Problem Set 4 Solutions

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I. PATH INTEGRAL FOR A PARTICLE IN A DOUBLE WELL POTENTIAL

In this problem, we consider a particle with coordinate $q$ and mass $m$ moving in the double well potential $V(q)$,

$$V(q) = \lambda(q^2 - q_0^2)^2.$$  

1. Imaginary time path integral

Using the method in the lecture note, one can find that the Euclidean transition amplitude is

$$A \left( q_0, T, -q_0, -T \right) = \int_{q(-T) = -q_0} Dq e^{-\frac{1}{\hbar} \int_{-T}^{T} dr \left[ \frac{1}{2} m \dot{q}^2 + V(q) \right]} ,$$  

where $V(q) = \lambda(q^2 - q_0^2)^2$ is the potential. Compared with the Minkowski formula, the double-well potential is inverted.

2. Euler-Lagrange equation

Solution: The Euler-Lagrange equation is

$$0 = \partial_r \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

$$= m \ddot{q}_c - \frac{\partial V}{\partial q} \bigg|_{q=q_c}$$

$$= m \ddot{q}_c - 4\lambda (q_c^2 - q_0^2) q_c.$$  

or

\[ \dot{q}_c = 4 \frac{\lambda}{m} (q_c^2 - q_0^2) q_c, \]

(3)

where we have denoted the solution as \( q_c \) and the subscript “c” stands for the classical trajectory.

Notice that the “energy” in the Euclidean spacetime is

\[ E = \frac{1}{2} m \dot{q}_c^2 - V (q), \]

(4)

which is a constant of motion. At \( \tau = \frac{\pm T}{2} \), there is \( q_c (\pm \frac{T}{2}) = \pm q_0 \), \( \dot{q}_c (\pm \frac{T}{2}) = 0 \) and thus \( E = 0 \). Hence, from the energy defined in Eq. (4), one can obtain

\[ \frac{1}{2} m \dot{q}_c^2 = \lambda (q_c^2 - q_0^2)^2 \]

\[ \dot{q}_c = \pm \sqrt{\frac{2\lambda}{m} (q_c^2 - q_0^2)}. \]

The corresponding solution is

\[ q_c (\tau) = \mp q_0 \tanh \left[ q_0 \sqrt{\frac{2\lambda}{m}} (\tau - \tau_0) \right]. \]

(5)

Notice that the boundary condition is

\[ q_c \left( \pm \frac{T}{2} \right) = \pm q_0, \]

(6)

so the solution should be

\[ q_c (\tau) = q_0 \tanh \left[ q_0 \sqrt{\frac{2\lambda}{m}} (\tau - \tau_0) \right]. \]

(7)

**Physical interpretation:** The classical trajectory in Eq. (7) shows that the particle has tunneled from \(-q_0\) at \(-\frac{T}{2}\) to \(q_0\) at \(\frac{T}{2}\), which happens at time \(\tau_0\). However, \(\tau_0\) can not be fixed from the equation of motion. Notice that our action is invariant under translation along the time axis, but the classical trajectory obtained here has clearly broken the translational symmetry. In order to restore the symmetry, we shall sum over all possible tunneling time.

3. **Action for \( q_c \)**

By inserting \( q_c \) back to the action, there is

\[ S_c = \int d\tau \left[ \frac{1}{2} m \dot{q}_c^2 + \lambda (q_c^2 - q_0^2)^2 \right] \]

\[ = \int dq_c \left[ 2\lambda (q_c^2 - q_0^2)^2 / \dot{q}_c \right] \]

\[ = \int_{-q_0}^{q_0} dq_c \left[ \sqrt{2\lambda m} (q_c^2 - q_0^2) \right] \]

\[ = \frac{4\sqrt{2\lambda m}}{3} q_0^3. \]

(8)

4. **Transition amplitude at leading order**

Now we decompose \( q (\tau) \) to \( q (\tau) = q_c (\tau) + \xi (\tau) \). By keeping terms to quadratic order of \( \xi (\tau) \) and rescaling \( \xi (\tau) \) as \( \xi (\tau) \rightarrow \sqrt{\hbar} \xi (\tau) \) so as to absorb the coefficient \( \hbar^{-1} \) in front of the action, there is
\[ A \left( q_0, \frac{T}{2}; -q_0, -\frac{T}{2} \right) = e^{-\frac{1}{\hbar} \hat{S}} \int d\xi \exp \left\{ -\frac{1}{2} \int \frac{dT}{T} \left[ m\dot{\xi}^2 + \lambda \left( 12q_0^2 - 4q_0^2 \xi^2 \right) \right] + \mathcal{O}(\hbar) \right\} \]
\[ = e^{-\frac{1}{\hbar} \hat{S}} \det \left[ \hat{O} \right] + \ldots, \] 
\[ \text{(9)} \]

where
\[ \hat{O} = -m \frac{d^2}{dT^2} + 4\lambda q_0^2 \left\{ 3 \tanh^2 \left[ q_0 \sqrt{\frac{2\lambda}{m}} (\tau - \tau_0) \right] - 1 \right\} \]
\[ \text{(10)} \]
is the fluctuation determinant.

**II. PATH INTEGRAL FOR A CHARGED PARTICLE MOVING ON A PLANE IN THE PRESENCE OF A PERPENDICULAR MAGNETIC FIELD**

Let us consider following Hamiltonian for a electron under magnetic fields
\[ H(\hat{q}, \hat{p}) = \frac{1}{2m} \left[ \hat{p} + eA(\hat{q}) \right]^2, \]
\[ \text{(11)} \]
where
\[ A_1(\hat{q}) = -\frac{1}{2} B x_2, \]
and
\[ A_2(\hat{q}) = \frac{1}{2} B x_1. \]

1. **Transition amplitude** \( \langle r_0, t_f | r_0, t_i \rangle \) with \( |t_f - t_i| \to \infty \)

The transition amplitude is
\[ A(\mathbf{r}_0, t_f; \mathbf{r}_0, t_i) = \langle \mathbf{r}_0, t_f | e^{-i\frac{\hbar}{\pi} H T} | \mathbf{r}_0, t_i \rangle, \]
\[ \text{(12)} \]
where \( T = t_f - t_i \). Now we divide \( T \) into \( N \) slice and label \( t_i \) as \( t_0, t_f \) as \( t_N \). Then, the transition amplitude can be recast as
\[ A(\mathbf{r}_0, t_f; \mathbf{r}_0, t_i) = \langle \mathbf{r}_0, t_f | e^{-i\frac{\hbar}{\pi} H \Delta t} \ldots e^{-i\frac{\hbar}{\pi} H \Delta t} | \mathbf{r}_0, t_i \rangle, \]
where \( \Delta t = \frac{t_f - t_i}{N} \) and \( N \to \infty, \Delta t \to 0 \). Notice that
\[ 1 = \int \frac{d^dp}{(2\pi\hbar)^2} |p\rangle \langle p| = \int d^dr |r\rangle \langle r|, \]
so by inserting this identity back to the transition amplitude, there is
\[ A(\mathbf{r}_0, t_f; \mathbf{r}_0, t_i) = \left( \int \prod_{j=1}^{N} \frac{d^dp_{j+1}}{(2\pi\hbar)^2} \right) \left( \int \prod_{j=1}^{N-1} d^dr \right) \langle \mathbf{r}_0, t_f | e^{-i\frac{\hbar}{\pi} H \Delta t} \ldots | \mathbf{p}_{i+1} \rangle e^{-i\frac{\hbar}{\pi} H \Delta t} | \mathbf{r}_i \rangle \langle \mathbf{r}_i | \mathbf{p}_i \rangle \ldots \langle \mathbf{r}_1 | \mathbf{p}_1 \rangle | e^{-i\frac{\hbar}{\pi} H \Delta t} | \mathbf{r}_0, t_i \rangle. \]
Now let us focus on
\[ \langle p_{i+1} | e^{-i\frac{1}{\hbar}H\Delta t} | r_i \rangle \langle r_i | p_i \rangle, \]
which is
\[ \langle p_{i+1} | e^{-i\frac{1}{\hbar}H\Delta t} | r_i \rangle \langle r_i | p_i \rangle = e^{i\frac{1}{\hbar}[p_{i+1} - p_i, r_i] - H(r_i, p_{i+1})\Delta t}. \]

Hence, the transition amplitude can be recast as
\[
A(r_0, t_f; r_0, t_i) = \int \mathcal{D}p\mathcal{D}r \exp\left\{ \frac{i}{\hbar} \sum_{i=0}^{N} \left[ (p_i - p_{i+1}) \cdot r_i - H(r_i, p_{i+1}) \Delta t \right] \right\}
\]
\[ = \int \mathcal{D}p\mathcal{D}r \exp\left\{ \frac{i}{\hbar} \sum_{i=0}^{N-1} \Delta t \left[ (\Delta t)^{-1} p_i \cdot (r_i - r_{i-1}) - H(r_i, p_{i+1}) \right] \right\}
\]
\[ = \int \mathcal{D}p\mathcal{D}r \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ p \cdot \dot{r} - H(r, p) \right] \right\} |_{r(t_i)=r(t_f)=r_0} \]
\[ = \prod_{j=1}^{N} \frac{d^3p_j}{(2\pi\hbar)^3} \prod_{j=1}^{N-1} d^3r_j, \] (14)

where \( \mathcal{D}p = \prod_{j=1}^{N} \frac{d^3p_j}{(2\pi\hbar)^3} \) and \( \mathcal{D}r = \prod_{j=1}^{N-1} d^3r. \)

Then, we shall integrate over the momentum to obtain the action in the configuration space, i.e.,
\[ A(r_0, t_f; r_0, t_i) = \int \mathcal{D}k\mathcal{D}r \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ k \cdot \dot{r} - \frac{1}{2m} \left( p - \frac{e}{c}A \right)^2 \right] \right\} |_{r(t_i)=r(t_f)=r_0}
\]
\[ = \int \mathcal{D}k\mathcal{D}r \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ k \cdot \dot{r} - \frac{1}{2m} \left( p - \frac{e}{c}A \right)^2 \right] \right\} |_{r(t_i)=r(t_f)=r_0}
\]
\[ = \int \mathcal{D}r \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left( \frac{1}{2} m |\dot{r}|^2 - \frac{e}{c} A \cdot \dot{r} \right) \right\} |_{r(t_i)=r(t_f)=r_0}, \]
(14)

where in the second line, we have defined the kinetic momentum \( k \) as \( k = p + \frac{e}{c}A \) and in the third line, we have integrated over \( k. \)

Hence, the action is
\[ \int_{t_i}^{t_f} dt \left( \frac{1}{2} m |\dot{r}|^2 - A \cdot \dot{r} \right), \]
(15)
which is the kinetic energy \( \left( \frac{1}{2} m |\dot{r}|^2 \right) \) minus the potential energy \( A \cdot \dot{r} \) as we expected.

2. “Ultra-quantum” limit

In the limit \( m \to 0 \), the action in Eq. (14) becomes
\[ \frac{1}{\hbar} S \to -\frac{e}{c\hbar} \oint_{\partial M} A \cdot dl, \]
where \( \partial M \) is a loop and it can be regraded as the boundary of a two-dimensional surface \( M \). Notice that
\[ \oint_{\partial M} A \cdot dl = \int_{M} B \cdot dS \equiv \Phi, \]
(16)
where $\Phi$ is the magnetic flux through the surface $M$. Hence, the action is
\[
\frac{1}{\hbar}S \rightarrow -2\pi \frac{\Phi}{\Phi_0},
\] (17)
where $\Phi_0 = \frac{2\pi\hbar}{e}$ is the magnetic flux quantum.

3. Ambiguities

First of all, let us consider such a case that $B$ at the boundary of the $L \times L$ plane takes the same value. In this case, this plane is compactified to a sphere. Then, there is ambiguity in the action defined above. Namely, the choice of the surface $M$ is not fixed uniquely by its boundary, $\partial M$. One can choose either the upper surface $M^+$, or the bottom surface $M^-$ bounded by $\partial M$. To fix this ambiguity, the action should satisfy
\[
2\pi n = \frac{e}{\hbar} \left( \int_{M^+} B \cdot dS - \int_{M^-} B \cdot dS \right) = \frac{e}{\hbar} \int_{M^+ + M^-} B \cdot dS,
\] (18)
where $n \in \mathbb{Z}$ and $2\pi$ on the left-handed side is because that the action appears as $e^{i\frac{1}{\hbar}S}$. Notice that $M^+ + M^-$ is a compact manifold, so this is actually the Dirac quantization condition. This also implies that the magnetic flux is quantized in unit of $\Phi_0 = \frac{2\pi\hbar}{e}$. In addition, because at the overlap of $M^+$ and $M^-$, the vector potential defined on these two patches, can be different only by a gauge transformation $e^{i\theta}$ and thus Eq. (18) becomes
\[
2\pi n = \oint_{\partial M} e^{-i\theta} \partial_i e^{i\theta} dx^i,
\]
which is the fundamental homotopy group, i.e., $\pi_1(U(1)) = \mathbb{Z}$.

Secondly, let us consider such a case that the $L \times L$ plane is not compactified to a closed manifold. Then, there is no ambiguity in the action defined above. Here, $L$ is taken to $\infty$, so it is natural to assume that the magnetic fields at the boundary take the same value, i.e., $B = 0$. Then, the plane is effectively compactified to a two-dimensional sphere. Following the argument above, the magnetic flux through the plane must be quantized.

III. PATH INTEGRAL FOR A SCALAR FIELD THEORY

In this problem, we consider complex scalar fields $\phi$ coupling with complex sources $J$ in $(3+1)$-dim spacetime, i.e.,
\[
S = \int d^4x \left( |\partial\phi|^2 - m^2 |\phi|^2 - J^* \phi - J\phi^* \right).
\] (19)
Note that for later convenience, the natural units are used here, i.e., $\hbar = c = 1$.

1. Vacuum persistence amplitude

Following the derivation in the lecture note, in the Minkowski spacetime, there is
\[
j_0(0|0) = Z[J, J^*] = \int \mathcal{D}\phi^* \mathcal{D}\phi e^{iS},
\] (20)
where
\[
S = \int d^4x \left( |\partial\phi|^2 - m^2 |\phi|^2 - J^* \phi - J\phi^* \right).
\] (21)
Hence, by performing the Wick rotation, the Euclidean vacuum persistence amplitude is
\[
Z_E[J, J^*] = \int \mathcal{D}\phi^* \mathcal{D}\phi e^{-\int d^4x \left( |\partial\phi|^2 + m^2 |\phi|^2 + J^* \phi + J\phi^* \right)}.
\] (22)
2. **Partition function**

In the Euclidean spacetime, the equation of motion is

\[ (-\partial^2 + m^2) \phi_e = -J, \]  

and

\[ (-\partial^2 + m^2) \phi^*_e = -J^*. \]  

Then, by decomposing \( \phi \) as

\[ \phi = \phi_e + \xi, \]

the partition function becomes

\[
Z_E[J, J^*] = e^{-\int d^4x E \left( |\partial \phi_e|^2 + m^2 |\phi_e|^2 + J^* \phi_e + J \phi_e^* \right) + \int D\xi^* D\xi e^{-\int d^4x E \left( |\partial \xi|^2 + m^2 |\xi|^2 \right)}}
\]

\[
= N_E e^{-\int d^4x E \left[ \phi_e^* \left( -\partial^2 + m^2 \right) \phi_e + J^* \phi_e + J \phi_e^* \right]} \det \left( -\partial^2 + m^2 \right)^{-1}
\]

\[
= N_E \det \left( -\partial^2 + m^2 \right)^{-1} e^{\int d^4x d^4y E \cdot J(x) \cdot \left( -\partial^2 + m^2 \right)^{-1} J(y)},
\]

where in the second line, we have used the equation of motion to rewrite \( \phi_e \) in terms of \( J \) and \( N_E \) is a normalization constant. Because that the correlation function can be derived from \( Z_E^{-1} (\frac{-\delta}{\delta J})^n \left( \frac{-\delta}{\delta J^*} \right)^n Z_E[J, J^*] |_{J=J^*=0} \), both \( N_E \) and \( \det \left( -\partial^2 + m^2 \right)^{-1} \) are canceled by \( Z_E^{-1} \).

Similarly, the Minkowski one is

\[
Z[J, J^*] = \mathcal{N} \det \left( -\partial^2 + m^2 \right)^{-1} e^{-\frac{i}{\hbar} \int d^4x d^4y \left( J(x) \cdot \left( -\partial^2 + m^2 \right)^{-1} J(y) \right)},
\]

where \( \mathcal{N} \) is an unimportant normalization constant.

3. **Two-point correlation functions**

In this section, we shall focus on the Minkowski spacetime.

\( G_2(x - x') = \langle 0 | T \phi(x) \phi^*(x') | 0 \rangle \): This two-point correlation function can be obtained from the partition function as

\[
G_2(x - x') = Z^{-1} \left[ \frac{\delta}{\delta J^* (x)} \right] \left[ \frac{\delta}{\delta J (x')} \right] \left| Z[J, J^*] \right|_{J=J^*=0}
\]

\[
= i \langle x | \left( -\partial^2 + m^2 \right)^{-1} \langle x' \rangle.
\]

\( G_2^*(x - x') = \langle 0 | T \phi^*(x) \phi (x') | 0 \rangle \): Similarly, this two-point correlation function is

\[
G_2(x' - x) = Z^{-1} \left[ \frac{\delta}{\delta J (x')} \right] \left[ \frac{\delta}{\delta J^* (x)} \right] \left| Z[J, J^*] \right|_{J=J^*=0}
\]

\[
= i \langle x' | \left( -\partial^2 + m^2 \right)^{-1} \langle x \rangle.
\]  

\( G_2'(x - x') = \langle 0 | T \phi(x) \phi (x') \rangle \) and \( G_2''(x - x') = \langle 0 | T \phi^* (x) \phi (x') \rangle | 0 \rangle \): By using the same derivation above, these two terms are zero, i.e.,

\[
G_2'(x - x') = G_2''(x - x') = 0.
\]

(28)
4. Equations satisfied by the Green function

**Euclidean spacetime:** The expectation value of \( \phi \) is

\[
\langle \phi (x) \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{-S_E(\phi, \phi^*)} \phi (x),
\]

which is invariant under reparametrization of \( \phi \), i.e.,

\[
0 = \left( - \int d^4 y_E \frac{\delta S_E}{\delta \phi (y)} \phi (x) \delta \phi (y) \right) + \langle \delta \phi (x) \rangle.
\]

This implies that

\[
\delta^4 (x - y) = (- \partial^2 + m^2) \langle T \phi^* (y) \phi (x) \rangle,
\]

so the Green function satisfies

\[
(- \partial^2 + m^2) G_E (y - x) = \delta^4 (x - y).
\]

(29)

By performing Fourier’s transformation, one can obtain

\[
G_E (x - y) = \int \frac{d^4 p_E}{(2\pi)^4} e^{ip_E \cdot (x-y)} \frac{1}{p_E^2 + m^2}.
\]

(30)

This is already calculated in the lecture note and it is given as

\[
G_E (x - y) = \frac{1}{2\pi^2} \frac{m}{|x - y|} K_1 (m |x - y|),
\]

(31)

where \( K_\nu (x) \) is the modified Bessel function. For \( m |x - y| \gg 1 \), there is

\[
G_E (x - y) = \sqrt{\frac{\pi}{2}} \frac{m^2}{4\pi^2 |x - y|^2} e^{-m |x - y|} \left[ 1 + \mathcal{O} \left( \frac{1}{m |x - y|} \right) \right],
\]

(32)

which decays exponentially. By contrast, in the limit \( m |x - y| \ll 1 \), there is

\[
G_E (x - y) = \frac{1}{4\pi^2 |x - y|^2} + \mathcal{O} (m |x - y|),
\]

(33)

which decays in a power law.

**Minkowski spacetime:** Under a Wick rotation, the Green function becomes

\[
G (x - y) = \frac{i}{2\pi^2} \frac{m}{\sqrt{-s^2}} K_1 (m \sqrt{-s^2}),
\]

(34)

where \( s^2 \equiv (x - y)^\mu (x - y)_\mu \).

For space-like separation, there is \( s^2 < 0 \) and \( \sqrt{-s^2} > 0 \). Hence, the asymptotic behavior is similar to the Euclidean spacetime. Now let us turn to the time-like separation, there is \( s^2 > 0 \) and thus \( \sqrt{-s^2} = i |s| \). Then, for \( m |s| \gg 1 \), there is

\[
G (x - y) = \frac{\sqrt{\pi} m^2}{4\pi^2 (m |s|)^3} e^{im |s|} \left[ 1 + \mathcal{O} \left( \frac{1}{m |s|} \right) \right],
\]

(35)

which shows a oscillation behavior. For \( m |s| \ll 1 \), there is

\[
G (x - y) = \frac{1}{4\pi^2 |s|^2} + \mathcal{O} (m |s|),
\]

(36)

which decays in a power law.
5. Four-point correlation functions

\[ G^a_4(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2)^* \phi(x_3) \phi(x_4) | 0 \rangle : \]

By using the Wick theorem, there is

\[ G^a_4(x_1, x_2, x_3, x_4) = G(x_1 - x_3) G(x_2 - x_4) + G(x_1 - x_4) G(x_2 - x_3). \]  \hspace{1cm} (37)

\[ G^b_4(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle = 0 \]

\[ G^c_4(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2)^* \phi(x_3)^* \phi(x_4)^* | 0 \rangle = 0 \]

\[ G^d_4(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2) \phi(x_3)^* \phi(x_4) | 0 \rangle : \]

By using the Wick theorem, there is

\[ G^d_4(x_1, x_2, x_3, x_4) = G(x_1 - x_2) G(x_3 - x_4) + G(x_1 - x_4) G(x_3 - x_2). \]  \hspace{1cm} (38)

\[ G^e_4(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2) \phi(x_3) \phi(x_4)^* | 0 \rangle \]

By using the Wick theorem, there is

\[ G^e_4(x_1, x_2, x_3, x_4) = G(x_1 - x_2) G(x_4 - x_3) + G(x_1 - x_3) G(x_4 - x_2). \]  \hspace{1cm} (39)

Relation between these four-point functions: From the results above and the symmetry of the two-point correlation functions, there are

\[ G^a_4(x_1, x_2, x_3, x_4) = G^b_4(x_1, x_3, x_2, x_4) \]  \hspace{1cm} (40)

\[ G^a_4(x_1, x_2, x_3, x_4) = G^b_4(x_1, x_4, x_3, x_2) \]  \hspace{1cm} (41)

\[ G^a_4(x_1, x_2, x_3, x_4) = G^c_4(x_1, x_4, x_3, x_2) \]  \hspace{1cm} (42)

\[ G^a_4(x_1, x_2, x_3, x_4) = G^c_4(x_3, x_2, x_1, x_4) \]  \hspace{1cm} (43)

\[ G^a_4(x_1, x_2, x_3, x_4) = G^c_4(x_1, x_2, x_4, x_3) \]  \hspace{1cm} (44)

\[ G^a_4(x_1, x_2, x_3, x_4) = G^c_4(x_2, x_1, x_3, x_4). \]  \hspace{1cm} (45)