1. FERMIONS IN ONE DIMENSION

In this problem, we study the Su-Schrieffer-Heeger (SSH) model of polyacetylene. Consider a series of \( N \) carbon atoms in 1D at equilibrium positions \( x(s) = s a_0, s = 1, \ldots, N \), where \( a_0 \) is the lattice constant. These carbon atoms share conduction electrons \( c_\sigma(s) \), \( \sigma = \uparrow, \downarrow \), having kinetic energy \( t \) much larger than that of the atoms themselves. These electrons are therefore able to hop from atom to atom, while the atoms themselves remain essentially classical. The Hamiltonian of this model is

\[
H = - \sum_{s=0}^{N/2} \left[ (t - \alpha \Delta x(n)) (c_\sigma^\dagger(s) c_\sigma(s + 1) + h.c.) + \frac{P^2(s)}{2M} + \frac{D}{2} [\Delta x(s)]^2 \right] ,
\]  

(1.1)

where we have defined \( \Delta x(s) = x(s) - x(s + 1) \), \( D \) is the elastic constant, and \( \alpha \) is the electron-phonon coupling constant.

By limiting ourselves to considering scattering only near the Fermi momentum \( p_F = \pi/2a_0 \), we can rewrite this theory in the continuum limit as one of two Dirac fermions \( \psi_\sigma = (R_\sigma, L_\sigma) \) (\( L \) and \( R \) are the left moving and right moving fermions we met in Problem...
Set 1) coupled to a real scalar field $\Delta$ with canonical momentum $\Pi$ describing the phonons,

$$H = \int dx \left[ -i v_F \psi^\dagger_\sigma(x) \sigma^z \partial_x \psi_\sigma(x) + \frac{\Pi^2(x)}{8M a_0^2} + \frac{1}{2} \Delta^2(x) + \sqrt{2g} \Delta(x) \bar{\psi}_\sigma(x) \psi_\sigma(x) \right]. \quad (1.2)$$

where we use the chiral basis for the Dirac gamma matrices: $\gamma_0 = \sigma^y$, $\gamma_1 = i \sigma^x$, $\gamma_5 = \gamma_0 \gamma_1 = \sigma^z$ and define $\bar{\psi} = \psi^\dagger \gamma_0$. It is important to emphasize that in the above Hamiltonian, $\sigma^z$ is to be contracted with the spinor indices, denoting the left and right-moving components of the Dirac fermion. This is different from the “spin” $\sigma$ indices, which actually are species/flavor indices. These refer to a global $SU(2)$ “spin rotation” symmetry which is common in condensed matter problems and is separate from the spin associated with the (here emergent) Lorentz spinor structure.

1. Consider the limit $M \to \infty$. In this limit the $\Delta$ field becomes classical, in the sense that the kinetic energy term can be neglected, and the energy can be minimized for a uniform configuration $\Delta(x) = \Delta_0 \equiv$ constant. In other words, quantum fluctuations of $\Delta$ are strongly suppressed.\(^\text{1}\)

In the limit $M \to \infty$, the theory consists of two species/flavors of free Dirac fermions of mass $m = \sqrt{2g} \Delta_0$. The two species correspond to “spin” up and “spin” down. We know from lecture how to quantize this theory and obtain the single particle spectrum in 3+1D. The 1+1D case is directly analogous. We write

$$\psi_\sigma(x) = \int \frac{dp}{2\pi} \frac{m}{\epsilon(p)} \left[ b_\sigma(p) u(p) e^{-ipx} + d_\sigma^\dagger(p) v(p) e^{ipx} \right], \quad (1.3)$$

where $\epsilon(p) = \sqrt{(v_F p)^2 + 2g \Delta_0^2}$ is the single particle dispersion, and

$$\{ b_\sigma(p), b_{\sigma'}^\dagger(p') \} = \{ d_\sigma(p), d_{\sigma'}^\dagger(p') \} = 2\pi \frac{\epsilon(p)}{m} \delta_{\sigma\sigma'} \delta(p - p'). \quad (1.4)$$

The basis spinors $u_\alpha(p)$ and $v_\alpha(p)$ are positive and negative energy eigenspinors of the Dirac operator which we choose such that the canonical anticommutation relation,

$$\{ \psi^\dagger_{\alpha,\sigma}(x'), \psi_{\alpha',\sigma'}(x) \} = \delta_{\alpha\alpha'} \delta_{\sigma\sigma'} \delta(x - x'), \quad (1.5)$$

is satisfied. Finding $u$ and $v$ is easy: we just diagonalize a 2 $\times$ 2 matrix,

$$\omega(p) u(p) = \begin{pmatrix} v_F p & -im \\ im & -v_F p \end{pmatrix} u(p), \quad -\omega(p) v(p) = \begin{pmatrix} v_F p & -im \\ im & -v_F p \end{pmatrix} v(p) \quad (1.6)$$

\(^\text{1}\) Next semester, we will learn how to check the self consistency of setting $\Delta$ to be a constant in this limit.
Clearly $\omega(p) = \epsilon(p)$. If we define $\sin \theta = m/\epsilon, \cos \theta = v_F p/\epsilon$, we can rewrite these equations as

$$u(p) = \begin{pmatrix} \cos \theta & -i \sin \theta \\ i \sin \theta & -\cos \theta \end{pmatrix} u(p), \quad -v(p) = \begin{pmatrix} \cos \theta & -i \sin \theta \\ i \sin \theta & -\cos \theta \end{pmatrix} v(p)$$

We seek solutions which have normalization $u^\dagger u = v^\dagger v = 1/\sin \theta = \epsilon(p)/m$ so that $\psi$ and $b, d$ have the proper anticommutation relations,

$$u = \sqrt{\frac{1}{4 \sin^2(\theta/2) \sin \theta}} \begin{pmatrix} i \sin \theta \\ \cos \theta - 1 \end{pmatrix}, \quad v = \sqrt{\frac{1}{4 \cos^2(\theta/2) \sin \theta}} \begin{pmatrix} i \sin \theta \\ \cos \theta + 1 \end{pmatrix}$$

Plugging everything back into the Hamiltonian, one finds

$$H = \frac{\Delta^2_0}{2} L + \int dp \frac{m}{2\pi \epsilon(p)} \epsilon(p) \left[ b_\sigma^\dagger(p) b_\sigma(p) - d_\sigma(p) d_\sigma^\dagger(p) \right],$$

where $L$ is the “length” of space.

The vacuum is the state is the state $|\text{gnd}\rangle$ in which all of the negative energy states are filled and all of the positive energy states are empty. It is thus defined by $d_\sigma|\text{gnd}\rangle = b_\sigma|\text{gnd}\rangle = 0$. Normal ordering $H$ with respect to the vacuum gives

$$H := H := +E_0,$$

where the normal ordered Hamiltonian is

$$: H := \int dp \frac{m}{2\pi \epsilon(p)} \epsilon(p) \left[ b_\sigma^\dagger(p) b_\sigma(p) + d_\sigma(p) d_\sigma^\dagger(p) \right].$$

This looks the same as the the Hamiltonian derived in the lecture notes for a four-component Dirac spinor in 3+1D, but this is only superficial: we emphasize again that the $\sigma$ index is a global flavor/species index, and has nothing to do with the Dirac spinor index.

To determine $\Delta_0$, we minimize the vacuum energy density $E_0$, which is

$$\mathcal{E}[\Delta_0] = \frac{E_0}{L} = -\int_{-\Lambda}^{\Lambda} dp \frac{1}{2\pi} \sqrt{(v_F p)^2 + 2g \Delta^2_0 + \frac{1}{2} \Delta_0^2},$$

where $\Lambda \sim \pi/a_0 = 2p_F$ is the ultraviolet (UV) cutoff. The equation we need to solve for $\Delta_0$ is therefore

$$0 = \frac{\partial \mathcal{E}}{\partial \Delta_0} = \Delta_0 - \int_{-\Lambda}^{\Lambda} dp \frac{2g \Delta_0}{2\pi \sqrt{(v_F p)^2 + 2g \Delta_0^2}}.$$
This equation has both a trivial solution $\Delta_0 = 0$ and a nontrivial solution obtained from
\[
\frac{v_F}{2g} = \int_{-\Delta_0}^{\Delta_0} \frac{d\tilde{p}}{2\pi \sqrt{\tilde{p}^2 + 2g\Delta_0^2}} = \frac{1}{\pi} \log \left( \frac{v_F}{g\Delta_0} \Lambda \right) \tag{1.14}
\]
for $\Lambda \gg 2g\Delta_0$. Thus,
\[
\Delta_0 = \frac{v_F}{g} e^{-\pi v_F/2g} \Lambda. \tag{1.15}
\]
corresponding to a spontaneously generated mass which is exponentially small compared to the cutoff! This phenomenon of mass generation in what looks to be an otherwise scale invariant theory is referred to as *dimensional transmutation*. In the language of the original SSH model, this spontaneous mass generation corresponds to dimerization.

2. The vacuum discussed above is defined by $d_{\sigma}|\text{gnd}\rangle = b_{\sigma}|\text{gnd}\rangle = 0$. The single particle states are
\[
b_{\sigma}^\dagger(p)|0\rangle, d_{\sigma}^\dagger(p)|0\rangle, \tag{1.16}
\]
corresponding respectively to a particle or antiparticle of momentum $p$ and “spin” polarization $\sigma$. Both states have energy above the vacuum,
\[
\langle \psi | : H : | \psi \rangle = \epsilon(p) = \sqrt{(v_F p)^2 + 2g\Delta_0^2}. \tag{1.17}
\]
The single-particle states therefore have a fourfold degeneracy corresponding to charge and spin polarization.

3. We now revisit the original continuum Hamiltonian without taking any limits. Consider the discrete symmetry transformation,
\[
\psi \mapsto \gamma_5 \psi, \Delta \mapsto -\Delta, \tag{1.18}
\]
where $\gamma_5 = \sigma^z$. In terms of left and right-moving components,
\[
L_\sigma \mapsto -L_\sigma, R_\sigma \mapsto R_\sigma. \tag{1.19}
\]
This clearly leaves the Hamiltonian invariant. Considering each term individually,
\[
\Pi^2, \Delta^2 \mapsto \Pi^2, \Delta^2 \tag{1.20}
\]
\[
-i v_F \psi_{\sigma}^\dagger \sigma^z \partial_x \psi_\sigma \mapsto -i v_F \psi_{\sigma}^\dagger (\sigma^z)^2 \sigma^z \partial_x \psi_\sigma = -i v_F \psi_{\sigma}^\dagger \sigma^z \partial_x \psi_\sigma \tag{1.21}
\]
\[
\sqrt{2g} \Delta \bar{\psi}_\sigma \psi_\sigma \mapsto -\sqrt{2g} \Delta \bar{\psi}_\sigma \sigma^z \sigma^y \psi_\sigma = \sqrt{2g} \Delta \bar{\psi}_\sigma \psi_\sigma. \tag{1.22}
\]
Notice that in deriving this we found that

\[ \bar{\psi}_\sigma \psi_\sigma \mapsto \bar{\psi}_\sigma \psi_\sigma \]  

(1.23)

making this operator a good order parameter for the symmetry.

How should we interpret this symmetry in the lattice model? The lattice fermions are related to the components of \( \psi_\sigma \) as follows,

\[ c_\sigma(s) = e^{ip_F a_0 s} R_\sigma(s) + e^{-ip_F a_0 s} L_\sigma(s), \]  

(1.24)

so \( c_\sigma \) transforms as

\[ c_\sigma(s) \mapsto e^{ip_F a_0 s} R_\sigma(s) - e^{-ip_F a_0 s} L_\sigma(s) = i c_\sigma(s + 1) \]  

(1.25)

since \( p_F a_0 = \frac{\pi}{2} \). Similarly, \( x(s) \) is shifted by a lattice site,

\[ x(s) = \delta(s) + (e^{2ip_F a_0 s} \Delta_+(s) + e^{-2ip_F a_0 s} \Delta_-(s)) \]  

(1.26)

\[ \mapsto \delta(s) - (e^{2ip_F a_0 s} \Delta_+(s) + e^{-2ip_F a_0 s} \Delta_-(s)) \]  

(1.27)

\[ = \Delta(s + 1). \]  

(1.28)

Thus, this transformation represents an ambiguity in the dimerized structure, i.e. how the fermions are arranged on the lattice.

4. Let us again consider the limit \( M \to \infty \). Our interest is in the vacuum expectation value of the order parameter we found in the previous sub-problem,

\[ \langle O \rangle = \langle \text{gnd} | \bar{\psi}_\sigma \psi_\sigma(x) | \text{gnd} \rangle. \]  

(1.29)

We can calculate this in canonical quantization by plugging in the eigenmode expansion for \( \psi \) in Eq. (1.3). To do this, we note the following identities

\[ \bar{u} u = u^\dagger \sigma^y u = 1, \quad \bar{v} v = v^\dagger \sigma^y v = -1, \]  

(1.30)

which can be derived through rigorous application of trig identities. We also recall

\[ \sin \theta = \frac{m}{\epsilon(p)} = \frac{\sqrt{2g \Delta_0}}{\sqrt{(v_F p)^2 + 2g \Delta_0^2}}, \]  

(1.31)
so we are now prepared to compute $\langle O \rangle$. Using the fact that vacuum expectation values of normally ordered operators vanish, we get

$$\langle O \rangle = \int \frac{dp dq}{(2\pi)^2} \left( \frac{m}{\epsilon(p)} \right) \left( \frac{m}{\epsilon(q)} \right) \bar{v}(q)v(p) \langle d_\sigma(q)d_\sigma^+(p) \rangle \ e^{i(q-p)x}$$

$$= \int \frac{dp dq}{2\pi} \frac{m}{\epsilon(p)} \delta(q-p)\bar{v}(q)v(p)$$

$$= -\int \frac{dp}{2\pi} \frac{m}{\epsilon(p)}$$

$$= -\int \frac{dp}{2\pi} \frac{\sqrt{2g\Delta_0}}{\sqrt{v_Fp}^2 + 2g\Delta_0^2} \quad (1.32)$$

$$= -\frac{\Delta_0}{\sqrt{2g}} \quad (1.33)$$

Indeed, this vanishes only if $\Delta_0 = 0$. Consequently, this order parameter signals spontaneous mass generation or, in the language of the original lattice model, dimerization.

2. GRASSMANN STUFF

1. Consider “analytic” functions of a single Grassmann variable $a^*$, $f(a^*)$ and $g(a^*)$. We can define the inner product

$$\langle f|g \rangle = \int da^*da e^{-a^*a}[f(a^*)]^*_a g(a^*) = \int da^*da \ (1 - a^*a) \ (\bar{f}_0 + \bar{f}_1a)(g_0 + g_1a^*)$$

$$= \int da^*da \ [\bar{f}_0g_0 + \bar{f}_1g_1] \ aa^*$$

$$= \bar{f}_0g_0 + \bar{f}_1g_1, \quad (2.1)$$

where we recall the definition

$$\int da^*da \ aa^* = 1. \quad (2.2)$$

2. Let $A(a^*, a)$ be some function such that

$$(Af)(a^*) = \int da^*a^\alpha A(a^*, \alpha)f(a^*)e^{-a^*a} = g(a^*) \quad (2.3)$$

Then

$$g_0 + g_1a^* = \int da^*da \ [A_{00} + A_{10}a^* + A_{01}a^* + A_{11}a^*a^*](f_0 + f_1a^*)(1 - a^*a) \quad (2.4)$$

$$= \int da^*da \ [(A_{00}f_0 + A_{10}f_1) + (A_{01}f_0 + A_{11}f_1)a^*]a^* \quad (2.5)$$

$$= (A_{00}f_0 + A_{10}f_1) + (A_{01}f_0 + A_{11}f_1)a^*. \quad (2.6)$$
Thus,
\[
\begin{pmatrix}
g_0 \\
g_1
\end{pmatrix} =
\begin{pmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{pmatrix}
\begin{pmatrix}
f_0 \\
f_1
\end{pmatrix}.
\]

(2.7)

3. Now let us define operators
\[
\hat{a}^* f(a^*) = a^* f(a^*), \quad \hat{a} f(a^*) = \partial_a f(a^*)
\]

Then clearly \((\hat{a}^*)^2 = (\hat{a})^2 = 0\) by the Grassmann-ness of \(a^*\), and
\[
\{\hat{a}^*, \hat{a}\} [f(a^*)] = a^* \partial_a f + \partial_{a^*}(a^* f) = a^* f_1 + \partial_{a^*}(f_0 a^*) = f_0 + f_1 a^* = f(a^*),
\]

i.e.
\[
\{\hat{a}^*, \hat{a}\} = 1.
\]

(2.10)

4. Consider the partition function for a Grassmann field \(\xi_i\) defined on \(i = 1, \ldots, N\) “sites”
\[
Z = \int \prod_i \bar{\xi}_i \, d\xi_i \, \exp \left( -\bar{\xi}_j M_{jk} \xi_k \right).
\]

(2.11)

We can compute this Gaussian path integral as follows. Since the \(\xi\)'s and \(\bar{\xi}\)'s are Grassmann variables, we can rewrite \(Z\) as
\[
Z = \int \prod_i \bar{\xi}_i \, d\xi_i \, \frac{(-1)^N}{N!} (\bar{\xi}_j M_{jk} \xi_k)^N
\]

(2.12)

\[
= \int \prod_i \bar{\xi}_i \, d\xi_i \, \frac{(-1)^N}{N!} \varepsilon_{i_1 \cdots i_N} \bar{\xi}_{j_1} \cdots \bar{\xi}_{j_N} M_{i_1 j_1} \cdots M_{i_N j_N} \xi_{j_2} \cdots \xi_{j_N} \xi_{i_1}.
\]

(2.13)

since the only terms in the integrand which give a nonzero answer are those containing only one of each \(\xi_i\) and \(\bar{\xi}_i\) variable. If we think about the \(i\) index as a lattice site index, then these configurations correspond to all possible sets of closed loops (including those involving only a single site). The Levi-Civita symbols are obtained by noting that permuting a single pair of labels with respect to the reference \((1234 \cdots N)\) leads to an overall minus sign.

Evaluating the Grassmann integral gives another factor of \((-1)^N\), so we finally obtain
\[
Z = \frac{1}{N!} \varepsilon_{i_1 \cdots i_N} \varepsilon_{j_1 \cdots j_N} M_{i_1 j_1} \cdots M_{i_N j_N} = \det M.
\]

(2.14)
3. DIRAC FERMIONS

Consider a free Dirac fermion $\psi$ of mass $m$ in 3+1 spacetime dimensions

$$S[\bar{\psi}, \psi] = \int d^4x \mathcal{L}[\bar{\psi}, \psi] = \int d^4x \bar{\psi} (i\partial - m) \psi. \quad (3.1)$$

Our goal in this problem will be to compute the generating functional for this theory and then use it to compute correlation functions.

1. The generating functional for this theory is simply the Grassmann path integral,

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left[iS[\bar{\psi}, \psi]+i \int d^4x (\bar{\eta}\psi + \bar{\psi}\eta)\right], \quad (3.2)$$

where $\bar{\eta}_\alpha(x), \eta_\alpha(x)$ are Grassmann-valued sources ($\alpha$ is a spinor index). We can compute the path integral by completing the square, defining $\mathcal{D} \equiv i\partial - m$,

$$\bar{\psi} \mathcal{D} \psi + \bar{\eta}\psi + \bar{\psi}\eta = (\bar{\psi} + \bar{\eta} \mathcal{D}^{-1}) \mathcal{D} (\psi + \mathcal{D}^{-1}\eta) - \bar{\eta} \mathcal{D}^{-1} \eta \quad (3.3)$$

We can then change variables in the path integral to $\chi = \psi + \mathcal{D}^{-1}\eta, \bar{\chi} = \bar{\psi} + \bar{\eta} \mathcal{D}^{-1}$. The Jacobian for this transformation is trivial (there are no anomalies), so

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\chi \mathcal{D}\chi \exp\left[i \int d^4x (\bar{\chi} \mathcal{D} \chi - \bar{\eta} \mathcal{D}^{-1} \eta)\right]. \quad (3.4)$$

As we showed in the previous problem, the integral over all configurations of the Grassmann variables $\chi$ and $\bar{\chi}$ leads to a functional determinant, $\det \mathcal{D} = \det(i\partial - m)$, so

$$Z[\bar{\eta}, \eta] = \det(i\partial - m) \exp\left[-i \int d^4x \bar{\eta}(x) \mathcal{D}^{-1} \eta(x)\right] \quad (3.5)$$

$$= \det(i\partial - m) \exp\left[- \int d^4x d^4y \bar{\eta}_\alpha(x) i S_F^{\alpha\beta}(x-y) \eta_\beta(y)\right], \quad (3.6)$$

where we have defined

$$\mathcal{D}^{-1} \eta_\alpha(x) = i \int d^4y S_F^{\alpha\beta}(x-y) \eta_\beta(y) = i \int d^4y \langle x| (i\partial - m)^{-1}_\alpha^\beta| y\rangle \eta_\beta(y). \quad (3.7)$$

2. The propagator of the Dirac theory is

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle = \frac{1}{i} \frac{\delta}{\delta \bar{\eta}^\alpha(x)} \frac{\delta}{\delta \eta^\beta(y)} \log Z[\bar{\eta}, \eta] \bigg|_{\eta=\bar{\eta}=0} = i S_F^{\alpha\beta}(x-y). \quad (3.8)$$

Note that the difference in the factors of $i$ is due to the anticommuting nature of the Grassmann variables.
3. Consider the four point function

\[ S_F^{(4)}(x_1, x_2, x_3, x_4) = \langle \psi_\alpha(x_1) \psi_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4) \rangle. \]  

(3.9)

Since the theory we are considering is free, we know that this will simply be a product of Dirac propagators: two identical particles are created at positions \( x_3, x_4 \) and freely propagate to positions \( x_1, x_2 \). There are two such processes, which by Fermi statistics must contribute amplitudes with opposite signs. Indeed, we will see that this comes out from differentiating the Grassmann path integral,

\[ S_F^{(4)}(x_1, x_2, x_3, x_4) = \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_\alpha(x_1)} \frac{1}{i} \frac{\delta}{\delta \bar{\eta}_\beta(x_2)} \frac{i}{\delta \eta_\gamma(x_3)} \frac{i}{\delta \eta_\delta(x_4)} \log Z[\bar{\eta}, \eta] \bigg|_{\eta=\bar{\eta}=0} \]

\[ = \delta \delta \bar{\eta}_\alpha(x_1) \delta \delta \bar{\eta}_\beta(x_2) \left[ \int d^4 y d^4 z \bar{\eta}_{\alpha'}(z) iS_F^{\alpha'\gamma}(z-x_3) \eta_{\beta'}(y) iS_F^{\beta\delta}(y-x_4) \right] \bigg|_{\eta=0} \]

\[ = iS_F^{\beta\gamma}(x_2-x_3) iS_F^{\alpha\delta}(x_1-x_4) - iS_F^{\beta\delta}(x_2-x_4) iS_F^{\alpha\gamma}(x_1-x_3), \]  

(3.10)

where the relative minus sign comes from the anticommutation of derivatives of Grassmann variables.

4. FUNCTIONAL DETERMINANTS AND THE CASIMIR EFFECT

Consider a (free) real scalar field in 1+1D

\[ S = \int d^2 x \mathcal{L}[\phi] = \int d^2 x \left[ \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \]  

(4.1)

with periodic boundary conditions in the spatial direction, \( \phi(x+L,t) = \phi(x,t) \). Our goal is to compute the vacuum energy density, or Casimir energy density, of this theory, defined as

\[ \mathcal{E}_{\text{Casimir}} = -\frac{1}{L} \lim_{\beta \to \infty} \frac{1}{\beta} \log Z_E, \quad Z_E = \int \mathcal{D}\phi e^{-S_E[\phi]} \]  

(4.2)

where \( S_E \) is the Wick rotated action, with \( t = -i\tau \). Note that in order to be able to calculate anything, we will need to impose periodic boundary conditions in imaginary time, \( \phi(\tau, x) = \phi(\tau + \beta, x) \), although at the end of the calculation we will take the limit \( \beta \to \infty \).

1. The classical ground state energy can be obtained by considering the equations of motion

\[ (\partial^2 + m^2)\phi_{cl} = 0. \]  

(4.3)
This has a trivial solution \( \phi_{cl} = 0 \), leading to \( S[\phi_{cl}] = 0 \). To compute \( \mathcal{E}_{\text{Casimir}} \), we will expand the path integral about this saddle point. We thus obtain a vanishing classical contribution to the ground state energy,

\[
E_{0,cl} = \frac{1}{\beta} S_E[\phi_{cl}] = 0. \tag{4.4}
\]

2. We will consider fluctuations \( \varphi \) about \( \phi_{cl} = 0 \), i.e. we limit ourselves to field configurations

\[
\phi = \phi_{cl} + \varphi(x,t). \tag{4.5}
\]

Since \( \phi_{cl} = 0 \), the only contribution to the ground state energy comes from the fluctuations \( \varphi \), which also have the action of a free scalar field. Consequently, the partition function is just a Gaussian path integral

\[
Z_E = \int \mathcal{D}\varphi e^{-S_E[\varphi]} = \left[ \det \left( -\partial^2_E + m^2 \right) \right]^{-1/2}, \tag{4.6}
\]

where \( \partial_E = (\partial_\tau, \partial_x) \) and \( (\partial_E)^2 = \partial^2_\tau + \partial^2_x \) (i.e. we contract with \( \delta_{\mu\nu} \)). Thus,

\[
\mathcal{E}_{\text{Casimir}} = \frac{1}{2L} \lim_{\beta \to \infty} \frac{1}{\beta} \log \left[ \det \left( -\partial^2_E + m^2 \right) \right]. \tag{4.7}
\]

3. To calculate the Casimir energy density, we will use zeta function regularization. Let \( A = -\partial^2_E + m^2 \) and let \( \{a_{\ell n}\} \) be the set of eigenvalues of \( A \),

\[
a_{\ell n} = \omega_\ell^2 + k_n^2 + m^2 = \left( \frac{2\pi}{\beta} \ell \right)^2 + \left( \frac{2\pi}{L} n \right)^2 + m^2 \tag{4.8}
\]

These correspond to plane wave solutions with Matsubara frequency \( \omega_\ell \) and wave vector \( k_n \). Then we can define the generalized zeta function

\[
\zeta_A(s) = \sum_{\ell,n} \frac{1}{a_{\ell n}^s} \sum_{\ell,n} \frac{1}{[\omega_\ell^2 + k_n^2 + m^2]^s} \sim_{\beta \to \infty} \sum_n \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\beta}{[\omega^2 + k_n^2 + m^2]^s} \tag{4.9}
\]

This can be more easily computed using the identity

\[
\zeta_A(s) = \frac{1}{\Gamma(s)} \sum_{\ell,n} \int_0^{\infty} d\tilde{\tau} \tilde{\tau}^{s-1} e^{-a_{\ell n} \tilde{\tau}} = \frac{\beta}{\Gamma(s)} \sum_n \int d\omega d\tilde{\tau} \tilde{\tau}^{s-1} e^{-\tilde{\tau}(\omega^2 + k_n^2 + m^2)} \tag{4.10}
\]

Computing the Gaussian integral over \( \omega \) gives

\[
\zeta_A(s) = \frac{\beta}{\Gamma(s)} \sum_n \int \frac{d\tilde{\tau}}{2\pi} \tilde{\tau}^{s-1} \sqrt{\frac{\pi}{\tilde{\tau}}} e^{-\tilde{\tau}(k_n^2 + m^2)} \tag{4.11}
\]
To compute the sum over \( n \), we invoke the Poisson summation formula,
\[
\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, e^{2\pi imk} f(k),
\]  
(4.12)
so
\[
\zeta_A(s) = \frac{\beta L}{\Gamma(s)} \sum_n \int \frac{dk}{(2\pi)^2} \int d\tilde{\tau} \, e^{inkL} \tilde{\tau}^{s-1} \sqrt{\frac{\pi}{\tilde{\tau}}} e^{-\tilde{\tau}(k^2+m^2)}
\]  
(4.13)
\[
= \frac{\beta L}{4\pi} \frac{1}{\Gamma(s)} \sum_n \int_0^\infty d\tilde{\tau} \tilde{\tau}^{s-2} \exp \left[ -m^2\tilde{\tau} - \frac{(nL)^2}{4\tilde{\tau}} \right]
\]  
(4.14)
where we have used the fact
\[
\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-(k^2+m^2)\tilde{\tau} + ikx} = \sqrt{\frac{1}{4\pi\tilde{\tau}}} e^{-m^2\tilde{\tau} - x^2/\tilde{\tau}}.
\]  
(4.15)
We now consider the limit \( m \to 0 \). Taking this limit is subtle, since if we set \( m^2 \equiv 0 \), the zero mode \( (n = 0) \) contribution will diverge. Thus, we will need to keep \( m^2 \) finite in that term,
\[
\zeta_A(s) = \frac{\beta L}{4\pi} \frac{1}{\Gamma(s)} \left[ \frac{\Gamma(s-1)}{m^2(s-1)} + 2 \sum_{n=1}^{\infty} \int_0^\infty d\tilde{\tau} \tilde{\tau}^{s-2} e^{-(nL)^2/4\tilde{\tau}} \right]
\]  
(4.16)
where we have used the definition of one of the most ubiquitous functions in quantum field theory, the Euler gamma function,
\[
\Gamma(s) = \int_0^\infty dw \, w^{s-1} e^{-w}.
\]  
(4.17)
To compute the integral in the second term, we change variables to \( w = (nL)^2/4\tilde{\tau} \), i.e. \(-4dw/(nLw)^2 = d\tilde{\tau}\). Then
\[
\sum_{n=0}^{\infty} \int_0^\infty d\tilde{\tau} \tilde{\tau}^{s-2} e^{-(nL)^2/4\tilde{\tau}} = \sum_{n=1}^{\infty} \left( \frac{2}{nL} \right)^{2(1-s)} \int_0^\infty dw \, w^{-s} e^{-w}
\]  
(4.18)
\[
= \left( \frac{2}{L} \right)^{2(1-s)} \zeta(2-2s) \Gamma(1-s),
\]  
(4.19)
and so we finally obtain
\[
\zeta_A(s) = \frac{\beta L}{4\pi} \frac{1}{\Gamma(s)} \left[ \frac{\Gamma(s-1)}{m^{2(s-1)}} + 2 \left( \frac{2}{L} \right)^{2(1-s)} \zeta(2-2s) \Gamma(1-s) \right].
\]  
(4.20)
Now, to obtain the Casimir energy, we use the fact that
\[
- \lim_{s \to 0^+} \frac{d\zeta_A}{ds} = \log \det A = -2 \log Z_E.
\]  
(4.21)
One finds, introducing a reference mass scale $\mu$,

$$
\log \det A = - \lim_{s \to 0^+} \frac{d \zeta_A}{ds} = \frac{\beta L}{4\pi} \left[ m^2 \left( 1 - \log \left( \frac{m^2}{\mu^2} \right) \right) - \frac{4\pi^2}{3L^2} \right],
$$

meaning

$$
\mathcal{E}_{\text{Casimir}} = \frac{1}{8\pi} m^2 \left( 1 - \log \left( \frac{m^2}{\mu^2} \right) \right) - \frac{\pi}{6L^2}
$$

Indeed, this consists of an extensive (divergent) term plus a term which vanishes like $1/L^2$ as $L \to \infty$.

4. We can think of $\mathcal{E}_{\text{Casimir}}$ as the pressure exerted by zero point fluctuations on the walls enclosing the system (although we should really be using vanishing, rather than periodic, boundary conditions for this to make sense). Since in 1+1D pressure and force have the same units, we can define the “force” on the walls to be the difference between the Casimir energy inside and “outside” the region of interest (for us, the “outside” contribution is just the extensive, divergent piece – if we were using vanishing boundary conditions, the region outside the walls of the system would clearly be infinite),

$$
F_{\text{Casimir}} = \frac{E_{\text{inside}} - E_{\text{outside}}}{L} = -\frac{\pi}{6L^2}
$$

The minus sign here indicates an attractive force on the walls. This makes sense because the zero point fluctuations “outside” the finite region can have any momentum, and so more modes can pop in and out of existence outside the finite region than inside, where the momenta are restricted to multiples of $2\pi/L$.

5. THE WEAKLY INTERACTING BOSE GAS

In this problem, we study a gas of non-relativistic bosons $\phi$ at finite chemical potential with density-density interactions in three spatial dimensions

$$
H = \int d^3x \left[ \phi^\dagger(x) \left( -\frac{1}{2m} \nabla^2 - \mu \right) \phi(x) + n(x)V(x - x')n(x') \right],
$$

where $n(x) = \phi^\dagger(x)\phi(x)$ is the local density operator. In this problem, we will choose a local, repulsive density-density interaction,

$$
V(x - x') = \lambda \delta^{(3)}(x - x'),
$$

where $\lambda > 0$.  

1. We can write down a coherent state path integral over the time dependent fields \( \phi(x,t), \phi^\dag(x,t) \) using the procedure discussed in the lecture notes,
\[
Z = \int_{\phi(x\to\infty) = \phi_0} D\phi^\dag D\phi \exp \left[ i \int dt d^3x \left( -i\phi^\dag \partial_t \phi - H[\phi^\dag, \phi] \right) \right].
\]
where we have assumed that \( \phi \) approaches a constant value at spatial infinity. To put the system at a finite temperature \( T \), we Wick rotate to imaginary time \( t = -i\tau \) and identify \( \tau \sim \tau + \beta \) \((\beta = 1/T)\),
\[
Z = \int_{\phi(\tau + \beta) = \phi(\tau), \phi(x\to\infty) = \phi_0} D\phi^\dag D\phi e^{-S_E[\phi^\dag, \phi]},
\]
where
\[
S_E[\phi^\dag, \phi] = \int_0^\beta d\tau \int d^3x \left( \phi^\dag \partial_\tau \phi + H[\phi^\dag, \phi] \right).
\]

2. A classical saddle point of the path integral is a solution equation of motion,
\[
\delta S = 0 \Rightarrow 0 = \frac{\delta L}{\delta \phi^\dag} = \partial_\tau \phi + \frac{\delta H}{\delta \phi^\dag},
\]
which can be rewritten as
\[
\partial_\tau \phi = \left( \frac{1}{2m} \nabla^2 + \mu \right) \phi - 2\lambda |\phi|^2 \phi
\]
we will consider a uniform solution \( \phi = \phi_c \), which clearly satisfies the boundary conditions we imposed in the previous sub-problem. This means the derivative terms vanish, leading to a family of solutions,
\[
\phi_c = \sqrt{\frac{\mu}{2\lambda}} e^{i\theta} = \rho_0^{1/2} e^{i\theta},
\]
where \( \theta \) is an arbitrary phase. In the limit \( T \to 0 \), this corresponds to the set of degenerate ground states of the system, since they minimize the Hamiltonian. Clearly, when we choose a particular value for \( \theta \), the global \( U(1) \) symmetry \( \phi \mapsto e^{i\alpha} \phi \) does not leave the ground state invariant: it is spontaneously broken!

3. We now wish to compute the two-point function
\[
G(x - x') = \langle T\phi(x)\phi^\dag(x') \rangle
\]
at temperature \( T = 0 \) at long distances \(|x - x'| \to \infty\). Since \( \phi = \phi_c + \) (small fluctuations) at low energies/long distances, we expect the asymptotic behavior of this Green’s function to go like
\[
G(x - x') \sim \rho_0 + \text{(fluctuation effects)}
\]
Since this is a constant, this indicates the presence of long-ranged order. Below, we will see that the fluctuation effects do not spoil this general behavior (so long as the spacetime dimension is $D > 2$).

4. Consider fluctuations about the saddle point configuration $\phi_c = \rho_0^{1/2}$,

$$\phi(x) = \sqrt{\rho_0 + \delta \rho(x)} e^{i \theta(x)}$$  

(5.11)

where $\delta \rho$ and $\theta$ are small fluctuation fields which vanish at spacetime infinity. The effective action at quadratic order in the fluctuations is

$$S_{\text{eff}} = \int d\tau d^3x \left[ \delta \rho i \partial_\tau \theta + \frac{1}{8m\rho_0} (\nabla \delta \rho)^2 + \lambda (\delta \rho)^2 + \frac{\rho_0}{2m} (\nabla \theta)^2 + \cdots \right],$$  

(5.12)

While the terms linear in the fluctuations coming from the Hamiltonian all nicely cancel, we note that there are some linear terms involving time derivatives which vanish on their own since we have required that $\theta$ and $\delta \rho$ vanish at spacetime infinity.

Since the effective action for $\delta \rho$ is quadratic, we can integrate it out, although it will be convenient to rescale $\delta \rho \mapsto \delta \rho/(4m\rho_0)^{1/2} \equiv \delta \phi$. Its (tree level) propagator can be computed by completing the square and differentiating the path integral in the usual way,

$$G_{\delta \phi}(x - x') = \langle \delta \phi(\tau, x) \delta \phi(\tau', x') \rangle = \delta(\tau - \tau') \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip(x-x')}}{|p|^2 + 2m\rho_0\lambda}$$  

(5.13)

$$= \delta(\tau - \tau') \frac{e^{-(mp)^{1/2}|x-x'|}}{4\pi|x-x'|} \equiv \delta(\tau - \tau') K(x - x')$$  

(5.14)

Thus, upon integrating out $\delta \rho$, a temporal kinetic term for $\theta$ is generated of the form

$$+ \int d\tau d^3x d^3x' 2m\rho_0 \partial_\tau \theta(\tau, x) U(x - x') \partial_\tau \theta(\tau, x'),$$  

(5.15)

So the final effective action for $\theta$ is

$$S_{\text{eff}} = \int d\tau d^3x d^3x' 2m\rho_0 \partial_\tau \theta(\tau, x) U(x - x') \partial_\tau \theta(\tau, x') + \int d\tau d^3x \frac{\rho_0}{2m} (\nabla \theta)^2 + \cdots,$$  

(5.16)

Notice that $\theta$ looks like a massless, relativistic scalar field, except for the fact that its temporal kinetic term appears nonlocal in space.
5. This non-locality is only important at length scales $R \lesssim (m\mu)^{-1/2}$, or energy scales of order the mass of the $\delta\phi$ field, $(m\mu)^{1/2}$. Consequently, we are justified in taking the limit as $m\mu \to \infty$. In this limit, the potential $U(x - x')$ is extremely sharply peaked, so we can expand in the integral,

$$\theta(\tau, x') = \theta(\tau, x) + (x - x') \cdot \nabla \theta(\tau, x) + \cdots \quad (5.17)$$

So long as $\theta$ is sufficiently slowly varying, we can neglect the higher derivative terms. We therefore obtain for the temporal derivative term,

$$\int d\tau d^3x \; d^3x' \; 2m\rho_0 \partial_\tau \theta(\tau, x) U(x - x') \partial_\tau \theta(\tau, x) \approx \int d\tau d^3x \; \frac{1}{m\mu} \partial_\tau \theta(\tau, x)^2 \quad (5.19)$$

The integral over $x'$ can be performed by inspecting Eq. (5.15). The final form of the low energy effective action is, in terms of the parameters $\mu, m$, and $\lambda$,

$$S_{\text{eff}} = \int d\tau d^3x \; \frac{1}{2} \left[ \frac{2}{\lambda} (\partial_\tau \theta)^2 + \frac{\mu}{2m\lambda} (\nabla \theta)^2 + \cdots \right] \quad (5.20)$$

Wick rotating back to real time, we finally obtain

$$S_{\text{eff}}[\theta] = \int d^4x \; \frac{1}{2} \lambda \left[ (\partial_t \theta)^2 - \frac{\mu}{4m} (\nabla \theta)^2 + \cdots \right] \quad (5.21)$$

This is looks like the action for a *massless* Klein-Gordon field! The only difference is the presence of different coefficients on the spatial and temporal parts of the kinetic term and the overall normalization. The extra coefficient on the spatial derivative term can be thought of as a “speed of light” $v = \sqrt{\mu/4m} = \sqrt{\lambda\rho_0/2m}$. The propagator of $\theta$ is

$$\langle \theta(-p)\theta(p) \rangle = \frac{\lambda}{2 \omega^2 - v^2 |p|^2} \quad (5.22)$$

Fourier transforming this propagator, we find power law decay,

$$G_{\theta}(x - x') = \langle T\theta(x)\theta(x') \rangle = \frac{1}{(2\pi)^2} \frac{\lambda}{2v} \frac{1}{2v (t - t')^2 - |x - x'|^2} \quad (5.23)$$

This justifies the result of the previous sub-problem: the corrections due to fluctuations to $\langle T\phi(x')\phi(x) \rangle$ decay at long distances. This can be seen as follows,

$$\langle T\phi(x)\phi^+(x') \rangle \approx \rho_0 e^{-\langle T[\theta(x) - \theta(x')]^2/2 \rangle} \to \rho_0 \text{ as } |x - x'| \to \infty. \quad (5.24)$$
Notice that if we were in 1+1D, the propagator \( G_{\theta}^{1+1}(x - x') = -C \log |x - x'| \), where \( C > 0 \) is some (non-universal) constant. As a result, in 1+1D, \( \langle T \phi(x) \phi^i(x') \rangle \) actually decays as a power law, indicating the absence of long-ranged order. In other words, continuous symmetries cannot be spontaneously broken in \( D \leq 2 \) spacetime dimensions (or \( d \leq 2 \) spatial dimensions at finite temperature), and there are no Goldstone modes. This is the statement of the famous Mermin-Wagner-Coleman theorem.