Here you will look again at problem 5 of Problem Set 1 in which you studied some of the properties of the dynamics of a charged (complex) scalar field $\phi(x)$ coupled to the electromagnetic field $A_\mu(x)$. Recall that the Lagrangian density $\mathcal{L}$ for this system is

$$\mathcal{L} = (D_\mu \phi(x))^* (D^\mu \phi(x)) - m_0^2 |\phi(x)|^2 - \frac{\lambda}{2} (|\phi(x)|^2)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$ \hspace{1cm} (1)$$

where $D_\mu$ is the covariant derivative

$$D_\mu \equiv \partial_\mu + ie A_\mu$$ \hspace{1cm} (2)$$

e is the electric charge and $^*$ denotes complex conjugation. In this problem set you will determine several important properties of this field theory at the classical level.

1. Derive an expression for the locally conserved current $j_\mu(x)$, associated with the global symmetry

$$\phi(x) \rightarrow \phi'(x) = e^{i\theta(x)} \phi(x)$$

$$\phi^*(x) \rightarrow \phi'^*(x) = e^{-i\theta(x)} \phi^*(x)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x)$$ \hspace{1cm} (3)$$

in terms of the fields of the theory.

2. Show that the conservation of the current $j_\mu$ implies the existence of a constant of motion. Find an explicit form for this constant of motion.

3. Consider now the case of the local (gauge) transformation

$$\phi(x) \rightarrow \phi'(x) = e^{i\theta(x)} \phi(x)$$

$$\phi^*(x) \rightarrow \phi'^*(x) = e^{-i\theta(x)} \phi^*(x)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$ \hspace{1cm} (4)$$

where $\theta(x)$ and $\Lambda(x)$ are two functions. What should be the relation between $\theta(x)$ and $\Lambda(x)$ for this transformation to be a symmetry of the Lagrangian of the system?
4. Show that, if the system has the local symmetry of the previous section, there is a locally conserved gauge current $J_\mu(x)$. Find an explicit expression for $J_\mu$ and discuss in which way it is different from the current $j_\mu$ of item 1). Find an explicit expression for the associated constant of motion and discuss its physical meaning.

5. Find the Energy-Momentum $T^{\mu\nu}$ tensor for this system. Show that it can be written as the sum of two terms

$$T^{\mu\nu} = T^{\mu\nu}(A) + T^{\mu\nu}(\phi, A)$$

where $T^{\mu\nu}(A)$ is the energy-momentum tensor for the free electromagnetic field and $T^{\mu\nu}(\phi, A)$ is the tensor which results by modifying the energy-momentum tensor for the decoupled complex scalar field $\phi$ by the minimal coupling procedure.

6. Find explicit expressions for the Hamiltonian density $H(x)$ and the linear momentum density $P(x)$ for this system. Give a physical interpretation for all of the terms that you found for each quantity.

7. Consider now the case of an infinitesimal Lorentz transformation

$$x_\mu \rightarrow x'_\mu + \omega_{\mu\nu} x'^\nu$$

where $\omega_{\mu\nu}$ infinitesimal and antisymmetric. Show that the invariance of the Lagrangian of this system under these Lorentz transformation leads to the existence of a conserved tensor $M_{\mu\nu\lambda}$. Find the explicit form of this tensor. Give a physical interpretation for its spacial components. Does the conservation of this tensor impose any restriction on the properties of the energy-momentum tensor $T^{\mu\nu}$? Explain.

**Warning:** Be very careful in how you treat the fields. Recall that not all of the fields are scalars!

8. In this section you will consider again the same system but in a polar representation for the scalar field $\phi$, i.e.

$$\phi(x) = \rho(x) e^{i\omega(x)}$$

In problem set 1, problem 2, you showed that for $m_0^2 < 0$ the lowest energy states of the system can be well approximated by freezing the amplitude mode $\rho$ to a constant value $\rho_0$ which you obtained by an energy minimization argument. In this section you are asked to find the form of

(a) the conserved gauge current $J_\mu$,
(b) the total energy $H$,
(c) the total linear momentum $P$ in this limit.
9. Consider now the analytic continuation to imaginary time of this theory. Find the energy functional of the equivalent system in classical statistical mechanics. Give a physical interpretation for each of the terms of this energy functional. If $D$ is the dimensionality of space-time for the original system, what is the dimensionality of space for the equivalent classical problem?

**Warning**: Be very careful in the way you perform the analytic continuation of the components of the vector potential.