1. PATH INTEGRAL QUANTIZATION OF THE FREE ELECTROMAGNETIC FIELD

N.B. The ordering of the sub-questions is a little confusing. We will accept solutions that answer the sub-questions in a different order.

1. In momentum space, the propagator is

\[ G_{\mu\nu}(p) = \frac{-1}{p^2 + i\epsilon} \left[ g_{\mu\nu} + (\alpha - 1)\frac{p^\mu p^\nu}{p^2} \right], \tag{1.1} \]

as will be derived in a subsequent part of the question. We have chosen the Feynman \( i\epsilon \) prescription which yields a time-ordered propagator.

2. Inverting the above matrix equation, we find

\[ \left[ -g_{\mu\nu}p^2 + \frac{\alpha - 1}{\alpha}p^\mu p^\nu \right] G_{\nu\lambda}(p) = \delta^\mu_\lambda. \tag{1.2} \]

One way to arrive at this expression without explicitly inverting the matrix is to note that the inverse of \( G_{\mu\nu} \) must be symmetric in \( \mu \) and \( \nu \) and so can only contain terms of the form \( g_{\mu\nu}p^2 \) and \( p^\mu p^\nu \). The coefficients of these terms are then fixed by checking that the above expression is satisfied. Now, inverting the Fourier transform, we find that the Green function satisfies the PDE

\[ \left[ g_{\mu\nu} \partial^2 - \frac{\alpha - 1}{\alpha} \partial^\mu \partial^\nu \right] G_{\nu\lambda}(x - y) = \delta^\mu_\lambda \delta(4)(x - y). \tag{1.3} \]

3. We can read off from part 1 that

\[ D_F(p) = -\frac{1}{p^2 + i\epsilon} \implies D_F(p) = -\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2 + i\epsilon} \tag{1.4} \]

which is the propagator for a massless, real scalar field. As calculated in a previous problem set and in lecture, in Minkowski space a massive, real scalar field has propagator

\[ G(x - x') = \frac{i}{4\pi} \frac{m}{\sqrt{-s^2}} K_1(m\sqrt{-s^2}). \tag{1.5} \]
In the $m \to 0$ limit we find that (using the asymptotic behavior of $K_1(z)$ discussed in chapter 5 of the lecture notes)

\[
D_F(x - x') = \begin{cases} 
\frac{i}{4\pi^2(-s^2)} & \text{(space-like separation, } (x - x')^2 = s^2 < 0) \\
\frac{i}{4\pi^2(s^2)} & \text{(time-like separation, } (x - x')^2 = s^2 > 0) 
\end{cases}
\]  

(1.6)

4. (a) The Fadeev-Popov determinant is given by

\[
\Delta_g[A_\mu] = \det \left( \frac{\delta g}{\delta U} \right) \bigg|_{g=0} 
\]  

(1.7)

where the generalized gauge fixing condition is

\[
g(A_\mu) = \partial_\mu A^\mu - c(x). 
\]  

(1.8)

Under a $U(1)$ gauge transformation $(U)$, we have that

\[
g(A_U^\mu) = \partial_\mu (A^\mu + \partial^\mu \phi) - c(x) = \partial_\mu A^\mu + \partial^2 \phi - c(x). 
\]  

(1.9)

Hence,

\[
\Delta_g[A_\mu] = \det \left( \frac{\delta g(x)}{\delta \phi(y)} \right) \bigg|_{g=0} = \det \partial^2 
\]  

(1.10)

is independent of the gauge field configuration.

(b) Under a gauge transformation, the field strength $F_{\mu\nu}$ is invariant, as was shown in a previous problem set. The coupling to the current transforms as follows:

\[
\delta (A_\mu J^\mu) = \partial_\mu \phi J^\mu + \partial_\mu (\partial^\mu \phi) - c \phi J^\mu 
\]  

(1.11)

where we have dropped total derivative terms. Hence the action is gauge invariant provided the currents are conserved.

However, in order to ensure that the full path integral is invariant, we must check that

\[
\Delta_g^{-1}[A_\mu] \equiv \int \mathcal{D}U \delta(g(A_U^\mu)) 
\]  

(1.12)

is also gauge invariant. Using the property of the Haar measure that $\mathcal{D}U = \mathcal{D}UU'$ for fixed $U'$, we have that

\[
\Delta_g^{-1}[A_U^\mu] = \int \mathcal{D}U \delta(g(A_U'^U)) = \int \mathcal{D}(U'U) \delta(g(A_U'^U)) = \int \mathcal{D}U \delta(g(A_U'^U)) = \Delta_g^{-1}[A_\mu] 
\]  

(1.13)
where we have set $U'' = U'U$. Altogether, the path integral

$$Z[J] = \int \mathcal{D}A_\mu \Delta g[A_\mu] \int \mathcal{D}U \delta(g(A_\mu)) e^{iS[A,J]}$$  \hspace{1cm} (1.14)$$

is gauge invariant.

5. We first massage the above path integral by changing variables $A_\mu \rightarrow A^V_\mu$ which is related to $A_\mu$ by the gauge transformation $V$. If we choose $V = U^{-1}$ then we obtain

$$Z[J] = \left[ \int \mathcal{D}U \right] \int \mathcal{D}A_\mu \Delta g[A_\mu] \delta(g(A_\mu)) e^{iS[A,J]}$$  \hspace{1cm} (1.15)$$

where $\int \mathcal{D}U \equiv N$ is an infinite constant. Now, we consider a generalized gauge-fixing condition of $g(x) = c(x)$, where $c(x)$ is some function, instead of $g(x) = 0$ (nothing in the above derivation changes). Since physical quantities and hence the path integral do not depend on the gauge-fixing condition, we can average over all $c(x)$'s, weighted by a Gaussian:

$$Z[J] = N \text{Det}[\partial^2] \int \mathcal{D}A_\mu \mathcal{D}c e^{-i \int d^4x \frac{c(x)^2}{2\alpha}} \delta(g(A_\mu) - c(x)) e^{iS[A,J]}$$  \hspace{1cm} (1.16)$$

$$= N \text{Det}[\partial^2] \int \mathcal{D}A_\mu e^{i \int d^4x \mathcal{L}_\alpha[A,J]}$$  \hspace{1cm} (1.17)$$

where

$$\mathcal{L}_\alpha = -\frac{1}{4} F_{\mu\nu}^2 - J_\mu A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2$$  \hspace{1cm} (1.18)$$

$$= \frac{1}{2} A_\mu \left[ g^{\mu\nu} \partial^2 - \frac{\alpha - 1}{\alpha} \partial_\mu \partial^\nu \right] A_\nu - J_\mu A^\mu$$  \hspace{1cm} (1.19)$$

(we integrated by parts and skipped some algebra to get to the second line). Performing the usual shift of variables and completing the square, we obtain

$$Z[J] = N \text{Det}[\partial^2] \text{Det} \left[ g^{\mu\nu} \partial^2 - \frac{\alpha - 1}{\alpha} \partial_\mu \partial^\nu \right] e^{-\frac{i}{2} \int d^4x \int d^4x' J_\mu(x) D_{\mu\nu}(x-x') J^\nu(x')}$$  \hspace{1cm} (1.20)$$

where $D_{\mu\nu}$ is the same $D_{\mu\nu}$ from part 1. This expression is independent of $\alpha$ since we started with a gauge-independent path integral. The parameter $\alpha$ was introduced when we averaged over different gauge-fixing conditions with a Gaussian; since the path integral was independent of the choice of gauge-fixing condition, it follows that it must also be independent of the choice of $\alpha$. Indeed, we can note that the exponential is explicitly independent of $\alpha$ since $\partial_\mu J^\mu(x) = 0$ and so the Fourier transform satisfies $p_\mu J^\mu(p) = 0$. Hence the term in the propagator dependent on $\alpha$ vanishes when
contracted with $J^\mu$. However, we cannot conclude that the path integral is gauge invariant by computing correlators of the gauge field, $\langle TA_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\rangle$ as these expression necessarily depend on the gauge.

2. PROPAGATORS AND CORRELATORS FOR THE ONE-DIMENSIONAL QUANTUM HEISENBERG ANITFERROMAGNET

We first recall the results of the spin-wave approximation from the previous problem set. We write the spin operators as:

\begin{align}
\hat{S}^+(j) &= \sqrt{2}\hat{a}(j) \\
\hat{S}^-(j) &= \sqrt{2}\hat{a}^\dagger(j) \\
\hat{S}_3(j) &= S - \hat{n}(j)
\end{align}

(2.1)

We can Fourier expand:

\begin{align}
\hat{a}(j) &= \sqrt{\frac{2}{N}} \sum_q e^{-iqj} \hat{a}(q), \\
\hat{b}(j) &= \sqrt{\frac{2}{N}} \sum_q e^{iqj} \hat{b}(j)
\end{align}

(2.2)

where the momentum sums are over the reduced Brillouin zone: $q \in [-\pi/2, \pi/2)$. In the spin-wave approximation, the Hamiltonian can be diagonalized via a Bogoliubov transformation

\begin{align}
\begin{pmatrix}
a \\
b^\dagger
\end{pmatrix} &= \begin{pmatrix}
cosh \theta & -\sinh \theta \\
-\sinh \theta & \cosh \theta
\end{pmatrix} \begin{pmatrix} c \\
d^\dagger
\end{pmatrix}
\end{align}

(2.3)

where

$$\cosh 2\theta_p = |\sin p|^{-1}.$$  

(2.4)

The ground state satisfies

$$\hat{c}_q |G\rangle = \hat{d}_q |G\rangle = 0 \quad \forall q.$$  

(2.5)

The single particle excitations satisfy

$$\hat{H}c_q |G\rangle = \varepsilon_q \hat{c}_q |G\rangle, \quad \hat{H}d_q |G\rangle = \varepsilon_q \hat{d}_q |G\rangle$$  

(2.6)
where

\[ \varepsilon_q = 2JS|\sin q|. \] (2.7)

In the spin-wave approximation we drop quantities of order less than 1/S. In particular, this means we can drop terms quadratic in \( \hat{n}(j) \) when computing the propagators below as, roughly speaking, \( \langle nn \rangle \sim 1/\varepsilon_q^2 \sim 1/S^2 \).

1. In the spin-wave approximation the unit cell is enlarged to two sites since we have assumed a Néel (i.e. antiferromagnetic) ordering. As such, when computing the propagators, we must consider different cases according to whether the sites are even or odd.

(a) Explicitly,

\[
D_{33}(nt,n't') = -i \langle G|T\hat{S}_3(n,t)\hat{S}_3(n',t')|G \rangle \\
= -i \langle G|\hat{S}_3(n,t)\hat{S}_3(n',t')|G \rangle \theta(t-t') - i \langle G|\hat{S}_3(n',t')\hat{S}_3(n,t)|G \rangle \theta(t'-t). 
\] (2.8)

Now,

\[
\langle G|\hat{S}_3(n,t)\hat{S}_3(n',t')|G \rangle = \langle G|[S - \hat{n}(n,t)][S - \hat{n}(n',t')]|G \rangle \\
\approx S^2 - S \langle G|\hat{n}(n,t)|G \rangle - S \langle G|\hat{n}(n',t')|G \rangle + \langle G|\hat{n}(n,t)\hat{n}(n',t')|G \rangle. 
\] (2.9)

Let us first consider the case where \( n \) and \( n' \) are both even. We then have that

\[
\langle G|\hat{n}(n,t)|G \rangle = \langle G|\hat{n}(n)|G \rangle \tag{2.11}
\]

\[
\begin{aligned}
\langle G|\hat{n}(n)|G \rangle &= 2N \sum_{q,q'} \langle G|e^{iqn}\hat{a}^\dagger(q)e^{-iq'n}\hat{a}(q')|G \rangle \\
&= 2N \sum_{q,q'} e^{in(q-q')} \langle G|\cosh \theta_q \hat{c}_q^\dagger - \sinh \theta_q \hat{d}_q|G \rangle \\
&= 2N \sum_{q,q'} \sinh^2 \theta_q. 
\end{aligned} 
\] (2.12)
We also find

\[ \langle G|\hat{n}(n, t)\hat{n}(n', t')|G\rangle \]

\[ = \frac{4}{N^2} \sum_{q,q',p,p'} \ e^{in(q-q')} e^{in'(p-p')} \langle G| [\cosh \theta_q \hat{c}_q^\dagger - \sinh \theta_q \hat{d}_q][\cosh \theta_{q'} \hat{c}_{q'} - \sinh \theta_{q'} \hat{d}_{q'}] \times e^{-\hat{H}(t-t')} [\cosh \theta_p \hat{c}_p^\dagger - \sinh \theta_p \hat{d}_p][\cosh \theta_{p'} \hat{c}_{p'} - \sinh \theta_{p'} \hat{d}_{p'}] |G\rangle \]

\[ = \frac{4}{N^2} \sum_{q,q',p,p'} \ e^{in(q-q')} e^{in'(p-p')} [\sinh \theta_{p'} \sinh \theta_{p'} \delta_{q,q'} | G\rangle \]

\[ + \sinh \theta_{p'} \cosh \theta_{p} e^{-i(\varepsilon_p + \varepsilon_{p'})(t-t')} \cosh \theta_{q'} \sinh \theta_{q'} \delta_{q,q'} \]

\[ = \frac{4}{N^2} \sum_{q,p} \sinh^2 \theta_{p} \sinh^2 \theta_{q} + \frac{4}{N^2} \sum_{q,p} e^{in(q-p)} e^{in'(p-q)} e^{-i(\varepsilon_p + \varepsilon_q)(t-t')} \sinh^2 \theta_{p} \cosh^2 \theta_{p} \]

\[ = \left( \frac{2}{N} \sum_{p} \sinh^2 \theta_{p} \right)^2 + \left( \frac{2}{N} \sum_{p} e^{ip(-n+n')} e^{-i\varepsilon_p (t-t')} \sinh^2 \theta_{p} \right) \times \left( \frac{2}{N} \sum_{p} e^{ip(-n+n')} e^{-i\varepsilon_p (t-t')} \cosh^2 \theta_{p} \right) \]

(2.16)

where, in the last line, we split the second double sum into a product of two sums over \( p \) and \( q \), and then relabelled \( q \to -p \), using the fact that \( \varepsilon_p \) and
$\sinh \theta_p, \cosh \theta_p$ are even under $p \rightarrow -p$. So,

$$D_{33}(nt, n't')$$

$$= -i \theta(t - t') \left\{ S^2 - 4 \frac{S}{N} \sum_q \sinh^2 \theta_q + \left( \frac{2}{N} \sum_p \sinh^2 \theta_p \right)^2 \right.$$  

$$+ \left( \frac{2}{N} \sum_p e^{ip(-n+n')e^{-i\epsilon_p(t-t')}} \sinh^2 \theta_p \right) \left( \frac{2}{N} \sum_p e^{ip(-n+n')e^{-i\epsilon_p(t'-t')}} \cosh^2 \theta_p \right) \right\}$$

$$+ (t \leftrightarrow t')$$

$$= -iS^2 + 4i \frac{S}{N} \sum_q \sinh^2 \theta_q + \left( \frac{2}{N} \sum_p \sinh^2 \theta_p \right)^2$$

$$+ \left( \frac{2}{N} \sum_p e^{ip(-n+n')e^{-i\epsilon_p(t-t')}} \sinh^2 \theta_p \right) \left( \frac{2}{N} \sum_p e^{ip(-n+n')e^{-i\epsilon_p(t'-t')}} \cosh^2 \theta_p \right) \theta(t - t')$$

$$+ \left( \frac{2}{N} \sum_p e^{ip(-n+n')e^{-i\epsilon_p(t-t')}} \sinh^2 \theta_p \right) \left( \frac{2}{N} \sum_p e^{ip(-n+n')e^{-i\epsilon_p(t'-t')}} \cosh^2 \theta_p \right) \theta(t' - t).$$

The first three terms are just constants and so their Fourier transforms are trivial to write down. As for the last two terms, we note that they are each the product of two functions written as inverse momentum-space Fourier transforms. By the convolution theorem, their momentum-space Fourier transforms will be the convolutions of the Fourier transforms of the factors in the products. So, we obtain,

$$iD_{33}(q, T) = \left( S^2 - 4 \frac{S}{N} \sum_p \sinh^2 \theta_p + \left( \frac{2}{N} \sum_p \sinh^2 \theta_p \right)^2 \right) \delta_{q,0}$$

$$+ \frac{2}{N} \sum_p \sinh^2 \theta_{p-q}e^{-i\epsilon_{p-q}T} \cosh^2 \theta_{p}e^{-i\epsilon_{p}T} \theta(T)$$

$$+ \frac{2}{N} \sum_p \sinh^2 \theta_{p-q}e^{i\epsilon_{p-q}T} \cosh^2 \theta_{p}e^{i\epsilon_{p}T} \theta(-T).$$

where we have set $T = t - t'$. By performing a Fourier transform in time or, equivalently, using the representation of the Heaviside step function as a contour
integral, \( \theta(T) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega T}}{\omega - i\epsilon} \), we can write

\[
D_{33}(q, \omega) = \int_{-\infty}^{\infty} D_{++}(q; T) e^{-i\omega T} dT
\]  
(2.21)

\[
= \left(-iS^2 + 4i \frac{S}{N} \sum_p \sinh^2 \theta_p - i \left( \frac{2}{N} \sum_p \sinh^2 \theta_p \right)^2 \right) \delta_{q,0} \delta(\omega) \]  
(2.22)

\[
- \frac{2}{N} \sum_p \sinh^2 \theta_{p-q} \cosh^2 \theta_p \left( \frac{1}{\omega + \epsilon_{p-q} + \epsilon_p - i\epsilon} - \frac{1}{\omega - \epsilon_{p-q} - \epsilon_p + i\epsilon} \right)
\]  
(2.23)

\[
= \left(-S^2 + 4 \frac{S}{N} \sum_p \sinh^2 \theta_p - \left( \frac{2}{N} \sum_p \sinh^2 \theta_p \right)^2 \right) \delta_{q,0} \left( \frac{1}{\omega - i\epsilon} - \frac{1}{\omega + i\epsilon} \right)
\]  
(2.24)

\[
- \frac{2}{N} \sum_p \sinh^2 \theta_{p-q} \cosh^2 \theta_p \left( \frac{1}{\omega + \epsilon_{p-q} + \epsilon_p - i\epsilon} - \frac{1}{\omega - \epsilon_{p-q} - \epsilon_p + i\epsilon} \right).
\]  
(2.25)

The equality of the first and second line is checked by inverting the Fourier transform and treating the \( \omega \) integral as a contour integral. When \( T > 0 \) we must close the contour in the upper half-plane and so we pick up the poles at \( \omega = i\epsilon \) and \( \omega = -\epsilon_{p-q} - \epsilon_p + i\epsilon \). When \( T < 0 \) we must close the contour in the lower half-plane and so we pick up the poles at \( \omega = -i\epsilon \) and \( \omega = \epsilon_{p-q} + \epsilon_p - i\epsilon \).

By identical calculations, one finds that for \( n \) and \( n' \) both even, the propagator takes the same form. In contrast, when one of \( n \), \( n' \) is odd and the other even, the propagator takes the form

\[
D_{33}(q, \omega) = \left(S^2 - 4 \frac{S}{N} \sum_p \sinh^2 \theta_p + \left( \frac{2}{N} \sum_p \sinh^2 \theta_p \right)^2 \right) \delta_{q,0} \left( \frac{1}{\omega - i\epsilon} - \frac{1}{\omega + i\epsilon} \right)
\]  
(2.26)

\[
+ \frac{1}{2N} \sum_p \sinh(2\theta_{p-q}) \sinh(2\theta_p) \left( \frac{1}{\omega + \epsilon_{p-q} + \epsilon_p - i\epsilon} - \frac{1}{\omega - \epsilon_{p-q} - \epsilon_p + i\epsilon} \right).
\]  
(2.27)
(b) Explicitly,

\[
D_{+-}(nt, n't') = -i \langle G | T \hat{S}^+(n, t) \hat{S}^-(n', t') | G \rangle \
= -i \langle G | \hat{S}^+(n, t) \hat{S}^-(n', t') | G \rangle \theta(t - t') - i \langle G | \hat{S}^-(n', t') \hat{S}^+(n, t) | G \rangle \theta(t' - t).
\]  

(2.28)

(2.29)

Let us first consider the case where both \( n \) and \( n' \) are even. We then have

\[
\langle G | \hat{S}^+(n, t) \hat{S}^-(n', t') | G \rangle = \langle G | \hat{S}^+(n) e^{-i\hat{H}(t-t')} \hat{S}^-(n') | G \rangle 
\approx 2S \langle G | \hat{a}(n) e^{-i\hat{H}(t-t')} \hat{a}^\dagger(n') | G \rangle 
= 2S \frac{2}{N} \sum_{q,q'} \langle G | e^{-iqn} \hat{a}(q) e^{-i\hat{H}(t-t')} \hat{a}^\dagger(q') e^{iq'n'} | G \rangle \times \cosh \theta_q \hat{c}_q - \sinh \theta_q \hat{d}_q \rangle | G \rangle
= 2S \frac{2}{N} \sum_{q,q'} e^{-iqn+iq'n'} e^{-i\varepsilon_q(t-t')} \cosh \theta_q \cosh \theta_q \delta_{q,q'}
= 2S \frac{2}{N} \sum_q e^{i\varepsilon_q(n+n')} e^{-i\varepsilon_q(t-t')} \cosh^2 \theta_q,
\]

(2.30)

(2.31)

(2.32)

(2.33)

(2.34)

(2.35)

where we used the fact that \( \hat{H} \hat{c}_q | G \rangle = \varepsilon_q \hat{c}_q | G \rangle \). By a similar calculation, we obtain

\[
\langle G | \hat{S}^-(n', t') \hat{S}^+(n, t) | G \rangle = 2S \frac{2}{N} \sum_q e^{i\varepsilon_q(n+n')} e^{-i\varepsilon_q(t'-t)} \sinh^2 \theta_q.
\]

(2.36)

Altogether,

\[
D_{+-}(nt, n't') = \left( -2iS \frac{2}{N} \sum_q e^{i\varepsilon_q(n+n')} e^{-i\varepsilon_q(t-t')} \cosh^2 \theta_q \right) \theta(t - t')
- \left( 2iS \frac{2}{N} \sum_q e^{i\varepsilon_q(n+n')} e^{-i\varepsilon_q(t'-t)} \sinh^2 \theta_q \right) \theta(t' - t).
\]

(2.37)

(2.38)

Setting \( T \equiv t - t' \), we can read off that

\[
D_{+-}(q; T) = -2iS \left[ e^{-i\varepsilon_q T} \cosh^2 \theta_q \theta(T) + e^{i\varepsilon_q T} \sinh^2 \theta_q \theta(-T) \right].
\]

(2.39)
By performing a Fourier transform in time or, equivalently, using the representation of the Heaviside step function as a contour integral, 
\[ \theta(T) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega T}}{\omega} \],
we can write
\[ D_{+-}(q; \omega) = \int_{-\infty}^{\infty} D_{+-}(q; T) e^{-i\omega T} dT = -2S \left[ \frac{\cosh^2 \theta_q}{\omega + \varepsilon_q - i\epsilon} - \frac{\sinh^2 \theta_q}{\omega_q - \varepsilon_q + i\epsilon} \right]. \tag{2.40} \]

One can check that this \( i\epsilon \) prescription satisfies the time-ordered boundary conditions. Indeed, when inverting the Fourier transform, we write the integral over frequency as a contour integral. When \( T > 0 \) we must close the contour in the upper half-plane and so we pick up the pole in the first term. When \( T < 0 \) we must close the contour in the lower half plane and so we pick up the pole in the second term.

Calculation of the propagator for \( n \) and \( n' \) both odd proceeds in much the same way. One finds
\[ D_{+-}(q; \omega) = -2S \left[ \frac{\sinh^2 \theta_q}{\omega + \varepsilon_q - i\epsilon} - \frac{\cosh^2 \theta_q}{\omega_q - \varepsilon_q + i\epsilon} \right]. \tag{2.41} \]

Lastly, when one of \( n \) and \( n' \) is odd and the other even, we find
\[ D_{+-}(q; \omega) = S \sinh 2\theta_q \left[ \frac{1}{\omega + \varepsilon_q - i\epsilon} - \frac{1}{\omega_q - \varepsilon_q + i\epsilon} \right]. \tag{2.42} \]

2. In the interaction picture, for an operator \( \hat{A}(x, t) \), the ground-state expectation value at time \( t \) is given by
\[ \langle G| U^\dagger(t) \hat{A}(x, t) U(t) |G \rangle \tag{2.43} \]

where
\[ U(t) = T e^{-\frac{i}{\hbar} \int_{-\infty}^{t} dt' H_{\text{ext}}(t')} \tag{2.44} \]
and \( \hat{A}(x, t) \) is time evolved using the unperturbed Hamiltonian (i.e. without \( H_{\text{ext}} \)). On expanding the above expectation value to first order in \( H_{\text{ext}} \), we find that the change in the expectation value of \( \hat{A} \) is given by
\[ \delta \langle \hat{A}(x, t) \rangle \approx \frac{i}{\hbar} \int_{-\infty}^{t} dt' \langle G|[H_{\text{ext}}(t'), \hat{A}(x, t)]|G \rangle. \tag{2.45} \]
In our case, the perturbation is

$$H_{\text{ext}} = \sum_n B(n) \cdot S(n) = \sum_n (B^+(n)S^+(n) + B^-(n)S^-(n) + S_z(n)B_z(n))$$

(2.46)

where

$$B_x = B^+ + B^-; \quad B_y = i(B^+ - B^-).$$

(2.47)

So,

$$\delta \langle \hat{A}(x, t) \rangle = \frac{i}{\hbar} \int_{-\infty}^{t} dt' \sum_{n'} \left[ B_z(n', t') \langle G|\hat{S}_z(n', t'), \hat{A}(x, t)|G \rangle \right.$$  

$$\left. + B^+(n', t') \langle G|\hat{S}^+(n', t'), \hat{A}(x, t)|G \rangle \right.$$  

$$\left. + B^-(n', t') \langle G|\hat{S}^-(n', t'), \hat{A}(x, t)|G \rangle \right].$$

(2.48)

Now, let us take $\hat{A}(n, t) = \hat{S}_3(n, t)$. In the spin-wave approximation, for $n$ and $n'$ both even,

$$\langle G|\hat{S}^+(n, t)\hat{S}_3(n', t')|G \rangle \approx \sqrt{2S} \left( \langle G|\hat{a}(n')e^{i\hat{H}(t-t')}|G \rangle - \langle G|\hat{a}(n')e^{i\hat{H}(t-t')}\hat{a}(n)|G \rangle \right).$$

(2.49)

It is immediate to see that this expression consists of expectation values of an odd number of $\hat{c}$ and $\hat{d}$ operators and so must vanish. The same holds for $n$ and $n'$ both odd or with one even and the other odd, as well as with $\hat{S}^+$ replaced with $\hat{S}^-$. Hence,

$$\langle G|\hat{S}^+(n', t'), \hat{S}_3(n, t)|G \rangle = \langle G|\hat{S}^-(n', t'), \hat{S}_3(n, t)|G \rangle = 0.$$  

(2.50)

So,

$$\delta \langle \hat{S}_3(n, t) \rangle = \frac{i}{\hbar} \int_{-\infty}^{t} dt' \sum_{n'} B_z(n', t') \langle G|\hat{S}_z(n', t'), \hat{S}_3(n, t)|G \rangle$$  

(2.51)

$$= -\frac{i}{\hbar} \int_{-\infty}^{t} dt' \sum_{n'} B_z(n', t') \langle G|\hat{S}_z(n', t'), \hat{S}_3(n, t)|G \rangle \theta(t - t')$$  

(2.52)

$$= \sum_{n'} \int_{-\infty}^{t} B_z(n', t') \chi_{33}(nt, n't')$$  

(2.53)

where we have defined the susceptibility

$$\chi_{33}(nt, n't') = -\frac{i}{\hbar} \langle G|\hat{S}_z(n', t'), \hat{S}_3(n, t)|G \rangle \theta(t - t')$$  

(2.54)

$$\equiv \frac{1}{\hbar} D_{33}^{R}(nt, n't').$$

(2.55)
Here $D_{33}^R(nt, n't')$ is a retarded propagator.

Let us now take $A(n, t) = \hat{S}^+(n, t)$. By logic similar to that above, we see that in the spin-wave approximation,

$$\langle G|\hat{S}^+(n', t'), \hat{S}^+(n, t)|G\rangle = 0. \quad (2.56)$$

Hence,

$$\delta \langle \hat{S}^+(n, t) \rangle = \frac{i}{\hbar} \int_{-\infty}^t dt' \sum_{n'} B^{-}(n', t') \langle G|\hat{S}^-(n', t'), \hat{S}^+(n, t)|G\rangle \theta(t - t') \quad (2.57)$$

$$= \sum_{n'} \int_{-\infty}^\infty B^{-}(n', t') \chi_{+-}(nt, n't') \quad (2.59)$$

where we have defined the susceptibility

$$\chi_{+-}(nt, n't') = \frac{1}{\hbar} \langle G|\hat{S}^+(n', t'), \hat{S}^-(n, t)|G\rangle \theta(t - t') \quad (2.60)$$

$$\equiv \frac{1}{\hbar} D_{+-}^R(nt, n't'). \quad (2.61)$$

Here $D_{+-}^R(nt, n't')$ is a retarded propagator.

Note that one must handle the cases of even and odd sites separately for the susceptibilities as well. Adding together the even-even, even-odd, and odd-odd propagators in momentum space does not yield a physically meaningful quantity.

3. We have already computed the time-ordered correlators in the spin-wave approximation in part 1.

4. We can deduce $D_{+-}^R$ from $D_{+-}$ by changing the integration contour, which amounts to changing the sign of one of the $i\epsilon$’s. Looking first at the even $n$–even $n'$ propagator, we have that ($\cosh(2\theta_p) = |\sin p|^{-1}$ and $\varepsilon_p = 2JS|\sin p|$)

$$\chi_{+-}(p, \omega) = \frac{1}{\hbar} D_{+-}^R(p, \omega) \quad (2.62)$$

$$= -\frac{2S}{\hbar} \left[ \frac{\cosh^2 \theta_p}{\omega + \varepsilon_p + i\epsilon} - \frac{\sinh^2 \theta_p}{\omega - \varepsilon_p + i\epsilon} \right] \quad (2.63)$$

$$= -\frac{2S}{\hbar} \left[ \frac{(\omega + i\epsilon) - \varepsilon_p \cosh 2\theta_p}{(\omega + i\epsilon)^2 - \varepsilon_p^2} \right] \quad (2.64)$$

$$= -\frac{2S}{\hbar} \left[ \frac{(\omega + i\epsilon) - 2JS}{(\omega + i\epsilon)^2 - (2JS)^2 \sin^2 p} \right]. \quad (2.65)$$
where we used some hyperbolic function identities in the third line. If we were to set \( \omega = 0 \), it is clear that we would get a pole at \( p = \pi \). However, this would be a pole of order two and so would have vanishing residue. In order to extract a finite, non-zero quantity, we must keep \( \omega \) finite which leads to two first-order poles near \( p = \pi \). Indeed, Taylor expanding the denominator about \( p = \pi \), we find

\[
\chi_{+-}(p, \omega) = -\frac{2S}{\hbar} \left[ \frac{(\omega + i\epsilon) - 2JS}{(\omega + i\epsilon)^2 - (2JS)^2(p - \pi)^2} \right] = -\frac{2S}{\hbar} \frac{1}{(2JS)^2} \left[ \frac{(\omega + i\epsilon) - 2JS}{(\omega + i\epsilon)^2/(2JS)^2 - (p - \pi)^2} \right]
\]

which has poles at \( p = \pi \pm (\omega + i\epsilon)/(2JS) \). Focusing on one of these poles, we find the residue

\[
\lim_{\omega \to 0} \text{Res}_{p = \pi - (\omega + i\epsilon)/(2JS)} \chi_{+-}(p, \omega) = \lim_{\omega \to 0} \frac{\omega}{\hbar} \frac{1}{(2JS)^2} \frac{\omega - 2JS}{-\omega/(JS)} = \frac{S}{\hbar}.
\]

(Note that added in a factor of \( \omega \) to make the limit convergent).

The odd \( n \)-odd \( n' \) susceptibility is given by

\[
\chi_{+-}(p, \omega) = \frac{-2S}{\hbar} \left[ \frac{\sinh^2 \theta_p}{\omega + \epsilon_p + i\epsilon} - \frac{\cosh^2 \theta_p}{\omega + \epsilon_p + i\epsilon} \right] = \frac{-2S}{\hbar} \left[ \frac{-(\omega + i\epsilon) - 2JS}{(\omega + i\epsilon)^2 - (2JS)^2 \sin^2 p} \right].
\]

So, by similar manipulations to those above, we find

\[
\lim_{\omega \to 0} \text{Res}_{p = \pi - (\omega + i\epsilon)/(2JS)} \omega \chi_{+-}(p, \omega) = \lim_{\omega \to 0} \frac{\omega}{\hbar} \frac{1}{(2JS)^2} \frac{\omega - 2JS}{-\omega/(JS)} = \frac{S}{\hbar}.
\]

This expression also has a second order pole at \( p = \pi \) with residue zero.

Lastly, for one of \( n, n' \) odd and the other even, the susceptibility is given by

\[
\chi_{+-}(p, \omega) = \frac{S}{\hbar} \sinh 2\theta_p \left[ \frac{1}{\omega + \epsilon_p + i\epsilon} - \frac{1}{\omega - \epsilon_p + i\epsilon} \right] = \frac{S}{\hbar} \sinh 2\theta_p \left[ \frac{-2\epsilon_p}{(\omega + i\epsilon)^2 - \epsilon_p^2} \right] = \frac{S}{\hbar} \frac{-4J}{(\omega + i\epsilon)^2 - \epsilon_p^2}
\]
where we used the fact that \( \sinh 2\theta_p = \cosh 2\theta_p \tanh 2\theta_p = \cos p/|\sin p| \). Expanding about \( p = \pi \),

\[
\chi_{-+}(p, \omega) = \frac{S}{\hbar (\omega + i\epsilon)^2 - (2JS)^2 \sin^2 p} - 4JS \\
= -\frac{S}{\hbar (2JS)^2 (p - \pi - (\omega + i\epsilon)/(2JS))(p - \pi + (\omega + i\epsilon)/(2JS))},
\]

(2.76)

(2.77)

This has the same poles as the other propagators. Again, focusing on one of them, we find

\[
\lim_{\omega \to 0} \text{Res}_{p=\pi-(\omega+i\epsilon)/(2JS)} \omega \chi_{-+}(p, \omega) = -\frac{S}{\hbar}.
\]

(2.78)

3. SPECTRAL FUNCTION FOR THE DIRAC PROPAGATOR

1. By definition, we have that

\[
\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle = \theta(t) \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle - \theta(-t) \langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0 \rangle
\]

(3.1)

where \( t \equiv x_0 - y_0 \). Let us consider the first of the two point functions (i.e. take \( x_0 > y_0 \)). Inserting a complete basis of states with definite momenta, we find

\[
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle = \int \sum_n \langle 0|\psi_\alpha(x)|n_p \rangle \langle n_p|\bar{\psi}_\beta(y)|0 \rangle \frac{1}{2E_p} \frac{d^3p}{(2\pi)^3}.
\]

(3.2)

Using the fact that \( P_\mu \) is the generator of translations and choosing the \( |n_p \rangle \) to be eigenstates of momentum, we find that

\[
\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle = \int \sum_n \langle 0|\psi_\alpha(0)|n_p \rangle \langle n_p|\bar{\psi}_\beta(0)|0 \rangle e^{-ip(x-x')} \frac{1}{2E_p} \frac{d^3p}{(2\pi)^3}
\]

(3.3)

\[
= \int \sum_n A_{\alpha\beta}(p) e^{-ip(x-x')} \frac{d^3p}{(2\pi)^3}
\]

where we have set \( A_{\alpha\beta}(p) = \langle 0|\psi_\alpha(0)|n_p \rangle \langle n_p|\bar{\psi}_\beta(0)|0 \rangle \). We can rewrite this as a contour integral over \( p_0 \):

\[
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle = i \int \sum_n \frac{A_{\alpha\beta}(p)}{p^2 - m_n^2 + i\epsilon} e^{-ip(x-x')} \frac{d^4p}{(2\pi)^4}.
\]

(3.4)

Finally, on inserting \( 1 = \int_0^{\infty} d\mu^2 \delta(\mu^2 - m_n^2) \), we obtain

\[
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \int_0^{\infty} d\mu^2 \sum_n \delta(\mu^2 - m_n^2) A_{\alpha\beta}(p) e^{-ip(x-x')}, \quad (x_0 > y_0)
\]

(3.5)
If we repeat the same analysis for the $y_0 > x_0$ term, $\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0 \rangle$, it would seem that we would get a different set of matrix elements, $\langle 0|\bar{\psi}_\beta(0)|n_p \rangle \langle n_p|\psi_\alpha(0)|0 \rangle$ and hence a different spectral function. However, if one were to repeat the same calculation for the anti-commutator, instead of the time-ordered propagator, one would find that it vanishes at space-like separations only if these two spectral densities are the same. So, it must be the case that the $y_0 > x_0$ term, $\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0 \rangle$, provides the same contribution as the the $x_0 > y_0$ term.

Hence, we can read off that in momentum space,

$$S_{\alpha\alpha'}^F(p) = -i \int_0^\infty d\mu^2 \frac{\rho_{\alpha\alpha'}(p)}{p^2 - \mu^2 + i\epsilon} \tag{3.6}$$

where

$$\rho_{\alpha\alpha'}(p) = \sum_n \delta(\mu^2 - m_n^2) \langle 0|\psi_\alpha(0)|n_p \rangle \langle n_p|\bar{\psi}_\beta(0)|0 \rangle. \tag{3.7}$$

2. The spectral function, $\rho(p)$, is a $4 \times 4$ matrix and must be Lorentz invariant. Moreover, it can only be a function of the four-momentum $p$ and must have the same transformation properties as the Dirac fields under boosts. Given these constraints, it follows that $\rho(p^2)$ must be a linear combination of $\gamma$ matrices and the identity of the following form

$$\rho(p) = \rho_1(p^2)\not{\!\!1} + \rho_2(p^2)\not{\!\!1} + \bar{\rho}_1(p^2)\gamma_5\not{\!\!1} + \bar{\rho}_2(p^2)\gamma_5. \tag{3.8}$$

Now, we recall that under a parity transformation, $\psi(x_0, \mathbf{x}) \to \gamma_0\psi(x_0, -\mathbf{x})$ and $\bar{\psi}(x_0, \mathbf{x}) \to \bar{\psi}(x_0, -\mathbf{x})\gamma_0$. So, the spectral function will transform as

$$P \rho(p) P^{-1} = \gamma_0\rho(-p)\gamma_0. \tag{3.9}$$

If the vacuum is invariant under parity, then the spectral function must be as well. Inspection of the most general, Lorentz invariant form given above reveals that the only non-invariant terms are those containing $\gamma_5$. So, we obtain

$$\rho(p) = \rho_1(p^2)\not{\!\!1} + \rho_2(p^2)\not{\!\!1}, \tag{3.10}$$

This may be interpreted as being a consequence of CPT invariance. See, for instance, the discussion in *Weinberg, Vol. I* on the Lehmann representation.
as required.

Now, we have that

\[ S^F(p) = i \int_0^\infty dm'^2 \rho_1(m'^2) \frac{1 + \rho_2(m'^2)}{p^2 - m'^2 + i\epsilon}. \]  \hspace{1cm} (3.11)

But we know that in the free Dirac theory,

\[ S^F(p) = i \frac{\bar{\psi} + m}{p^2 - m^2 + i\epsilon}. \]  \hspace{1cm} (3.12)

So, we can read off

\[ \rho_1(\mu^2) = \delta(\mu^2 - m^2), \quad \rho_2(\mu^2) = m\delta(\mu^2 - m^2). \]  \hspace{1cm} (3.13)

4. WICK'S THEOREM

1. Due to the global $O(3)$ symmetry and the fact that the theory is free, the ground state is invariant under $O(3)$ rotations and so any $n$-point function which is not $O(3)$ invariant must necessarily vanish. Consider the following correlation functions:

(a) \( \langle 0 | T\phi_a(x)\phi_a(x') | 0 \rangle \)

(b) \( \langle 0 | T\phi_a(x)\phi_b(x')\phi_b(x'') | 0 \rangle \)

(c) \( \langle 0 | T\phi_a(x)\phi_a(x')\phi_b(x'')\phi_b(x''') | 0 \rangle \).

It is immediate to see that (a) and (c) are invariant under $O(3)$ rotations since all the flavor indices are contracted. Explicitly,

\[ \langle 0 | T\phi_a(x)\phi_a(x') | 0 \rangle \rightarrow \langle 0 | TO_{ab}\phi_b(x)O_{ad}\phi_d(x') | 0 \rangle = \langle 0 | T\phi_a(x)\phi_a(x') | 0 \rangle \]  \hspace{1cm} (4.4)

where we used the fact that for $O \in O(3)$, \( O_{ab}O_{ad} = O_{ba}^T O_{ad} = \delta_{bd} \). In contrast, (b) is not invariant under $O(3)$ rotations since it has one free index; indeed, it transforms as a vector:

\[ \langle 0 | T\phi_a(x)\phi_b(x')\phi_b(x'') | 0 \rangle \rightarrow \langle 0 | TO_{ac}\phi_c(x)O_{bd}\phi_d(x')O_{be}\phi_e(x'') | 0 \rangle \]

\[ = O_{ac} \langle 0 | T\phi_c(x)\phi_b(x')\phi_b(x'') | 0 \rangle. \]  \hspace{1cm} (4.5)

So, if (b) were to be non-zero, this would mean there is a preferred direction in flavor-space, contradicting the fact that the ground state is invariant under $O(3)$ rotations. Hence (b) must vanish.
2. Since \((b)\) vanishes, we only need to look at \((c)\). Using Wick’s theorem, we obtain

\[
\langle 0 | T \phi_a(x) \phi_b(x') \phi_b(x'') \phi_b(x'''') | 0 \rangle = \langle 0 | T \phi_a(x) \phi_b(x') | 0 \rangle \langle 0 | T \phi_b(x'') \phi_a(x''') | 0 \rangle \\
+ \langle 0 | T \phi_a(x) \phi_b(x''') | 0 \rangle \langle 0 | T \phi_b(x') \phi_a(x'') | 0 \rangle \\
+ \langle 0 | T \phi_a(x) \phi_b(x'') | 0 \rangle \langle 0 | T \phi_b(x') \phi_b(x''') | 0 \rangle .
\]

Due to the global \(O(3)\) symmetry, we have that

\[
\langle 0 | T \phi_1(x) \phi_1(x') | 0 \rangle = \langle 0 | T \phi_2(x) \phi_2(x') | 0 \rangle = \langle 0 | T \phi_3(x) \phi_3(x') | 0 \rangle = \frac{1}{3} \langle 0 | T \phi_a(x) \phi_a(x') | 0 \rangle .
\]

Conversely, terms of the form \(\langle 0 | T \phi_a(x) \phi_b(x') | 0 \rangle\), \(a \neq b\) must vanish as they are not invariant under \(O(3)\) rotations. We can thus write

\[
\langle 0 | T \phi_a(x) \phi_a(x') \phi_b(x'') \phi_b(x''') | 0 \rangle = \frac{1}{3} \langle 0 | T \phi_a(x) \phi_a(x') | 0 \rangle \langle 0 | T \phi_a(x'') \phi_a(x''') | 0 \rangle \\
+ \frac{1}{3} \langle 0 | T \phi_a(x) \phi_a(x'') | 0 \rangle \langle 0 | T \phi_a(x') \phi_a(x''') | 0 \rangle \\
+ \langle 0 | T \phi_a(x) \phi_a(x'') | 0 \rangle \langle 0 | T \phi_a(x') \phi_a(x''') | 0 \rangle \\
+ \langle 0 | T \phi_a(x) \phi_a(x'') | 0 \rangle \langle 0 | T \phi_a(x') \phi_a(x''') | 0 \rangle
\]

**5. REDUCTION FORMULAS**

First, we must construct the ‘in’ state creation and annihilation operators. A free, complex scalar field has the mode expansion

\[
\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} (a(k)e^{-ik \cdot x} + b^\dagger(k)e^{ik \cdot x})
\]

where \(k^\mu = (\omega(k), \mathbf{k})\). We also have

\[
\Pi_\phi(x) = \partial_0 \phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} (-i\omega(k))(-a^\dagger(k)e^{ik \cdot x} + b(k)e^{-ik \cdot x}).
\]

In order to solve for \(a(k)\) and \(b(k)\), we compute

\[
\int d^3x e^{ik \cdot x} \phi(x) = \frac{1}{2\omega(k)} a(k) + \frac{1}{2\omega(k)} b(-k)e^{2i\omega(k)x_0}
\]

\[
\int d^3x e^{ik \cdot x} \partial_0 \phi(x) = -\frac{i}{2} a(k) + \frac{1}{2} b(-k)e^{2i\omega(k)x_0}.
\]

We can invert these expressions to find \(a(k) = \int d^3x [\omega(k)\phi(x) + i\partial_0 \phi(x)]e^{ik \cdot x}\). A similar expression for \(b(k)\) with \(\phi \leftrightarrow \phi^\dagger\) can be found by considering the Fourier transforms of \(\phi^\dagger\).
and \( \partial_0 \phi \) instead. So we have

\[
a_{\text{in}}(k) = i \int d^3 x e^{i k \cdot x} \partial_0 \phi_{\text{in}}(x) \tag{5.5}
\]

\[
b_{\text{in}}(k) = i \int d^3 x e^{i k \cdot x} \partial_0 \phi_{\text{in}}(x), \tag{5.6}
\]

where we have used the notation \( f \leftrightarrow \partial g = f \partial g - \partial f g \). As for the gauge field, we recall that in the \( A_0 = 0 \) gauge the transverse components have the mode expansion

\[
A(x) = \frac{d^3 k}{(2\pi)^3 2|k|} \sum_{\alpha=1,2} \varepsilon(\alpha)[a_{\alpha}(k)e^{-i k \cdot x} + a_{\alpha}^\dagger(k)e^{i k \cdot x}] \tag{5.7}
\]

where \( \varepsilon_{\alpha} \) are the polarization vectors. In analogy with the computation for a real scalar field and making use of the orthonormality of the polarization vectors, \( \varepsilon_{\alpha} \cdot \varepsilon_{\beta} = \delta_{\alpha\beta} \), we find for the gauge field

\[
a_{\alpha,\text{in}}(k) = i \int e^{i k \cdot x} \partial_0 A_{\alpha,\text{in}}^\mu(x)(\varepsilon_{\alpha})_\mu(k)d^3 x. \tag{5.8}
\]

Here we have defined \( (\varepsilon_{\alpha})_0 = 0 \) to make use of four-vector notation.

We now proceed to the computation of the \( S \)-matrix element:

\[
\langle p_+, p_-; \text{out} | p_i, \alpha; \text{in} \rangle = \langle p_+, p_- | S | p_i, \alpha \rangle \tag{5.9}
\]

\[
= \langle p_+, p_-; \text{out} | a_{\alpha,\text{in}}^\dagger(p_i)|0; \text{in} \rangle \tag{5.10}
\]

\[
= -i \lim_{t \to -\infty} Z_{A}^{-1/2} \int d^4 x e^{-ip_i \cdot x} \partial_0 \langle p_+, p_-; \text{out} | A^\mu(x)|0; \text{in} \rangle (\varepsilon_{\alpha})_\mu(p_i) \tag{5.11}
\]

where \( Z_{A} \) is the wavefunction renormalization for \( A_\mu \) which takes into account the fact that \( A_{\mu,\text{in}}(x) \) creates single particle states whereas \( A_\mu(x) \) also creates multi-particle states. Following the steps in the notes, we recall that for some function \( F(x) \)

\[
(\lim_{t_+ \to \infty} - \lim_{t_- \to -\infty}) \int d^3 x F(x) = \int d^4 x \partial_0 F(x), \tag{5.12}
\]

and so taking \( F(x) = e^{-ip_i \cdot x} \partial_0 \langle p_+, p_-; \text{out} | A^\mu(x)|0; \text{in} \rangle (\varepsilon_{\alpha})_\mu(p_i) \), we can write

\[
\langle p_+, p_-; \text{out} | p_i, \alpha; \text{in} \rangle = \langle p_+, p_-; \text{out} | a_{\alpha,\text{out}}^\dagger(p_i)|0; \text{in} \rangle
\]

\[
+ \frac{i}{Z_A^{1/2}} \int d^4 x \partial_0 \left[ e^{-ip_i \cdot x} \partial_0 \langle p_+, p_-; \text{out} | A^\mu(x)|0; \text{in} \rangle (\varepsilon_{\alpha})_\mu(p_i) \right]. \tag{5.13}
\]
where we have used the fact that
\[
\langle 0|A_\mu(x)|1 \rangle = Z_A^{1/2} \langle 0|A_{\mu,\text{in}}(x)|1 \rangle = Z_A^{1/2} \langle 0|A_{\mu,\text{out}}(x)|1 \rangle.
\] (5.14)

The first term clearly vanishes since it is an overlap of states with different particle species. To simplify the second term, we note that the plane wave \( e^{-ip_x x} \) must satisfy the gauge field EOM:
\[
\partial^2 e^{-ip_x x} = 0.
\] (5.15)

So, we have that
\[
\int d^4x \partial_0 \left[ e^{-ip_x x} \partial_0 (p_+, p_-; \text{out} | A_\mu(x) | 0; \text{in}) (\varepsilon_\alpha)_\mu(p_i) \right]
= \int d^4x \left[ e^{-ip_x x} \partial_0^2 (p_+, p_-; \text{out} | A_\mu(x) | 0; \text{in}) (\varepsilon_\alpha)_\mu(p_i) - (\partial_0^2 e^{-ip_x x}) (p_+, p_-; \text{out} | A_\mu(x) | 0; \text{in}) (\varepsilon_\alpha)_\mu(p_i) \right]
= \int d^4x \left[ e^{-ip_x x} \partial_0^2 (p_+, p_-; \text{out} | A_\mu(x) | 0; \text{in}) (\varepsilon_\alpha)_\mu(p_i) \right]
= \int d^4x \left[ e^{-ip_x x} \partial^2 (p_+, p_-; \text{out} | A_\mu(x) | 0; \text{in}) (\varepsilon_\alpha)_\mu(p_i) \right]
\] (5.16) (5.17) (5.18)

where in the penultimate line we used the above EOM and in the final line we integrated by parts. Repeating the same calculation for the outgoing \( p_\pm \) complex scalar states, we obtain
\[
\langle p_+, p_-; \text{out} | p_i, \alpha; \text{in} \rangle = \frac{i^3}{Z_\alpha Z_A^{1/2}} \int d_4y_+ d_4y_- d_4x \left[ e^{-ip_x x + p_+ y_+ + p_- y_-} \times (\partial_{y_+}^2 + m^2)(\partial_{y_-}^2 + m^2)\partial_x^2 (0; \text{out} | T \phi(y_+) \phi^\dagger(y_-) A_\mu(x) | 0; \text{in}) (\varepsilon_\alpha)_\mu(p_i) \right].
\] (5.20) (5.21)

Note that the correlator is time-ordered because the in and out states are already arranged in a time-ordered fashion to begin with.