{i\int p \cdot (q - \frac{e}{m}A(q)) - \frac{p^2}{2m} - \frac{e^2}{2mc^2} A^2} = \left( \frac{m}{2\pi i \hbar \Delta t} \right)^{\frac{1}{2}} e^{i \frac{\Delta t}{\hbar} \left[ \frac{1}{2}m(q - \frac{e}{m}A)^2 - \frac{e^2}{2mc^2} A^2 \right]}.$$

Thus,\[ (2.12) \]

$$F(q_f, t_f | q_i, t_i) = \int_{q(t_f) = q_{f}}^{q(t_i) = q_i} e^{\frac{i}{\hbar} \int_{t_i}^{t_f} L(q, \dot{q}) dt}$$\[ (2.13) \]

where

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 - \frac{e}{c} \dot{q} \cdot A$$\[ (2.14) \]

as expected.

2. As derived in the previous part,

$$S = \int_{t_i}^{t_f} \frac{1}{2} m \dot{r}^2 - \frac{e}{c} \dot{r} \cdot A \cdot dt.$$\[ (2.15) \]

In the ultra-quantum limit ($m \to 0$), this becomes

$$S \to - \int_{t_i}^{t_f} \frac{e}{c} \dot{r} \cdot A \cdot \frac{dr}{dt} dt = - \frac{e}{c} \oint \dot{r} \cdot A \cdot dr$$\[ (2.16) \]

where $\Gamma$ is the trajectory of the particle which begins and ends at $r_0$. Using Stokes' theorem, we obtain

$$S = - \frac{e}{c} \int_{\Omega} \nabla \times A \cdot dS = - \frac{e}{c} \int_{\Omega} B \cdot dS = \frac{c}{e} \Phi$$\[ (2.17) \]

where $\Omega$ is the region enclosed by $\Gamma$ and $\Phi$ is the magnetic flux through this region.

3. There is no ambiguity involved in evaluating the above formula in the path integral for a particle moving in a uniform magnetic field on the plane of size $L \times L$. However, if we take $L \to \infty$, then the plane plus the point at infinity is topologically equivalent to the two-sphere. Now consider a closed trajectory on the sphere. Using the result of the previous part, the amplitude of this trajectory is given by the Aharonov-Bohm flux it encloses. However, on the sphere, a closed trajectory bounds two regions and so there is an ambiguity as to what is considered to be “inside”. In particular, if we denote the surface area of the sphere as $A$ and that of the smaller of the two regions bounded by the trajectory as $a$, we have that

$$\int_{\Omega} \nabla \times A \cdot dS = Ba \quad \text{OR} \quad -B(A - a).$$\[ (2.18) \]
The extra minus for the second possibility comes from the orientation of the trajectory.

If we want the path integral to be well defined, we must require that the actions corresponding to these two regions differ by a multiple of $2\pi\hbar$ so that $e^{iS_1/\hbar} = e^{iS_2/\hbar + 2\pi im} = e^{iS_2/\hbar}$. So, we must have

$$\frac{e}{c} B(A-a) - \left(-\frac{e}{c}\right) Ba = 2\pi m\hbar \implies BA = \frac{hc}{e} m \equiv \Phi_0 m. \quad (2.19)$$

Hence the total flux must be quantized to an integer multiple of the fundamental flux quantum $\Phi_0$.

### 3. PATH INTEGRALS FOR A SCALAR FIELD

1. The vacuum persistence amplitude in Minkowski space is

$$J(0|0)J = N \int \mathcal{D}\phi \mathcal{D}\phi^* e^\frac{i}{\hbar} \int d^4x \left[ \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + J^* \phi + J \phi^* \right]. \quad (3.1)$$

We obtain the Euclidean space expression by Wick rotating the action (sending $t \to -ix_D$) so that

$$iS \to -S_E = -\int \nabla_\mu \phi^* \nabla^\mu \phi + m^2 \phi \phi^* d^4x. \quad (3.2)$$

So,

$$J(0|0)J = N \int \mathcal{D}\phi \mathcal{D}\phi^* e^{-\frac{1}{\pi}} \int d^4x \left[ \nabla_\mu \phi^* \nabla^\mu \phi + m^2 \phi \phi^* - J^* \phi - J \phi^* \right]. \quad (3.3)$$

2. We will first do the calculation in Euclidean space and then Wick rotate back to Minkowski space. We first define a shifted field $\xi$ as

$$\phi(x) = \phi_c(x) + \xi(x) \implies \phi^*(x) = \phi^*_c(x) + \xi^*(x) \quad (3.4)$$

where $\phi_c$ is a fixed field configuration (which we fix below). On integrating by parts, we can write the Lagrangian in terms of these fields:

$$\mathcal{L}_E = \phi^* \left[ -\nabla^2 + m^2 \right] \phi - J^* \phi - J \phi^* \quad (3.5)$$

$$= \phi^*_c \left[ -\nabla^2 + m^2 \right] \phi_c - J^* \phi_c - J \phi^*_c + \xi^* \left[ -\nabla^2 + m^2 \right] \xi + \xi^* \left[ -\nabla^2 + m^2 \right] \phi_c \quad (3.6)$$

$$+ \xi \left[ -\nabla^2 + m^2 \right] \phi^*_c - J^* \xi - J \xi^*. \quad (3.7)$$
In order to make the cross terms vanish, we fix $\phi_c$ to satisfy
\[ [-\nabla^2 + m^2] \phi_c = J \implies [-\nabla^2 + m^2] \phi_c^* = J^* \] (3.8)
and so
\[ \phi_c(x) = \int d^4x' G_E(x - x') J(x'), \quad \phi_c^*(x) = \int d^4x' G_E(x - x') J^*(x') \] (3.9)
where $G_E(x - x')$ is the Euclidean Green function. Hence the Euclidean action becomes
\[ S_E = \int d^4x (\phi_c^*[-\nabla^2 + m^2] \phi_c - J^* \phi_c - J \phi_c^*) + \int d^4x (\xi^*[-\nabla^2 + m^2] \xi) \] (3.10)
\[ = -\int d^4xd^4x' [J^*(x) G_E(x - x') J(x')] + \int d^4x (\xi^*[-\nabla^2 + m^2] \xi) \] (3.11)
where we have used the explicit forms of $\phi_c$ and $\phi_c^*$ in terms of the Green function and we have used the fact that the Green function is symmetric: $G_E(x - x') = G_E(x' - x)$.

Hence,
\[ Z_E[J] = Z_E[0] \exp \left( \int d^4xd^4x' [J^*(x) G_E(x - x') J(x')] \right) \] (3.12)
where
\[ Z_E[0] \equiv \int \mathcal{D}\xi \mathcal{D}\xi^* e^{-\int d^4x \xi^*[-\nabla^2 + m^2] \xi}. \] (3.13)

In order to evaluate $Z_E[0]$, we expand $\xi$ in orthonormal eigenfunctions, $\psi_n$, of $\hat{A} \equiv -\nabla^2 + m^2$ with eigenvalues $A_n$:
\[ \xi(x) = \sum_n c_n \psi_n(x) \implies \xi^*(x) = \sum_n c^*_n \psi^*_n(x). \] (3.14)

We define the measure as
\[ \mathcal{D}\xi \mathcal{D}\xi^* = \mathcal{N} \prod_n \frac{dc_n dc^*_n}{\sqrt{2\pi} \sqrt{2\pi}} = \mathcal{N} \prod_n \frac{dRe(c_n) dIm(c_n)}{\sqrt{2\pi} \sqrt{2\pi} i} \] (3.15)
where $Re(x)$ and $Im(x)$ denote the real and imaginary parts of $x$ and $\mathcal{N}$ is a normalization constant which we will use to absorb unimportant constant factors. Evaluating the integrals in the usual way, we find
\[ Z_E[J] = \mathcal{N} \left[ \text{Det}(-\nabla^2 + m^2) \right]^{-1} \exp \left( \int d^4xd^4x' [J^*(x) G_E(x - x') J(x')] \right). \] (3.16)
In order to find the corresponding expression in Minkowski space, we Wick rotate back:

\[-\nabla^2 + m^2 \rightarrow \partial^2 + m^2\]  \hspace{1cm} (3.17)

\[G_E(x - x') \rightarrow -iG(x - x').\]  \hspace{1cm} (3.18)

Hence,

\[Z[J] = \mathcal{N} \left[ \text{Det}(-\nabla^2 + m^2) \right]^{-1} \exp \left( i \int d^4x d^4x' [J^*(x)G(x - x')J(x')] \right). \hspace{1cm} (3.19)\]

Note that the functional determinant is raised to the power \(-1\) in contrast to the single real scalar field case in which the determinant is raised to the power \(-1/2\). This is a reflection of the fact that a single complex scalar is equivalent to two real scalars.

3. By inspection of the Minkowski space path integral (before performing the field shift and completing the square), we see that

\[G_2'(x - x') = \langle 0| T\phi(x)\phi(x')|0 \rangle = \frac{1}{i^2} \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J^*(x)\delta J^*(x')} \bigg|_{J=J^*=0}. \hspace{1cm} (3.20)\]

Using the form of the path integral in Eq. (3.19), we find

\[G_2'(x - x') = \langle 0| T\phi(x)\phi(x')|0 \rangle \hspace{1cm} (3.21)\]

\[\quad = -\frac{1}{Z[J]} \frac{\delta}{\delta J^*(x')} \left[ i \left( \int d^4y G(x - y)J(y) \right) Z[J] \right] \bigg|_{J=0} \hspace{1cm} (3.22)\]

\[\quad = \frac{1}{Z[J]} \left( \int d^4y G(x - y)J(y) \right)^2 Z[J] \bigg|_{J=0} \hspace{1cm} (3.23)\]

\[\quad = 0. \hspace{1cm} (3.24)\]

By an identical calculation, we also find

\[G_2''(x - x') = \langle 0| T\phi^*(x)\phi^*(x')|0 \rangle = \frac{1}{i^2} \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x)\delta J(x')} \bigg|_{J=J^*=0} = 0. \hspace{1cm} (3.25)\]
On the other hand,

\[ G_2(x - x') = \langle 0 | T \phi^*(x) \phi(x') | 0 \rangle \]
\[ = -i G(x - x') \]

\[ = -i G(x - x') \]

By a similar calculation, one finds

\[ G_2^*(x - x') = \langle 0 | T \phi(x) \phi^*(x') | 0 \rangle = -i G(x - x') \]

4. The propagator for the complex scalar field is identical to that of a real scalar field and so all the results in the notes carry apply to the current situation. Here we will simply restate these results as they are derived in the notes (but the reader is expected to work through the derivations in detail).

**Euclidean space:** The Euclidean Green function satisfies

\[ [-\nabla_x^2 + m^2] G_E(x - x') = \delta^{(4)}(x - x') \implies G_E(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-x')} \]

\[ \frac{1}{4\pi^2} \left( \frac{m}{|x - x'|} \right) K_1(m|x - x'|) \]

where \( K_\nu(z) \) is the modified Bessel function. For large distances \((m|x - x'|) \gg 1\), we can use the asymptotic behaviour of the Bessel function to find that

\[ G_E(x - x') \approx \frac{\sqrt{\pi/2m^2} e^{-m|x-x'|}}{(2\pi)^2 (m|x - x'|)^{3/2}} \left[ 1 + O((m|x - x'|)^{-1}) \right] . \]

Hence the Green function decays exponentially. On the other hand, for short distances \((m|x - x'|) \ll 1\), we find that

\[ G_E(x - x') \approx \frac{1}{4\pi^2} \frac{1}{|x - x'|^2} \]
and so the Green function decays as a power law.

**Minkowski space:** The Minkowski and Euclidean space propagators are related as follows:

\[ G(x - x') = iG_E(x - xi')|_{x_4 \rightarrow ix_0}. \tag{3.37} \]

Under a Wick rotation, the Euclidean and Minkowski metrics are related as

\[ |x - x'| \rightarrow \sqrt{-s^2}. \tag{3.38} \]

So, in our case,

\[ G(x - x') = \frac{i}{4\pi^2} \frac{m}{\sqrt{-s^2}} K_1(m\sqrt{-s^2}). \tag{3.39} \]

In order to understand the asymptotic behaviour, we must consider two cases corresponding to when \( s \) represents two space-like or time-like separated points. For the space-like separated case, \( s^2 < 0 \), we have that \( \sqrt{-s^2} \) is a real, positive number and so the argument of \( K_\nu(z) \) in the Green function is also a positive number. Hence the Minkowski and Euclidean Green functions are identical (up to an overall factor of \( i \)) under the replacement \( |x - x'| \rightarrow \sqrt{-s^2} \). Hence the long-distance behaviour of the Green function is

\[ G(x - x') \approx \frac{\sqrt{\pi/2}m^2e^{-m\sqrt{-s^2}}}{(2\pi)^2(m\sqrt{-s^2})^{3/2}} \left[ 1 + O((m\sqrt{-s^2})^{-1}) \right]. \tag{3.40} \]

whereas the short-distance behaviour is

\[ G(x - x') \approx \frac{i}{4\pi^2} \frac{1}{-s^2}. \tag{3.41} \]

In the time-like separated case \( (s^2 > 0) \), we have that \( \sqrt{-s^2} = i\sqrt{s^2} \) is imaginary. As in the notes, we use the relation \( K_1(iz) = -\frac{\pi}{2}H_1^{(1)}(-z) \) where \( H_1(z) \) is a Hankel function. Using the asymptotic behaviour of the Hankel function, we find that in the long-distance regime,

\[ G(x - x') \approx \frac{\sqrt{\pi/2}}{4\pi^2} \frac{m^2}{(m\sqrt{s^2})^{3/2}}e^{im\sqrt{s^2}}. \tag{3.42} \]

Hence the Green function exhibits oscillatory behaviour and power-law decay. Conversely, in the short-distance regime,

\[ G(x - x') \approx \frac{1}{4\pi^2 s^2}, \tag{3.43} \]

and so we again obtain power-law decay.
For brevity, we will write \( \phi \) in place of \( \phi(x_i) \).

5. We can express the four-point functions in terms of the two-point functions using Wick’s theorem. For instance,

\[
G_4^a(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2)^* \phi(x_3) \phi(x_4) | 0 \rangle 
\]

\[
= \langle 0 | T \phi_1^* \phi_2^* | 0 \rangle \langle 0 | T \phi_3 \phi_4 | 0 \rangle + \langle 0 | T \phi_1^* \phi_3^* | 0 \rangle \langle 0 | T \phi_2^* \phi_4 | 0 \rangle
\]

\[
+ \langle 0 | T \phi_1^* \phi_4^* | 0 \rangle \langle 0 | T \phi_2^* \phi_3 | 0 \rangle
\]

\[
= G_2^a(x_1 - x_3)G_2^b(x_2 - x_4) + G_2^b(x_1 - x_4)G_2^a(x_2 - x_3)
\]

\[
= G_2(x_1 - x_3)G_2(x_2 - x_4) + G_2(x_1 - x_4)G_2(x_2 - x_3),
\]

where we have used the fact that \( G_2^a(x - x') = G_2(x - x') \). Following the same procedure, we can compute

\[
G_4^b(x_1, x_2, x_3, x_4) = 0
\]

\[
G_4^c(x_1, x_2, x_3, x_4) = 0
\]

and

\[
G_4^b(x_1, x_2, x_3, x_4) = G_2(x_1 - x_2)G_2(x_3 - x_4) + G_2(x_1 - x_4)G_2(x_2 - x_3)
\]

\[
G_4^c(x_1, x_2, x_3, x_4) = G_2(x_1 - x_2)G_2(x_3 - x_4) + G_2(x_1 - x_3)G_2(x_2 - x_4).
\]

Comparing the above expressions, we find that

\[
G_4^a(x_3, x_2, x_1, x_4) = G_4^c(x_1, x_2, x_3, x_4)
\]

\[
G_4^a(x_1, x_4, x_3, x_2) = G_4^b(x_1, x_2, x_3, x_4)
\]

\[
G_4^a(x_1, x_3, x_2, x_4) = G_4^b(x_1, x_2, x_3, x_4)
\]

\[
G_4^a(x_4, x_2, x_3, x_1) = G_4^b(x_1, x_2, x_3, x_4)
\]

\[
G_4^b(x_1, x_2, x_4, x_3) = G_4^c(x_1, x_2, x_3, x_4)
\]

\[
G_4^b(x_2, x_1, x_3, x_4) = G_4^c(x_1, x_2, x_3, x_4).
\]

These relations follow from using the symmetry of the two-point function. We can also note that the four-point functions are symmetric under permutations of the spacetime positions of the \( \phi \)'s or \( \phi^* \)'s. For instance, \( G_4^a(x_1, x_2, x_3, x_4) = G_4^b(x_2, x_1, x_3, x_4) \). Using this fact, the above set of relations are equivalent to the single relation

\[
G_4^a(x_1, x_4, x_3, x_2) = G_4^c(x_1, x_2, x_3, x_4) = G_4^b(x_1, x_2, x_4, x_3).
\]