1 Spin waves in a quantum Heisenberg Antiferromagnet

1. First let’s recall the Heisenberg equation of motion for an operator $\hat{A}$:

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A}, \hat{H}] + \partial_t \hat{A}$$

In our case we have operators which do not depend explicitly on time; thus, the above equation can be written as:

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}]$$

The Hamiltonian for the Heisenberg Antiferromagnet is:

$$\hat{H} = J \sum_{j=-N/2+1}^{N/2} \hat{S}_k(j) \cdot \hat{S}_k(j+1)$$

We are interested in the case that $\hat{A}$ is a spin-operator. Since $\hat{S}_j$ belongs to the spin-$S$ representation irrespective of the parity of $j$, we do not have to consider special cases here. Furthermore, we can simplify our calculation if we rewrite our Hamiltonian using ladder operators:

$$\hat{H} = J \sum_j \left[ \hat{S}_3(j)\hat{S}_3(j+1) + \frac{1}{2} \left( \hat{S}^+(j)\hat{S}^-(j+1) + \hat{S}^-(j)\hat{S}^+(j+1) \right) \right]$$

It is easy to show for any pair of sites, $j$ and $j'$:

- $\left[ \hat{S}_3(j), \hat{S}_3(j') \right] = \pm i\hbar \delta_{jj'}$
- $\left[ \hat{S}^\pm(j), \hat{S}^{\mp}(j') \right] = \pm 2\hbar \delta_{jj'}$

It is now easy to arrive at:
(a)  
\[ \frac{d\hat{S}_3(j)}{dt} = \frac{J}{2i\hbar} \sum_{j'} \left( [\hat{S}_3(j), \hat{S}^+(j')\hat{S}^-(j' + 1)] + [\hat{S}_3(j), \hat{S}^-(j')\hat{S}^+(j' + 1)] \right) \]
\[ = \frac{J}{2i\hbar} \sum_{j'} \left( [\hat{S}_3(j), \hat{S}^+(j')]\hat{S}^-(j' + 1) + \hat{S}^+(j')[\hat{S}_3(j), \hat{S}^- (j')\hat{S}^+(j' + 1)] \right) \]
\[ = \frac{J}{2i} \sum_{j'} \left( (\hat{S}^+(j)\hat{S}^-(j' + 1) - \hat{S}^-(j)\hat{S}^+(j' + 1))\delta_{jj'} + (\hat{S}^-(j')\hat{S}^+(j) - \hat{S}^+(j')\hat{S}^-(j))\delta_{jj' + 1} \right) \]
\[ = \frac{J}{2i} \left( (\hat{S}^-(j + 1) + \hat{S}^-(j - 1))\hat{S}^+(j) - (\hat{S}^+(j + 1) + \hat{S}^+(j - 1))\hat{S}^-(j) \right) \]
\[ \tag{1} \]

Note: I used the fact spin operators on distinct lattice points commute to get the EOM in the above form.

(b) Proceeding in an analogous manner to a.):
\[ \frac{d\hat{S}^\pm(j)}{dt} = \pm \frac{J}{2i} \left( \hat{S}_3(j) \left( \hat{S}^\pm(j + 1) + \hat{S}^\pm(j - 1) \right) - \hat{S}^\pm(j) \left( \hat{S}_3(j + 1) + \hat{S}_3(j - 1) \right) \right) \]

So we have a system of first order differential equations for operators which depend on time. Since our equations contain products of operators which also depend on time, we can not express these differential equation as a differential operator which acts linearly; that is, they are not linear.

2. Note: I probably did this problem different than the way you decided to do it. Letting the operators provided speak for themselves is all you needed to do. I motivate the alternative form handed to you.

We recall how the raising and lowering operators act on our 2S + 1 basis states:
\[ S^\pm |S, M(j)\rangle = \sqrt{(s \mp m)(s \pm m + 1)} |S, M(j) \pm 1\rangle \]

Here I used lower case s’s and m’s in the radical; this just saves on clutter.

In this problem we are interested in how our system deviates from perfect antiferromagnetism, so we consider the following operator used to label our eigenstates in terms of its eigenvalues:
\[ \hat{n}(j) = S + (-1)^{j-1}\hat{S}_3(j) \]

Which has the corresponding quantum numbers:
\[ M(j) = S + (-1)^{j-1}n(j) \]

Suppressing the j dependence on our eigenvalues, \( m = s + (-1)^{j-1}n \), we will substitute this into our above equation. Starting with even j:
\[ S^+ |S, M(j)\rangle = \sqrt{(s - m)(s + m + 1)} |S, M(j) + 1\rangle \]
\[ = \sqrt{2sn(1 - \frac{n - 1}{2s})} |S, M(j) + 1\rangle \]
\[ := \sqrt{2sn(1 - \frac{n - 1}{2s})}|n - 1\rangle \]
\[ \tag{2} \]
\[ S^{-}|S, M(S)\rangle = \sqrt{(s + m)(s - m + 1)}|S, M(j) - 1\rangle \\
= \sqrt{2s(n + 1)(1 - \frac{n}{2s})}|S, M(j) - 1\rangle \\
:= \sqrt{2s\left(1 - \frac{n - 1}{2s}\right)}|n + 1\rangle \quad (3) \]

Notice the lowering ladder operator increases \(n\) here. This makes sense since even sites where elected to be spin up and so the spin deviation for these states are increased when we decrease \(M\).

Now for odd \(j\):
\[ S^{+}|S, M(S)\rangle = \sqrt{(s - m)(s + m + 1)}|S, M(j) + 1\rangle \\
= \sqrt{2s(n + 1)(1 - \frac{n}{2s})}|S, M(j) + 1\rangle \\
:= \sqrt{2s(n + 1)(1 - \frac{n}{2s})}|n + 1\rangle \quad (4) \]

Notice the raising ladder operator sends \(n \to n + 1\) here. Similarly:
\[ S^{-}|S, M(S)\rangle = \sqrt{(s + m)(s - m + 1)}|S, M(S) - 1\rangle \\
= \sqrt{2sn\left(1 - \frac{n - 1}{2s}\right)}|S, M(S) - 1\rangle \\
:= \sqrt{2sn\left(1 - \frac{n - 1}{2s}\right)}|n - 1\rangle \quad (5) \]

The form of our eigenstates are reminiscent of the Harmonic oscillator’s, which inspires us to introduce the following creation and annihilation operators. First, for even sites:
\[ \hat{a}^\dagger |n\rangle = \sqrt{n + 1}|n + 1\rangle \]
\[ \hat{a}|n\rangle = \sqrt{n}|n - 1\rangle \]

and for odd sites,
\[ \hat{b}^\dagger |n\rangle = \sqrt{n + 1}|n + 1\rangle \]
\[ \hat{b}|n\rangle = \sqrt{n}|n - 1\rangle \]

Clearly, \([\hat{a}, \hat{b}] = 0\). All the remaining commutation relations we suspect from the harmonic oscillator are immediately satisfied from the above equations.

Using the above definitions we express our ladder operators in the following form:
\[ S^{+}|n\rangle = \sqrt{2sn\left(1 - \frac{n - 1}{2s}\right)}|n - 1\rangle \\
= \sqrt{2s\left(1 - \frac{\hat{n}}{2s}\right)}\sqrt{n}|n - 1\rangle \\
= \sqrt{2s\left(1 - \frac{\hat{n}}{2s}\right)}\hat{a}|n\rangle \\
\Rightarrow S^{+} = \sqrt{2s\left(1 - \frac{\hat{n}}{2s}\right)}\hat{a} \quad (6) \]
This is the relation we are after!

We can easily justify the second line above. Indeed, if we can expand a function, \( f \), as a Taylor series, then it is easy to prove with induction: for Hermitian operators, \( \hat{A} \), with it’s corresponding eigenstates, \( |\alpha\rangle \), we have: \( \hat{A}^k |\alpha\rangle = \alpha^k |\alpha\rangle \). We use this to send \( n - 1 \rightarrow \hat{n} \) since the eigenstate is currently \( |n - 1\rangle \), and the square-root is analytic. We then use the fact \( n \) is a real number to permute it with \( \hat{n} \).

We can find \( S^- \) in a similar fashion:

\[
S^- |n\rangle = \sqrt{2s(n + 1)(1 - \frac{n}{2s})}|n + 1\rangle
\]
\[
= \sqrt{2s(1 - \frac{n}{2s})}\hat{a}^\dagger |n\rangle
\]
\[
= \hat{a}^\dagger \sqrt{2s(1 - \frac{n}{2s})}|n\rangle
\]
\[
= \hat{a}^\dagger \sqrt{2s(1 - \frac{n}{2s})}|n\rangle
\]
\[
\Rightarrow S^- = \hat{a}^\dagger \sqrt{2s(1 - \frac{n}{2s})}
\]

We can rinse and repeat the above procedure, but instead, we recall the odd sites act counter to the even, so we just swap the roles of \( S^+ \leftrightarrow S^- \). For completeness:

\[
S^+ = \hat{b}^\dagger \sqrt{2s(1 - \frac{n}{2s})}
\]
\[
S^- = \sqrt{2s(1 - \frac{n}{2s})}\hat{b}
\]

This illustrates we can rewrite our ladder operators with the spin excess operator in place of the run of the mill spin operators!

3. We now rewrite the Hamiltonian in terms of these operators. First recall:

\[
\hat{H} = J \sum_{j=-N/2+1}^{N/2} \left[ \hat{S}_3(j)\hat{S}_3(j + 1) + \hat{S}_1(j)\hat{S}_1(j + 1) + \hat{S}_2(j)\hat{S}_2(j + 1) \right]
\]
\[
= J \sum_{j=-N/2+1}^{N/2} \left[ \hat{S}_3(j)\hat{S}_3(j + 1) + \frac{1}{2} \left( \hat{S}^+(j)\hat{S}^-(j + 1) + \hat{S}^-(j)\hat{S}^+(j + 1) \right) \right]
\]

Note: these operators are lattice site dependent; i.e. even or odd, so we need to be careful here! First let’s note:

\[
\hat{S}_3(j)\hat{S}_3(j + 1) = \left[ (-1)^{j-1}(n(j) - S) \right] \left[ (-1)^{(j+1)-1}(n(j + 1) - S) \right]
\]
\[
= -(n(j) - S)(n(j + 1) - S)
\]
\[
= -S^2 + S(n(j) + n(j + 1)) - n(j)n(j + 1)
\]

Thus, this term is independent of the parity of \( j \). We can easily see:

\[
J \sum_{j=-N/2+1}^{N/2} \hat{S}_3(j)\hat{S}_3(j + 1) = -NS^2 + J \sum_{j=-N/2+1}^{N/2} \left( S(n(j) + n(j + 1)) - n(j)n(j + 1) \right)
\]
The ladder operators are where we should be careful, so let’s consider even sites alone. First, we define $\sqrt{1 - \frac{n(j)}{2S}} = \hat{f}(j)$ to reduce clutter. This results in:

$$J \sum_{j \text{ even}} \frac{1}{2} \left( \hat{S}^+(j)\hat{S}^-(j+1) + \hat{S}^-(j)\hat{S}^+(j+1) \right) = SJ \sum_{j \text{ even}} \left( \hat{f}(j)\hat{f}(j+1)\hat{a}(j)\hat{b}(j+1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j+1)\hat{f}(j)\hat{f}(j+1) \right)$$

(10)

All that is different when summing the odd sites is the $j + 1$ terms correspond to $\hat{a}$ and the $j$ terms to $\hat{b}$. Performing the substitutions:

$$\hat{H} = -JNS^2 + J \sum_{j} \left( S(\hat{n}(j) + \hat{n}(j+1)) - \hat{n}(j)\hat{n}(j+1) \right) + SJ \sum_{j \text{ even}} \left( \hat{f}(j)\hat{f}(j+1)\hat{a}(j)\hat{b}(j+1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j+1)\hat{f}(j)\hat{f}(j+1) \right)$$

(11)

4. Now that we have an expression for our Hamiltonian in terms of our creation and annihilation operators we take the limit $S \to \infty$.

Looking at our above sum the leading order term is clearly $S^2$, so let’s consider terms linear in $S$ and neglect those of lower order (i.e. $S^0$, $S^{-1}$, ...). Following this line of reasoning the summation resulting from the $\hat{S}_3$’s is trivial: drop $\hat{n}(j)\hat{n}(j+1)$. Notice the order of our creation and annihilation operators has been reduced for this part of the Hamiltonian!

The portion containing the ladder operators takes a bit more work. $S$ shows up as an overall multiplicative factor and inside our function $\hat{f}(j)$. Thus, we are interested in:

$$S\hat{f}(j)\hat{f}(j+1) = S\sqrt{1 - \frac{\hat{n}(j)}{2S}} \sqrt{1 - \frac{\hat{n}(j+1)}{2S}}$$

$$= S\left[ 1 - \frac{\hat{n}(j)}{2S} + \mathcal{O}\left( \frac{1}{S^2} \right) \right] \left[ 1 - \frac{\hat{n}(j+1)}{2S} + \mathcal{O}\left( \frac{1}{S^2} \right) \right]$$

(12)

$$= S - \frac{1}{2} \left[ \hat{n}(j) + \hat{n}(j+1) \right] + \mathcal{O}\left( \frac{1}{S} \right)$$

So, within our approximation we simply replace these products with $S$! Thus, within the so called *spin-wave approximation* we find:

$$\hat{H} = -JNS^2 + SJ \sum_{j} \left[ \hat{n}(j) + \hat{n}(j+1) \right] + SJ \sum_{j \text{ even}} \left[ \hat{a}(j)\hat{b}(j+1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j+1) \right]$$

$$+ SJ \sum_{j \text{ odd}} \left[ \hat{a}(j+1)\hat{b}(j) + \hat{a}^\dagger(j+1)\hat{b}^\dagger(j) \right] + \mathcal{O}\left( S^0 \right)$$

(13)

This is clearly *quadratic in our creation and annihilation operators*.

5. Let’s formally apply the $S \to \infty$ limit to our equations of motion.

Note: since we have rewritten our Hamiltonian in terms of our creation and annihilation operators, we will be looking at the EOM for said operators.
Thus, we are about to see that the dynamics of these operators are also simplified in the spin-wave approximation since they are linear. We just calculate the following commutator:

\[
[\hat{a}^\dagger(k), \hat{H}] = SJ \left[ \sum_{j \text{ even}} \left[ \hat{a}^\dagger(j) \hat{a}^\dagger(k), \hat{a}(j) \right] + \left[ \hat{a}^\dagger(k), \hat{a}(j) \right] \hat{b}(j+1) \right] \\
+ \sum_{j \text{ odd}} \left[ \hat{a}^\dagger(j+1) \hat{a}^\dagger(k), \hat{a}(j+1) \right] + \left[ \hat{a}^\dagger(k), \hat{a}(j+1) \right] \hat{b}(j) \right] + O(S^0)
\]

(14)

\[
= -SJ \left[ 2\hat{a}^\dagger(k) + \hat{b}(k+1) + \hat{b}(k-1) \right] + O(S^0)
\]

From this result we can see:

\[
i\hbar \frac{d\hat{a}(k)}{dt} = -SJ \left[ 2\hat{a}^\dagger(k) + \hat{b}(k+1) + \hat{b}(k-1) \right] + O(S^0)
\]

To leading order in \( S \) we have a linear equation of motion. The time evolution of the annihilation operator can be found from the Hermitian conjugate of this above equation:

\[
i\hbar \frac{d\hat{a}(k)}{dt} = SJ \left[ 2\hat{a}(k) + \hat{b}^\dagger(k+1) + \hat{b}^\dagger(k-1) \right] + O(S^0)
\]

We can rinse and repeat for \( \hat{b}(k) \), but this is the same as \( \hat{b}(k) \rightarrow \hat{a}(k) \) and \( \hat{a}(k) \rightarrow \hat{b}(k) \) in the above expressions. Thus, all the EOM for the relevant operators are linear (up to order \( S \)).

6. Let’s now rewrite our creation and annihilation operators using the following Fourier Transforms:

\[
\hat{a}(q) = \sqrt{\frac{2}{N}} \sum_{j \text{ even}} e^{iqj} \hat{a}(j)
\]

\[
\hat{b}(k) = \sqrt{\frac{2}{N}} \sum_{j \text{ odd}} e^{-ikj} \hat{b}(j)
\]

Using, \( \sum_{j \text{ even/odd}} e^{\pm(q-k)j} = \frac{N}{2} \delta_{q,k} \), we can invert this expression:

\[
\hat{a}(j) = \sqrt{\frac{2}{N}} \sum_{q \in B.Z.} e^{-iqj} \hat{a}(q)
\]

\[
\hat{b}(j) = \sqrt{\frac{2}{N}} \sum_{k \in B.Z.} e^{ikj} \hat{b}(k)
\]

Quickly note for \( j \text{ even} \):

\[
\sum_{j \text{ even}} \hat{n}(j) = \frac{2}{N} \sum_{q,k \in B.Z.} \hat{a}^\dagger(q) \hat{a}(k) \sum_{j \text{ even}} e^{i(q-k)j} \\
= \sum_{q,k \in B.Z.} \hat{a}^\dagger(q) \hat{a}(k) \delta_{qk} \\
= \sum_{q \in B.Z.} \hat{a}^\dagger(q) \hat{a}(q) \\
= \sum_{q \in B.Z.} \hat{n}_a(q)
\]

(15)
A similar result works for $j$ odd. The penultimate term above is preferable for the Bogoliubov transformation we will soon perform:

$$\sum_j \left[ \hat{n}(j) + \hat{n}(j + 1) \right] = 2 \sum_{q \in \text{B.Z.}} \left[ \hat{a}^\dagger(q)\hat{a}(q) + \hat{b}^\dagger(q)\hat{b}(q) \right]$$

Recall, we are Fourier transforming:

$$\hat{H} = -JNS^2 + SJ \sum_j \left[ \hat{n}(j) + \hat{n}(j + 1) \right]$$

$$+ SJ \sum_{j \text{ even}} \left[ \hat{a}(j)\hat{b}(j + 1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j + 1) \right]$$

$$+ SJ \sum_{j \text{ odd}} \left[ \hat{b}(j)\hat{a}(j + 1) + \hat{b}^\dagger(j)\hat{a}^\dagger(j + 1) \right] + O(S^0)$$

so, all that is left is the even/odd sums:

$$\sum_{j \text{ even}} \left[ \hat{a}(j)\hat{b}(j + 1) + \hat{a}^\dagger(j)\hat{b}^\dagger(j + 1) \right] + \sum_{j \text{ odd}} \left[ \hat{b}(j)\hat{a}(j + 1) + \hat{b}^\dagger(j)\hat{a}^\dagger(j + 1) \right]$$

$$= \frac{2}{N} \sum_{k,q \in \text{B.Z.}, j \text{ even}} e^{\pm i(q-k)j} \left[ e^{i\theta} \hat{a}(q)\hat{b}(k) + e^{-i\theta} \hat{a}^\dagger(q)\hat{b}^\dagger(k) \right]$$

$$+ \frac{2}{N} \sum_{k,q \in \text{B.Z.}, j \text{ odd}} e^{\pm i(q-k)j} \left[ e^{-i\theta} \hat{a}(q)\hat{b}(k) + e^{i\theta} \hat{a}^\dagger(q)\hat{b}^\dagger(k) \right]$$

$$= \sum_{k,q \in \text{B.Z.}} \delta_{k,q} \left[ e^{i\theta} \hat{a}(q)\hat{b}(k) + e^{-i\theta} \hat{a}^\dagger(q)\hat{b}^\dagger(k) + e^{-i\theta} \hat{a}(q)\hat{b}(k) + e^{i\theta} \hat{a}^\dagger(q)\hat{b}^\dagger(k) \right]$$

$$= 2 \sum_{k,q \in \text{B.Z.}} \cos(q) \left[ \hat{a}(q)\hat{b}(q) + \hat{a}^\dagger(q)\hat{b}^\dagger(q) \right]$$

Applying this transformation we end up with:

$$\hat{H} = -JNS^2 + 2JS \sum_{q \in \text{B.Z.}} \left[ \hat{a}^\dagger(q)\hat{a}(q) + \hat{b}^\dagger(q)\hat{b}(q) + \cos(q)\hat{a}(q)\hat{b}(q) + \cos(q)\hat{a}^\dagger(q)\hat{b}^\dagger(q) \right]$$

$$= -JNS^2 - 2JS \sum_{q \in \text{B.Z.}} 1 + 2SJ \sum_{q \in \text{B.Z.}} \left[ \hat{a}^\dagger(q)\hat{a}(q) + \hat{b}(q)\hat{b}^\dagger(q) + \cos(q)\hat{a}(q)\hat{b}(q) + \cos(q)\hat{a}^\dagger(q)\hat{b}^\dagger(q) \right]$$

$$= -JNS^2 - JS + 2JS \sum_{q \in \text{B.Z.}} \left[ \hat{a}^\dagger(q) \begin{bmatrix} \hat{a}(q) \\ \hat{b}^\dagger(q) \end{bmatrix} \right] \begin{bmatrix} 1 & \cos(q) \\ \cos(q) & 1 \end{bmatrix} \begin{bmatrix} \hat{a}(q) \\ \hat{b}^\dagger(q) \end{bmatrix}$$

(19)

See Appendix 1 if you prefer foiling out the expressions over matrix multiplication; my opinion is you shouldn’t! The latter method is much easier.

We can diagonalize the Hamiltonian by introducing the following Bogoliubov transformation:

$$\begin{bmatrix} c(q) \\ d^\dagger(q) \end{bmatrix} = \begin{bmatrix} \cosh(\theta(q)) & \sinh(\theta(q)) \\ \sinh(\theta(q)) & \cosh(\theta(q)) \end{bmatrix} \begin{bmatrix} \hat{a}(q) \\ \hat{b}(q) \end{bmatrix}$$

so then:

$$\begin{bmatrix} \hat{a}(q) \\ \hat{b}^\dagger(q) \end{bmatrix} = \begin{bmatrix} \cosh(\theta(q)) & -\sinh(\theta(q)) \\ -\sinh(\theta(q)) & \cosh(\theta(q)) \end{bmatrix} \begin{bmatrix} c(q) \\ d^\dagger(q) \end{bmatrix}$$
Thus, we are interested in:

\[
\begin{bmatrix}
\cosh(\theta(q)) & -\sinh(\theta(q)) \\
-\sinh(\theta(q)) & \cosh(\theta(q))
\end{bmatrix}
\begin{bmatrix}
1 & \cos(q) \\
\cos(q) & 1
\end{bmatrix}
\begin{bmatrix}
\cosh(\theta(q)) & -\sinh(\theta(q)) \\
-\sinh(\theta(q)) & \cosh(\theta(q))
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cosh(2\theta(q)) - \sinh(2\theta(q)) \cos(q) & \cosh(2\theta(q)) \cos(q) - \sinh(2\theta(q)) \\
\cosh(2\theta(q)) \cos(q) - \sinh(2\theta(q)) & \cosh(2\theta(q)) - \sinh(2\theta(q)) \cos(q)
\end{bmatrix}
\]

To diagonalize the Hamiltonian we choose:

\[
\theta(q) = \frac{1}{2} \tanh^{-1}(\cos(q))
\]

Notice this means:

\[
\text{sech}^2(2\theta(q)) = 1 - \tanh^2(2\theta(q)) = 1 - \cos^2(q) = \sin^2(q)
\]

and,

\[
\sinh(2\theta(q)) = \cosh(2\theta(q)) \tanh(2\theta(q)) = \cosh(2\theta(q)) \cos(q)
\]

The diagonal terms are now a breeze to find:

\[
cosh(2\theta(q)) - \sinh(2\theta(q)) \cos(q) = \frac{1}{\text{sech}(2\theta(q))} \left(1 - \cos^2(q)\right)
\]

\[
= |\sin(q)|
\]

In terms of our quasi-particles our Hamiltonian becomes:

\[
\hat{H} = -JNS^2 - JS + 2SJ \sum_{q \in B.Z.} \left[\hat{a}^\dagger(q) \hat{b}(q)\right] \begin{bmatrix} 1 & \cos(q) \\ \cos(q) & 1 \end{bmatrix} \left[\hat{a}(q) \hat{b}^\dagger(q)\right]
\]

\[
= -JNS^2 - JS + 2SJ \sum_{q \in B.Z.} |\sin(q)| \left[\hat{c}^\dagger(q) \hat{d}(q)\right] \begin{bmatrix} \hat{c}(q) \\ \hat{d}(q) \end{bmatrix}
\]

\[
= -JNS^2 - JS + 2SJ \sum_{q \in B.Z.} |\sin(q)| \left(\hat{c}^\dagger(q)\hat{c}(q) + \hat{d}(q)\hat{d}^\dagger(q)\right)
\]

Defining \(\omega(q) = 2JS|\sin(q)|\), \(E_o = -JNS^2 - JS + \sum_q \omega(q)\), and sending the sum to an integral we get:

\[
\hat{H} = E_o + \int_{-\pi/2}^{\pi/2} \frac{dq}{2\pi} \omega(q) \left[\hat{n}_c(q) + \hat{n}_d(q)\right]
\]
Note:

$$\sum_{q \in B.Z.} \omega(q) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dq}{2\pi} \omega(q)$$  \hspace{1cm} (27)

$$= JS \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dq}{\pi} |\sin(q)|$$  \hspace{1cm} (28)

$$= \frac{JS}{\pi} \left[ \left| \cos(q) \right|_{-\frac{\pi}{2}}^{0} - \left| \cos(q) \right|_{0}^{\frac{\pi}{2}} \right]$$  \hspace{1cm} (29)

$$= \frac{2JS}{\pi}$$  \hspace{1cm} (30)

Thus,

$$E_o = -JNS^2 - JS \left[ 1 - \frac{2}{\pi} \right]$$

The leading order is what we might expect the answer to be, and this is indeed what we would get if we did not worry about the commutation relations above: it is a ”classical” result. The remaining terms are Quantum fluctuations, which we drop here on out.

7. We can define the ground state, $|0\rangle$, to be the state which satisfies: $\hat{n}_c(q)|0\rangle = \hat{n}_d(q)|0\rangle = 0$. Thus, the energy of the ground state is:

$$E_o = -NJS^2$$

8. For a single particle state we have:

$$|n_c(k) = 1\rangle = \hat{c}^\dagger(k)|0\rangle$$

or,

$$|n_d(k) = 1\rangle = \hat{d}^\dagger(k)|0\rangle$$

Let’s find the energy of these states using our above Hamiltonian. All we need to note is our Kronecker delta becomes a Dirac delta times $2\pi$. This let’s us rewrite the following terms:

$$\hat{n}_c(q)\hat{c}^\dagger(k)|0\rangle = \hat{c}^\dagger(q)\hat{c}(q)\hat{c}^\dagger(k)|0\rangle = \hat{c}^\dagger(q) \left[ 2\pi \delta_{k,-q} + \hat{c}^\dagger(k)\hat{c}(q) \right] |0\rangle$$

The same exact relation holds for $\hat{d}(k)$, so, after subtracting off the ground state energy, the excitation energies of our single particle states are of the form:

$$E(k) = 2JS|\sin(k)|$$

Note that $k = 0 \Rightarrow E(k = 0) = 0$ (restricting ourselves to the first Brillouin zone, so neglect $q = n\pi$ with $n \neq 0$), so these single particle states have energies equal to the ground state.

Furthermore, we can expand our energy in the immediate neighborhood of $k = 0$:

$$E(k) \approx 2JS|k|$$

Thus, we see the energy vanishes linearly with the momentum $k$! Finally, the spin-wave velocity at $k = 0$ is given by:

$$v_s = \left. \frac{dE(k)}{dk} \right|_{k=0} = 2JS$$
Two component complex scalar field

Consider the Lagrangian:

$$\mathcal{L} = \left( \partial_{\mu} \phi_{a} \right)^{*} \partial^{\mu} \phi_{a} - V(|\phi_{a}|^2)$$

Where: $V(|\phi_{a}|^2) = m^2 \phi_{a} \phi_{a}^{*}$

1. Let’s begin by finding:

   (a) The canonical momentum $\Pi_{a}$ conjugate to $\phi_{a}$. This is simple enough:

   $$\Pi_{a} = \frac{\delta \mathcal{L}}{\delta \partial_{0} \phi_{a}} = \partial^{0} \phi_{a}^{*}$$

   Since the field is complex, we can not forget about:

   $$\Pi_{a}^{*} = \frac{\delta \mathcal{L}}{\delta \partial_{0} \phi_{a}^{*}} = \partial^{0} \phi_{a}$$

   As a quick note, we see that we neglect any sort of index which indicates if we are talking about a co or contra variant canonical momentum. We just make the correct choice where it matters, and forgo any additional labels to avoid too much clutter.

   (b) With $\sigma$ representing the extra degree of freedom we get from the complex fields we see the Hamiltonian of this system is:

   $$\mathcal{H} = \sum_{a,\sigma} (\partial_{\mu} \phi_{a}^{\sigma}) \Pi_{a}^{\sigma} - \mathcal{L} = 2 \sum_{a} \Pi_{a}^{*} \Pi_{a} - \mathcal{L}$$

   From here on out I drop the summation symbol with the understanding we are summing over the components of the field. With this in mind we see:

   $$\mathcal{H} = \left( \Pi_{a}^{*} \Pi_{a} + \nabla \phi_{a} \cdot \nabla \phi_{a}^{*} \right) + V(|\phi|^{2})$$

   So then:

   $$H = \int d^{3}x \mathcal{H}$$

   (c) Now for the total linear momentum; the momentum density is given by:

   $$P^{k} = \Pi_{a} \partial^{k} \phi_{a} + h.c.$$  

   We could work this out from the stress tensor as we did in the last homework, but it’s not hard to see we’d need the Hermitian conjugate on top of what we already know. From this we have:

   $$P^{k} = \int d^{3}x P^{k}$$

2. We begin by considering a two component complex scalar field:

   $$\Phi = \begin{bmatrix} \phi_{1} \\ \phi_{2} \end{bmatrix}$$

   clearly:

   $$\forall U \in SU(2), |U\Phi|^2 = |\Phi|^2$$
Furthermore, in the case of a global symmetry: $\mathcal{L}(U\Phi) = \mathcal{L}(\Phi)$ (see Appendix 2 for the case of a local symmetry). We know from Noether’s theorem that a symmetry implies the existence of conserved currents. To find these currents we express the elements of our group in the following form, where the specific representation is not yet determined:

$$(U(x))_{ab} = [e^{i\lambda^k(x)}]_{ab} \approx \delta_{ab} + i\lambda^k_{ab}\theta^k(x)$$

We take $\theta^k(x)$ to be small above and here on out.

Taking the Pauli matrices as the generators of $SU(2)$, i.e. $\lambda^k_{ab} = \sigma^k_{ab}$, we see that our fields transform in the following way:

$$\delta\phi_a = [(U(x))_{ab} - \delta_{ab}]\phi_b \approx i\sigma^k_{ab}\phi_b\theta^k(x)$$

Note, we are summing over $b$. Furthermore, since we are considering a 2-component complex scalar field we also need:

$$\delta\phi^*_a = \delta\phi^*_a \approx -i\sigma^k_{ba}\phi^*_b\theta^k(x)$$

We are interested in a global symmetry; thus, our $\theta^k$’s are constant. Applying this variation to our Lagrangian, then exploiting the Euler-Lagrange equation, and finally taking the necessary functional derivatives we find $^1$:

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\partial_\nu\phi_a + \phi_a \leftrightarrow \phi^*_a$$

$$= \partial_\nu \left[ \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\phi_a + \delta\phi^*_a \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi^*_a} \right]$$

$$= i\partial_\nu \left[ \sigma^k_{ab}(\partial_\nu\phi_a)^* \phi_b - \phi^*_b\sigma^k_{ba}(\partial_\nu\phi_a) \right] \theta^k$$

$$= i\partial_\nu \left[ (\partial^\nu\Phi^T)^* \sigma^k\Phi - (\Phi^T)^* \sigma^k\partial^\nu\Phi \right] \theta^k$$

So we have 3 conserved 4-currents (one for each Pauli Matrix) indexed with $k$. Because we want to (eventually) show our corresponding conserved charges are the generators of $SU(2)$, we introduce an additional minus sign, and define:

$$j^{\nu k} := -i\left( (\partial^\nu\Phi^T)^* \sigma^k\Phi - (\Phi^T)^* \sigma^k\partial^\nu\Phi \right)$$

This is clearly conserved by the above relation.

We can express our constants of motion in terms of these conserved currents in the usual way:

$$Q^k = \int d^3x j^{0k} = -i \int d^3x \left( (\partial^0\Phi^T)^* \sigma^k\Phi - (\Phi^T)^* \sigma^k\partial^0\Phi \right)$$

$$= -i \int d^3x \left( \Pi^T \sigma^k\Phi - (\Phi^T)^* \sigma^k\Pi^* \right)$$

$$= -i\sigma^k_{ab} \int d^3x \left( \Pi_a\phi_b - \phi^*_a\Pi^*_b \right)$$

$^1$Notice the order of product in the second term of the second line. This can be seen in a few ways. For one, the h.c. of $\phi$ lives in the dual space, so it must left multiply. You can also write this line as: $\partial_\nu \left[ \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a} \phi_a + \text{h.c.} \right]$. Finally, just plugging in $\phi + \delta\phi$ into our action achieves the same ordering of terms.
3. Now let’s go ahead and Quantize our theory. All we need to do is promote our fields to operators, and then impose equal time commutation relations. The result of which is:

\[
\hat{H} = \int d^3x \left( \hat{\Pi}_a \hat{\phi}_a^* + \nabla \hat{\phi}_a \cdot \nabla \hat{\phi}_a^* + V(|\hat{\phi}|^2) \right)
\]

and we impose: \([\hat{\phi}_a(x,t), \hat{\Pi}_b(x',t)] = [\hat{\phi}_a(x,t), \hat{\Pi}_a^*(x',t)] = i\delta_{ab}\delta^3(x - x')\). All other commutators are zero.

We can also see that the total momentum operator becomes:

\[
\hat{P}^k = \int d^3x \hat{\Pi}_a \partial^k \hat{\phi}_a + h.c.
\]

4. The quantum mechanical generators of global infinitesimal SU(2) symmetry are related to the classical conserved quantities above \(Q^k\) in a very intimate way; indeed, they are just the offsprings of our classical conserved charge. We promote the conserved charge to an operator by using the quantized field operators; thus:

\[
Q^k = \int d^3x j^{0k} = -i \int d^3x \left( (\hat{\sigma}^0 \hat{\Phi}^T)^* \sigma^k \hat{\Phi} - (\hat{\Phi}^T)^* \sigma^k \hat{\Phi} \right)
\]

\[
= -i \int d^3x \left( \hat{\Pi}^T \sigma^k \hat{\Phi} - (\hat{\Phi}^T)^* \sigma^k \hat{\Pi}^* \right)
\]

\[
= -i \sigma^k_{ab} \int d^3x \left( \hat{\Pi}_a \hat{\phi}_b - \hat{\phi}_a^* \hat{\Pi}_b^* \right)
\]

These guys are indeed the generators we are after which is evident from their commutation relations: \([\hat{Q}^i, \hat{Q}^j] = 2i\varepsilon_{ijk} \hat{Q}^k\). These are clearly generators of our SU(2) symmetry since we have the same structure constant as the Pauli matrices!

To work out these commutation relations we note at equal times:

\[
[\hat{\Pi}_a \hat{\phi}_b(\vec{x}), \hat{\Pi}_c \hat{\phi}_d(\vec{x}')] = i \left( \hat{\Pi}_a \hat{\phi}_d(\vec{x}) \delta_{bc} - \hat{\Pi}_c \hat{\phi}_b(\vec{x}) \delta_{ad} \right) \delta(\vec{x} - \vec{x}')
\]

Which implies:

\[
- [\hat{\Pi}_a \sigma^i_{ab} \hat{\phi}_b(\vec{x}), \hat{\Pi}_c \sigma^j_{cd} \hat{\phi}_d(\vec{x}')] = -i \left( \hat{\Pi}_a \hat{\phi}_d(\vec{x}) \sigma^i_{ab} \sigma^j_{cd} - \hat{\Pi}_c \hat{\phi}_b(\vec{x}) \sigma^i_{ab} \sigma^j_{cd} \right) \delta(\vec{x} - \vec{x}')
\]

\[
= -i \left( \hat{\Pi}_a \hat{\phi}_d(\vec{x}) [\sigma^i \sigma^j]_{ad} - \hat{\Pi}_c \hat{\phi}_b(\vec{x}) [\sigma^i \sigma^j]_{bc} \right) \delta(\vec{x} - \vec{x}')
\]

\[
= -i \hat{\Pi}_a \left[ [\sigma^j, \sigma^i] \right]_{ad} \hat{\phi}_d(\vec{x}) \delta(\vec{x} - \vec{x}')
\]

\[
= 2i\varepsilon^{ijk} \left( -i \hat{\Pi}_a \sigma^j_{ad} \hat{\phi}_d(\vec{x}) \right) \delta(\vec{x} - \vec{x}')
\]

We can finally show our desired result:

\[
[\hat{Q}^i, \hat{Q}^j] = \int d^3x d^3x' 2i\varepsilon^{ijk} \left[ i \left( \hat{\phi}_a^* \sigma^k \hat{\Pi}_a^*(\vec{x}) - \hat{\Pi}_a \sigma^k \hat{\phi}_d(\vec{x}) \right) \right] \delta(\vec{x} - \vec{x}')
\]

\[
= 2i\varepsilon^{ijk} \hat{Q}^k
\]

Again, these conserved charges satisfy the same algebra as the Pauli matrices; therefore, they are generators of SU(2) transformations!
5. Let’s now use the Heisenberg equation of motion to find the equation of motion for our field operators:

(a) \[ \partial_o \hat{\phi}_a(x, t) = \frac{1}{i}[\hat{\phi}_a(x, t), \hat{H}(x', t)] \]

Clearly \( \hat{\phi}_a \) only doesn’t commute with the momentum operator; thus, we just need to compute:

\[ [\hat{\phi}_a(x, t), \hat{\Pi}_b(x', t)\hat{\Pi}_b^*(x', t)] = [\hat{\phi}_a(x, t), \hat{\Pi}_b(x', t)]\hat{\Pi}_b^*(x', t) = i\delta_{ab}\delta^3(x - x')\hat{\Pi}_b^*(x', t) \]

It follows:

\[ \partial_o \hat{\phi}_a(x, t) = \hat{\Pi}_b^*(x, t) \]

As it should be!

(b) The equation of motion for the complex conjugate scalar field follows the exact same procedure, and is just the complex conjugate of the above equation:

\[ \partial_o \hat{\phi}_a^*(x, t) = \hat{\Pi}_b(x, t) \]

(c) \[ \partial_o \hat{\Pi}_a(x, t) = \frac{1}{i}[\hat{\Pi}_a(x, t), \hat{H}(x', t)] \]

The commutator with the potential is the exact same procedure as (a) and (b) except the canonical commutation relation is reversed, so there will be an overall minus sign. Because of this, I will not do it explicitly.

The following commutator requires an integration by parts; I throw out all boundary terms immediately because our field operators evaluate to zero there. We find:

\[ \int d^3x'[\hat{\Pi}_a(x, t), \nabla \hat{\phi}_b(x', t)\nabla \hat{\phi}_b^*(x', t)] = \int d^3x'[\hat{\Pi}_a(x, t), \nabla \hat{\phi}_b(x', t)]\nabla \hat{\phi}_b^*(x', t) \]

\[ = \int d^3x'\left( - [\hat{\Pi}_a(x, t), \hat{\phi}_b(x', t)]\nabla^2 \hat{\phi}_b^*(x', t) \right) \]

\[ = i\nabla^2 \hat{\phi}_a^*(x, t) \]

The last line comes from throwing in the values of the commutators, and then using both of the delta functions. As a result, we find:

\[ \partial_o \hat{\Pi}_a(x, t) = \nabla^2 \hat{\phi}_a^*(x, t) - m^2 \hat{\phi}_a^* \]

(d) Now we can rinse and repeat... Or take a complex conjugate of the above expression; I opt for the latter:

\[ \partial_o \hat{\Pi}_a^*(x, t) = \frac{1}{i}[\hat{\Pi}_a^*(x, t), \hat{H}(x', t)] \]

\[ = \nabla^2 \hat{\phi}_a(x, t) - m^2 \hat{\phi}_a(x, t) \]

Finally, we can use our two equations for each of the fields above (four in total) and write:

\[ (\partial^2 + m^2)\hat{\phi}_a(x, t) = 0 \]

\[ (\partial^2 + m^2)\hat{\phi}_a^*(x, t) = 0 \]
6. We’re now in a position to introduce a set of creation and annihilation operators, which will show our analogy between fields and harmonic oscillators still holds in the complex case. Since we have already quantized the real scalar field, we quickly note some similarities and some differences.

**We first consider the case of a single component complex scalar field**, then we generalize our result to multiple components. Even with a single component however we will see a complex scalar field requires two sets of creation and annihilation operators opposed to one for the case of a real scalar field; this has to do with the a set of two canonical fields opposed to just one.

Following Peskin and Schroeder we expand our field operator in its Fourier modes, treating each mode as an independent oscillator, which is the same idea we had for the real scalar field:

\[
\hat{\phi}(\bar{x},x_o) = \int \frac{d^3k}{(2\pi)^3} \left[ \hat{\phi}_+(\vec{k},x_o) e^{i\vec{k} \cdot \bar{x}} + \hat{\phi}_-(\vec{k},x_o) e^{-i\vec{k} \cdot \bar{x}} \right]
\]

From our equations of motion in 2.5 we know that each part must satisfy the Klein-Gordon equation:

\[
\partial_o^2 \hat{\phi}_\pm(\vec{k},x_o) + (k^2 + m^2) \hat{\phi}_\pm(\vec{k},x_o) = 0
\]

We deduce the following time dependence:

\[
\hat{\phi}_\pm(\vec{k},x_o) = \hat{\phi}_\pm(\vec{k}) e^{\pm i\omega(k) x_o}
\]

where \(\omega(k) = \sqrt{k^2 + m^2}\). We keep the solutions which are Lorentz invariant, so after writing, \(\omega x_o - \vec{k} \cdot \vec{x} = k \cdot x\) we see:

\[
\hat{\phi}(\bar{x},x_o) = \int \frac{d^3k}{(2\pi)^3} \left[ \hat{\phi}_+(\vec{k}) e^{-i\vec{k} \cdot \bar{x}} + \hat{\phi}_-(\vec{k}) e^{i\vec{k} \cdot \bar{x}} \right]
\]

We now recall that for the case of a real scalar field we had the conditions: \(\hat{\phi}_+(\vec{k}) = \hat{\phi}_\dagger_-(\vec{k})\) and \(\hat{\phi}_-(\vec{k}) = \hat{\phi}_\dagger_+(\vec{k})\). These conditions are no longer necessary because our complex scalar field is not necessarily Hermitian. **This reaffirms the prediction that the complex field will have a set of creation and annihilation operators!** Furthermore, we can again see the necessity of a set of two operators after considering the mode expansion of the complex conjugate of the above field:

\[
\hat{\phi}^*(\bar{x},x_o) = \int \frac{d^3k}{(2\pi)^3} \left[ \hat{\phi}_+(\vec{k},x_o) e^{i\vec{k} \cdot \bar{x}} + \hat{\phi}_-(\vec{k},x_o) e^{-i\vec{k} \cdot \bar{x}} \right]
\]

and so,

\[
\partial_o^2 \hat{\phi}^*_\pm(\vec{k},x_o) + (k^2 + m^2) \hat{\phi}^*_\pm(\vec{k},x_o) = 0
\]

which leads us to:

\[
\hat{\phi}^*(\bar{x},x_o) = \int \frac{d^3k}{(2\pi)^3} \left[ \hat{\phi}^*_+(\vec{k}) e^{-i\vec{k} \cdot \bar{x}} + \hat{\phi}^*_-(\vec{k}) e^{i\vec{k} \cdot \bar{x}} \right]
\]

We are now in a position to define our creation and annihilation operators; **our constraint this time is our field operators must be complex conjugates of one another.** Exploiting orthogonality to equate the coefficients of our exponentials we see:

\[
\hat{\phi}_\dagger = \hat{\phi}^* \rightarrow \hat{\phi}_+^* = \hat{\phi}_-^\dagger, \hat{\phi}_-^* = \hat{\phi}_+^\dagger
\]

This motivates:
\[ \hat{a}(k) = 2\omega(k)\hat{\phi}_+ \]

and,

\[ \hat{b}^\dagger(k) = 2\omega(k)\hat{\phi}_- \]

We can show that if these guys satisfy the following commutation relations:

\[ [\hat{a}(k), \hat{a}^\dagger(k')] = (2\pi)^3\omega(k)\delta^3(k-k') \]

\[ [\hat{b}(k), \hat{b}^\dagger(k')] = (2\pi)^3\omega(k)\delta^3(k-k') \]

with all others zero, then our canonical commutation relations are preserved. With these the field operators become:

\[ \hat{\phi}(\vec{x}, x_o) = \int \frac{d^3k}{(2\pi)^32\omega(k)} (\hat{a}(k)e^{-ik\cdot x} + \hat{b}^\dagger(k)e^{ik\cdot x}) \]

and,

\[ \hat{\phi}^\dagger(\vec{x}, x_o) = \int \frac{d^3k}{(2\pi)^32\omega(k)} (\hat{b}(k)e^{-ik\cdot x} + \hat{a}^\dagger(k)e^{ik\cdot x}) \]

The generalization to a 2-component complex field, let alone a n-component one, is trivial: just introduce subscripts and impose \([\hat{a}_m, \hat{a}^\dagger_n]\) etc. are all proportional to \(\delta_{mn}\) (where appropriate). Then we find:

\[ \hat{\phi}_m(\vec{x}, x_o) = \int \frac{d^3k}{(2\pi)^32\omega(k)} (\hat{a}_m(k)e^{-ik\cdot x} + \hat{b}_m^\dagger(k)e^{ik\cdot x}) \]

and,

\[ \hat{\phi}^\dagger_m(\vec{x}, x_o) = \int \frac{d^3k}{(2\pi)^32\omega(k)} (\hat{b}_m(k)e^{-ik\cdot x} + \hat{a}_m^\dagger(k)e^{ik\cdot x}) \]

From these relations we can easily calculate our canonical momentum operators in terms of these creation and annihilation operators:

\[ \hat{P}_m(\vec{x}, x_o) = i \int \frac{d^3k}{2(2\pi)^3} (\hat{a}^\dagger_m(k)e^{ik\cdot x} - \hat{b}_m(k)e^{-ik\cdot x}) \]

\[ \hat{P}^\dagger_m(\vec{x}, x_o) = -i \int \frac{d^3k}{2(2\pi)^3} (\hat{a}_m(k)e^{-ik\cdot x} - \hat{b}^\dagger_m(k)e^{ik\cdot x}) \]

7. Now let’s go ahead and express \(\hat{H}, \hat{P}\) and \(\hat{Q}^k\) in terms of our creation and annihilation operators. We’ll start with \(\hat{Q}^k\). We calculate two types of terms from the definition of \(\hat{Q}^k\), \(\hat{P}_m\hat{\phi}_m(\vec{x}, x_o)\) and \(-\hat{\phi}_m^\dagger\hat{P}_m^\dagger(\vec{x}, x_o)\). We see:

\[ -\hat{\phi}_m^\dagger\hat{P}_m^\dagger(\vec{x}, x_o) = i \int \frac{d^3k}{2(2\pi)^3} \int \frac{d^3k'}{(2\pi)^32\omega(k')} (\hat{b}_m(k')e^{-ik'\cdot x} + \hat{a}_m^\dagger(k')e^{ik'\cdot x}) (\hat{a}_m(k)e^{-ik\cdot x} - \hat{b}_m^\dagger(k)e^{ik\cdot x}) \]

Let’s now foil everything out:

\[ i\left( \hat{b}_m\hat{\phi}_m(-k)\hat{\phi}_m(k) - \hat{b}^\dagger_m\hat{\phi}_m^\dagger(k)\hat{\phi}_m(k) + \hat{a}_m\hat{\phi}_m(k)\hat{\phi}_m^\dagger(k) - \hat{a}^\dagger_m\hat{\phi}_m^\dagger(-k)\hat{\phi}_m^\dagger(k) \right) \]

\[ \rightarrow i\left( \hat{b}_m\hat{\phi}_m(-k)e^{-i\omega(k)x_o} - \hat{b}_m\hat{\phi}_m^\dagger(k)\hat{\phi}_m(k) + \hat{a}_m\hat{\phi}_m(k)\hat{\phi}_m^\dagger(k) - \hat{a}^\dagger_m\hat{\phi}_m(k)\hat{\phi}_m^\dagger(-k)e^{2i\omega(k)x_o} \right) \]
This last step comes about from the fact conserved charge integrates over spatial coordinates, and the identity:
\[
\int d^3x e^{\pm i(k\cdot \vec{x})} = (2\pi)^3 \delta^3(k \pm \vec{k}')
\]
Notice this means we have informally collapsed the \(\vec{k}'\) integral in the above step.
In a similar fashion we find:
\[
\hat{\Pi}_m \hat{\phi}_n(\vec{x}, x_o) = i \int \frac{d^3k}{2(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3 2\omega(\vec{k}')} \left[ \left( \hat{a}^\dagger_m(\vec{k}) e^{i(k\cdot \vec{x})} - \hat{b}_m(\vec{k}) e^{-i(k\cdot \vec{x})} \right) \left( \hat{a}_n(\vec{k}') e^{-i(k'\cdot \vec{x})} + \hat{b}_n(\vec{k}') e^{i(k'\cdot \vec{x})} \right) \right]
\]
Expanding:
\[
i \left( \hat{a}^\dagger_m(\vec{k}) \hat{a}_n(\vec{k}') e^{i(k-k')\cdot \vec{x}} - \hat{b}_m(\vec{k}) \hat{a}_n(\vec{k}') e^{-i(k+k')\cdot \vec{x}} + \hat{a}^\dagger_m(\vec{k}) \hat{b}_n(\vec{k}') e^{-i(k+k')\cdot \vec{x}} - \hat{b}_m(\vec{k}) \hat{b}_n(\vec{k}') e^{-i(k-k')\cdot \vec{x}} \right)
\]
\[
\rightarrow i \left( \hat{a}^\dagger_m(\vec{k}) \hat{a}_n(\vec{k}) - \hat{b}_m(\vec{k}) \hat{a}_n(-\vec{k}) e^{-2i\omega(\vec{k}) x_o} + \hat{a}^\dagger_m(\vec{k}) \hat{b}_n(\vec{k}) e^{-2i\omega(\vec{k}) x_o} - \hat{b}_m(\vec{k}) \hat{b}_n(\vec{k}) \right) \]
(49)
(50)
Since there is a factor of \(i\) multiplying these above two results, adding these two expressions together is a cynch:
\[
\hat{Q}^k = -i \sigma^k_{mn} \int d^3x \left( \hat{\Pi}_m \hat{\phi}_n - \hat{\phi}_n^* \hat{\Pi}_m^* \right) = \sigma^k_{mn} \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})} \left( \hat{a}^\dagger_m(\vec{k}) \hat{a}_n(\vec{k}) - \hat{b}_m(\vec{k}) \hat{b}_n^*(\vec{k}) \right) \]
(51)
This is reminiscent of our conserved charge in the case of a \(U(1)\) symmetry, which described particle number conservation. Notice the fortuitous cancellation of the time dependent factors, so this conserved charge is indeed time independent! That is, it’s a constant of motion.

8. Let’s now express the Hamiltonian in terms of our creation and annihilation operators. For simplicity I’ll work out the case of a single component complex scalar field as we did in 2.6.

We begin by noting we’ll need the following sum of terms:
\[
\hat{\Pi}^*(\vec{x}, x_o) + \nabla \cdot \nabla^* (\vec{x}, x_o) = \int \int \frac{d^3k}{2\omega(\vec{k})(2\pi)^3} \frac{d^3k'}{2\omega(\vec{k}')(2\pi)^3} \left( \omega(\vec{k}) \omega(\vec{k}') + \vec{k} \cdot \vec{k}' \right)
\]
\[
\times \left( \hat{a}(\vec{k}) e^{i(k\cdot \vec{x})} - \hat{b}(\vec{k}) e^{-i(k\cdot \vec{x})} \right) \left( \hat{a}^\dagger(\vec{k}') e^{i(k'\cdot \vec{x})} - \hat{b}(\vec{k}') e^{-i(k'\cdot \vec{x})} \right)
\]
(52)

We find \(\nabla \hat{\phi}(\vec{x}, x_o)\) in the same way we found \(\hat{\Pi}(\vec{x}, x_o)\). The parentheses foil out to:
\[
\left( \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}') e^{i(k-k')\cdot \vec{x}} e^{-i(\omega(\vec{k})-\omega(\vec{k}')) x_o} - \hat{b}(\vec{k}) \hat{a}^\dagger(\vec{k}') e^{-i(k+k')\cdot \vec{x}} e^{i(\omega(\vec{k})+\omega(\vec{k}')) x_o} \right)
\]
\[
- \hat{a}(\vec{k}) \hat{b}(\vec{k}') e^{i(k+k')\cdot \vec{x}} e^{-i(\omega(\vec{k})+\omega(\vec{k}')) x_o} + \hat{b}(\vec{k}) \hat{b}(\vec{k}') e^{-i(k-k')\cdot \vec{x}} e^{i(\omega(\vec{k})-\omega(\vec{k}')) x_o}
\]
(54)
(55)
Let’s just put a pin in this guy for now, and move on to the next expansion:
\[
\hat{\phi}^*(\vec{x}, x_o) = \int \int \frac{d^3k}{2\omega(\vec{k})(2\pi)^3} \frac{d^3k'}{2\omega(\vec{k}')(2\pi)^3} \left( \hat{a}(\vec{k}) e^{-i(k\cdot \vec{x})} + \hat{b}(\vec{k}) e^{i(k\cdot \vec{x})} \right) \left( \hat{a}^\dagger(\vec{k}') e^{i(k'\cdot \vec{x})} + \hat{b}(\vec{k}') e^{-i(k'\cdot \vec{x})} \right)
\]
We foil out the terms to find:

\[
(\hat{a}(\vec{k})\hat{a}^*(\vec{k}'))e^{i(\vec{k} - \vec{k}').\vec{x}}e^{-i(\omega(k) - \omega(k'))x_o} + \hat{b}^\dagger(\vec{k})\hat{a}^*(\vec{k}')e^{-i(\vec{k} + \vec{k}').\vec{x}}e^{i(\omega(k) + \omega(k'))x_o}
\]

\[
(\hat{a}(\vec{k})\hat{a}^*(\vec{k}'))e^{i(\vec{k} - \vec{k}').\vec{x}}e^{-i(\omega(k) - \omega(k'))x_o} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}')e^{-i(\vec{k} - \vec{k}').\vec{x}}e^{i(\omega(k) - \omega(k'))x_o}
\]

(56) (57)

Now we recall to obtain the Hamiltonian we integrate the Hamiltonian density over spatial coordinates. Using the identity,

\[
\int d^3 x e^{i(\vec{k} + \vec{k}').\vec{x}} = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')
\]

we can seriously clean up this mess! Indeed, making the necessary substitutions, and performing the integral over \(d^3 x\) we find:

\[
\hat{H} = \int d^3 x \left( \hat{\Pi} \cdot \hat{\Pi}^* + \nabla \hat{\phi} \cdot \nabla \hat{\phi}^* + V(|\phi|^2) \right) = \int d^3 x \left( \hat{\Pi} \cdot \hat{\Pi}^* + \nabla \hat{\phi} \cdot \nabla \hat{\phi}^* + m^2 |\phi|^2 \right)
\]

(58)

\[
= \int \int \frac{d^3 k}{2\omega(k)(2\pi)^3} \frac{d^3 k'}{2\omega(k')}
\]

\[
[\delta^3(\vec{k} - \vec{k}') \left( \omega(k)\omega(k') + 2\vec{k}k' + m^2 \right) \left( \hat{a}(\vec{k})\hat{a}^*(\vec{k}')e^{i(\omega(k) - \omega(k'))x_o} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}')e^{-i(\omega(k) - \omega(k'))x_o} \right)
\]

\[
+ \delta^3(\vec{k} + \vec{k}') \left( -\omega(k)\omega(k') - 2\vec{k}k' + m^2 \right) \left( \hat{b}^\dagger(\vec{k})\hat{a}(\vec{k}')e^{-i(\omega(k) + \omega(k'))x_o} + \hat{a}(\vec{k})\hat{b}(\vec{k}')e^{i(\omega(k) + \omega(k'))x_o} \right)
\]

(59) (60) (61)

After collapsing the delta functions, the second term goes away. This is due to the fact the delta function sends \(k' \rightarrow -k\), and our frequency is just the energy, which we know from our equations of motion is given by: \(\omega^2(k) = k^2 + m^2 \rightarrow -\omega^2(k) + k^2 + m^2 = 0\). Similarly, for the first term we find that \(k' \rightarrow k\) so then \(\omega(k)\omega(k') + 2\vec{k}k' + m^2 \rightarrow 2\omega^2(k)\). Thus, we find:

\[
\hat{H} = \int \frac{d^3 k}{(2\pi)^3 2\omega(k)} \omega(k) \left[ \hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k}) \right]
\]

Finally, for a 2-component (or an \(n\)-component) complex scalar field we find:

\[
\hat{H} = \int d^3 x \left( \hat{\Pi}_m \cdot \hat{\Pi}_m^* + \nabla \hat{\phi}_m \cdot \nabla \hat{\phi}^*_m + V(|\phi_m|^2) \right)
\]

(62)

\[
= \int \frac{d^3 k}{(2\pi)^3 2\omega(k)} \omega(k) \left[ \hat{a}_m(\vec{k})\hat{a}^\dagger_m(\vec{k}) + \hat{b}^\dagger_m(\vec{k})\hat{b}_m(\vec{k}) \right]
\]

(63)

\[
= \int \frac{d^3 k}{(2\pi)^3 2\omega(k)} \omega(k) \left[ \hat{a}_m(\vec{k})\hat{a}^\dagger_m(\vec{k}) + \hat{b}^\dagger_m(\vec{k})\hat{b}_m(\vec{k}) \right] + \sum_m \int d^3 k \frac{\omega(k)}{2} \delta^3(0)
\]

(64)

Note: we sum over \(m\)!
Thus, the normal ordered Hamiltonian is:

\[
\hat{H} = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \omega(k) \left[ \hat{a}_m^\dagger(k)\hat{a}_m(k) + \hat{b}_m^\dagger(k)\hat{b}_m(k) \right]
\]  

(65)

The ground state is defined by:

\[
\hat{a}_m^\dagger(k)\hat{a}_m(k)|0\rangle = \hat{b}_m^\dagger(k)\hat{b}_m(k)|0\rangle = 0
\]

This tells us that the ground state Energy of our system is \( E_0 = 2 \int d^3k \frac{\omega(k)}{2} \delta^3(0) \), and has occupation numbers \( n_{ak} = n_{bk} = 0 \). With normal ordering we shift the ground state energy to zero.

What about the momentum? Well we go back through the riveting procedure of substituting in our creation and annihilation operators once more! Recall:

\[
\hat{P} = \int d^3x \hat{\Pi} \nabla \phi + h.c.
\]

(66)

So foiling again:

\[
\hat{a}(k)\hat{a}^\dagger(k')e^{i(k-k')x} - \hat{b}(k)\hat{b}^\dagger(k')e^{-i(k+k')x} - \hat{a}(k)\hat{b}(k')e^{i(k+k')x} + \hat{b}(k)\hat{b}^\dagger(k')e^{-i(k-k')x}
\]

Which becomes on substituting back into our integral:

\[
\hat{P} = \int \int \frac{d^3k}{2(2\pi)^3} \frac{d^3k'}{2(2\pi)^3} \frac{1}{2\omega(k')} \left[ \hat{a}(k)\hat{a}^\dagger(k')\delta^3(k-k')e^{i(\omega(k)-\omega(k'))x_o} - \hat{b}(k)\hat{b}^\dagger(k')\delta^3(k+k')e^{-i(\omega(k)+\omega(k'))x_o} \right]
\]

(68)

\[
\hat{P} = \int \int \frac{d^3k}{2(2\pi)^3} \frac{d^3k'}{2\omega(k)} \left[ \hat{a}(k)\hat{a}^\dagger(k') + \hat{b}(k)\hat{b}^\dagger(k') \right] + h.c.
\]

(69)

Since \( \omega \) depends on the square of \( k \), we can drop all the exponentials. Furthermore, when we do this, the remaining term:

\[
\kappa[\hat{a}(k)\hat{b}^\dagger(-k) + \hat{a}(-k)\hat{b}^\dagger(k)]
\]

is odd under \( \kappa \), so it integrates to zero. Thus, we finally find:

\[
\hat{P} = \int \frac{d^3k}{2(2\pi)^3 2\omega(k)} \left[ \hat{a}(k)\hat{a}^\dagger(k) + \hat{b}(k)\hat{b}^\dagger(k) \right] + h.c.
\]

(72)

\[
\hat{P} = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \left[ \hat{a}(k)\hat{a}^\dagger(k) + \hat{b}(k)\hat{b}^\dagger(k) \right] + h.c.
\]

(73)

\[
\hat{P} = : \hat{P} :
\]

(74)
The last equality follows from the creation and annihilation operator commutation relations and:

\[
\int \frac{d^3k}{(2\pi)^3 2\omega(k)} \vec{k}\delta^3(0) = 0
\]

as well as adding the Hermitian conjugate. Our momentum operator is already normal ordered!

The expected result follows:

\[\hat{P}|0\rangle = 0\]

Finally we normal order our \(SU(2)\) generators in respect to this very same ground state:

\[
\hat{Q}^k = \sigma^k_{mn} \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \left( \hat{a}_m^\dagger(\vec{k}) \hat{a}_n(\vec{k}) - \hat{b}_m(\vec{k}) \hat{b}_n^\dagger(\vec{k}) \right)
\]

\[\rightarrow: \hat{Q}^k : = \sigma^k_{mn} \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \left( \hat{a}_m^\dagger(\vec{k}) \hat{a}_n(\vec{k}) - \hat{b}_m(\vec{k}) \hat{b}_n^\dagger(\vec{k}) \right)\]

Clearly:

\[: \hat{Q}^k : |0\rangle = 0\]

9. Using our creation operators we can build up our single particle states from the vacuum in the usual fashion:

\[\hat{a}_i^\dagger(\vec{q})|0\rangle = |n_{ai}(\vec{q}) = 1 : = |a_i(\vec{q})\rangle\]

\[\hat{b}_i^\dagger(\vec{q})|0\rangle = |n_{bi}(\vec{q}) = 1 : = |b_i(\vec{q})\rangle\]

Using the commutation relations of our creation and annihilation operators we find the excitation spectrum for these single particles states to be:

\[\langle : \hat{H} : \rangle = \omega(\vec{q}) = \sqrt{|q|^2 + m^2}\]

Note this is independent of \(i\) (species) as well as if we are talking about particles or anti-particles; thus, there is a degeneracy of four altogether. If we focus on just the particles or the anti-particles alone, we see that the degeneracy is 2. This means the single particle states of a single "type" form a \(j = 1/2\) representation of \(SU(2)\). Indeed, one can show:

\[\langle a_i(\vec{q})| : \hat{Q}^k : |a_i(\vec{q})\rangle = \sigma^k_{ii}\]

\[\langle b_i(\vec{q})| : \hat{Q}^k : |b_i(\vec{q})\rangle = -\sigma^k_{ii}\]

Which is indeed, what you’d expect for the \(j = 1/2\) states! We could also show:

\[\langle a_i(\vec{q})| : \hat{P} : |a_i(\vec{q})\rangle = \vec{q}\]

\[\langle b_i(\vec{q})| : \hat{P} : |b_i(\vec{q})\rangle = \vec{q}\]

This completes our list of Quantum numbers as well as this assignment!
3 Appendix I

We begin by multiplying everything out:

\[ \hat{c}(q) = \cosh(\theta(q)) \hat{a}(q) + \sinh(\theta(q)) \hat{b}(q) \rightarrow \hat{c}^\dagger(q) = \cosh(\theta(q)) \hat{a}^\dagger(q) + \sinh(\theta(q)) \hat{b}(q) \]

\[ \hat{d}(q) = \cosh(\theta(q)) \hat{b}(q) + \sinh(\theta(q)) \hat{a}(q) \rightarrow \hat{d}^\dagger(q) = \cosh(\theta(q)) \hat{b}^\dagger(q) + \sinh(\theta(q)) \hat{a}(q) \]

Let’s invert these equations so that we have an explicit expression for \( \hat{a} \) and \( \hat{b} \) in terms of \( \hat{c} \) and \( \hat{d} \). This is easy enough if we recall: \( \cosh^2(x) - \sinh^2(x) = 1 \). Note this identity also preserves our desired commutation relations! We find:

\[ \hat{a}(q) = \cosh(\theta(q)) \hat{c}(q) - \sinh(\theta(q)) \hat{d}(q) \rightarrow \hat{a}^\dagger(q) = \cosh(\theta(q)) \hat{c}^\dagger(q) - \sinh(\theta(q)) \hat{d}(q) \]

\[ \hat{b}(q) = \cosh(\theta(q)) \hat{d}(q) - \sinh(\theta(q)) \hat{c}(q) \rightarrow \hat{b}^\dagger(q) = \cosh(\theta(q)) \hat{d}^\dagger(q) - \sinh(\theta(q)) \hat{c}(q) \]

An astute choice of \( \theta(q) \) will cancel all cross terms resulting in a diagonalized Hamiltonian. Let’s expand each term in our summation individually, then we can group like terms, and see where this brings us:

1. \( \hat{a}^\dagger(q) \hat{a}(q) = \left[ \cosh(\theta(q)) \hat{c}^\dagger(q) - \sinh(\theta(q)) \hat{d}(q) \right] \left[ \cosh(\theta(q)) \hat{c}(q) - \sinh(\theta(q)) \hat{d}(q) \right] \]
   \[ = \cosh^2(\theta(q)) \hat{c} \hat{c}^\dagger(q) + \sinh^2(\theta(q)) \hat{d} \hat{d}^\dagger(q) - \cosh(\theta(q)) \sinh(\theta(q)) \left( \hat{c}^\dagger(q) \hat{d}^\dagger(q) + \hat{d}(q) \hat{c}(q) \right) \]

2. \( \hat{b}^\dagger(q) \hat{b}(q) = \left[ \cosh(\theta(q)) \hat{d}^\dagger(q) - \sinh(\theta(q)) \hat{c}(q) \right] \left[ \cosh(\theta(q)) \hat{d}(q) - \sinh(\theta(q)) \hat{c}(q) \right] \]
   \[ = \cosh^2(\theta(q)) \hat{d} \hat{d}^\dagger(q) + \sinh^2(\theta(q)) \hat{c} \hat{c}^\dagger(q) - \cosh(\theta(q)) \sinh(\theta(q)) \left( \hat{c}^\dagger(q) \hat{d}^\dagger(q) + \hat{d}(q) \hat{c}(q) \right) \]

3. \( \hat{a}(q) \hat{b}(q) = \left[ \cosh(\theta(q)) \hat{c}(q) - \sinh(\theta(q)) \hat{d}(q) \right] \left[ \cosh(\theta(q)) \hat{d}(q) - \sinh(\theta(q)) \hat{c}(q) \right] \]
   \[ = \cosh^2(\theta(q)) \hat{c} \hat{d}^\dagger(q) + \sinh^2(\theta(q)) \hat{c} \hat{c}^\dagger(q) - \cosh(\theta(q)) \sinh(\theta(q)) \left( \hat{c}^\dagger(q) \hat{d}^\dagger(q) + \hat{d}(q) \hat{c}(q) \right) \]

4. \( \hat{a}^\dagger(q) \hat{b}(q) = \left[ \cosh(\theta(q)) \hat{c}^\dagger(q) - \sinh(\theta(q)) \hat{d}(q) \right] \left[ \cosh(\theta(q)) \hat{d}(q) - \sinh(\theta(q)) \hat{c}(q) \right] \]
   \[ = \cosh^2(\theta(q)) \hat{c}^\dagger \hat{d}^\dagger(q) + \sinh^2(\theta(q)) \hat{c}^\dagger \hat{c}(q) \hat{d}(q) - \cosh(\theta(q)) \sinh(\theta(q)) \left( \hat{c}^\dagger(q) \hat{c}(q) + \hat{d}(q) \hat{d}^\dagger(q) \right) \]

Substituting these expressions into our Hamiltonian and Normal ordering we find (this is why you should use a matrix!):

\[ \left[ \hat{a}^\dagger(q) \hat{a}(q) + \hat{b}^\dagger(q) \hat{b}(q) + \cos(q) \hat{a}(q) \hat{b}(q) + \cos(q) \hat{a}^\dagger(q) \hat{b}(q) \right] \]
\[ = 2 \left[ \cosh(2\theta(q)) - \sinh(2\theta(q)) \cos(q) \right] \left( 1 + \hat{c}^\dagger \hat{c}(q) + \hat{d}^\dagger \hat{d}(q) \right) \]
\[ + \left[ \cosh^2(\theta(q)) + \sinh^2(\theta(q)) \right] \cos(q) - 2 \cosh(\theta(q)) \sinh(\theta(q)) \left( \hat{c}(q) \hat{d}(q) + \hat{c}(q) \hat{d}^\dagger(q) \right) \]

This shows us what choice we need to make on our angle \( \theta(q) \); indeed, setting the coefficient of the cross term to zero gives:

\[
\cos(q) = \frac{2 \cosh(\theta(q)) \sinh(\theta(q))}{\cosh^2(\theta(q)) + \sinh^2(\theta(q))} = \frac{2 \tanh(\theta(q))}{1 + \tanh^2(\theta(q))} = \tanh(2\theta(q))
\]
This implies:

\[ \theta(q) = \frac{1}{2} \tanh^{-1}(\cos(q)) \]

This is what we got before with 10 times the work!

4 Appendix 2

We are interested in the following symmetry: \( \mathcal{L}(U\Phi) = \mathcal{L}(\Phi) \). Our Lagrangian density above is only invariant for constant transforms; i.e. \( \partial_\mu U_{ab}(x) = 0 \) \( \forall a, b \). In order to promote our global symmetry to a local one we will need to define a covariant derivative. Inspired by our simpler case of a \( U(1) \) symmetry we define:

\[ D_\mu = I \partial_\mu - igA_\mu \]

where \( I \) in the 2x2 identity matrix and \( A_\mu \) is a 2x2 matrix valued vector field. We can figure out how \( A_\mu \) transforms if we invoke:

\[ (D_\mu \Phi)' = U(D_\mu \Phi) \]

This tells us that the field itself transforms as the covariant derivative does provided we choose the transformation properties of our gauge field wisely. A simple calculation shows:

\[ (I \partial_\mu - igA'_\mu)U \Phi = U\left[I \partial_\mu \Phi + iU^{-1} (I(\partial_\mu U) - igA'_\mu U) \Phi \right] \]

\[ = U(\partial_\mu \Phi - igA_\mu) \Phi \]

From this we deduce, for our transformation properties to be realized, that:

\[ A'_\mu = UA_\mu U^{-1} - \frac{i}{g} (\partial_\mu U)U^{-1} = UA_\mu U^{-1} + \frac{i}{g} U(\partial_\mu U^{-1}) \]

We can actually expand this vector field using the generators of SU(2); with the above generators of our group we find:

\[ (A_\mu(x))_{ab} = A^k_\mu(x)\lambda^k_{ab} \]

Here \( a, b \) and \( k \) all run over \( N \). Following the work in problem 2.2 we see that under a local symmetry we’d have also need to consider how the gauge field varies. In the adjoint representation the gauge fields transforms like:

\[ \delta A^k_\mu(x) \approx i f^{ksj} A^j_\mu(x) \theta^s(x) + I \frac{1}{g} \partial_\mu \theta^k (x) \]

So since \( [\sigma^i, \sigma^j] = 2i \varepsilon_{ijk} \sigma^k \) for SU(2), we see:

\[ \delta \phi_a \approx i \sigma^k_{ab} \phi_b \theta^k(x) \]

and,

\[ \delta A^k_\mu(x) \approx 2i \varepsilon^{ksj} A^j_\mu(x) \theta^s(x) + I \frac{1}{g} \partial_\mu \theta^k (x) \]

We use these relations to derive conserved currents, and thus, charges. We know already:

\[ \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi_a} \delta \phi_a + \frac{\delta \mathcal{L}}{\delta \partial_\nu \phi_a} \delta \partial_\nu \phi_a + \phi_a \leftrightarrow \phi^*_a + \frac{\delta \mathcal{L}}{\delta A^k_\mu} \delta A^k_\mu + \frac{\delta \mathcal{L}}{\delta \partial_\nu A^k_\mu} \delta \partial_\nu A^k_\mu \]

Using our equations of motion we can see:
\[
\frac{\delta L}{\delta \phi_a} \delta \phi_a + \frac{\delta L}{\delta \phi_a} \delta \partial_\nu \phi_a + \phi_a \leftrightarrow \phi_a^* = \partial_\nu \left[ \frac{\delta L}{\delta \phi_a} \delta \phi_a + \frac{\delta L}{\delta \phi_a} \delta \phi_a^* \right]
\]

\[
= \frac{i}{2} \partial_\nu \left[ \left( \sigma_{ab}^k (D^\nu \phi_a)^* \phi_b - \sigma_{ba}^k (D^\nu \phi_a) \phi_b^* \right) \theta^k(x) \right]
\]

\[
:= \partial_\nu \left[ j^{\nu k} \theta^k(x) \right]
\]

We can see the Lagrangian does not depend on \( \partial_\nu A_\mu \); thus:

\[
\frac{\delta L}{\delta A^k_\mu} \delta A^k_\mu + \frac{\delta L}{\delta \partial_\nu A^k_\mu} \delta \partial_\nu A^k_\mu = \frac{\delta L}{\delta A^k_\mu} \delta A^k_\mu
\]

we put everything together and find:

\[
\delta L = \partial_\nu \left[ j^{\nu k} \theta^k(x) \right] + \left[ 2i \varepsilon^{kjs} A^j_\mu(x) \theta^s(x) + \frac{1}{g} \partial_\mu \theta^k(x) \right] \frac{\delta L}{\delta A^k_\mu}
\]

For \( \theta \) constant this reduces to the answer in problem 2.2 (after choosing a gauge where \( A^k_\mu = 0 \)).