582 Homework 1

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Contents

1 The Dirac Equation

Consider the Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

From this we get the following equations of motion:

$$(i\partial\!\!\!/ - m)\psi = 0$$

and

$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0$$

where the differential operator is acting to the right and left, respectively.

1. **[5/20]** Consider $j^\mu = \bar{\psi}\gamma^\mu\psi$. We will see *this 4-current is conserved* (i.e. $\partial_\mu j^\mu = 0$) if ψ satisfies the Dirac equation. There are several ways we could see this; let's begin with the straight forward way:

$$\begin{aligned}\partial_\mu j^\mu &= (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi \\ &= \bar{\psi}\overleftarrow{\partial_\mu}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi \\ &= \bar{\psi}\overleftarrow{\partial\!\!\!/}\psi + \bar{\psi}\partial\!\!\!/\psi\end{aligned}$$

   Using our equations of motion we see:

$$\partial\!\!\!/\psi = -im\psi$$

   and,

$$\bar{\psi}\overleftarrow{\partial\!\!\!/} = im\bar{\psi}$$

   Substituting these into our above equations we see:

$$\partial_\mu j^\mu = im\Big(\bar{\psi}\psi - \bar{\psi}\psi\Big) = 0$$

   We could have reached this conclusion via Noether's theorem: there is a global $U(1)$ symmetry present in the Dirac Lagrangian. Of course, this concept was not met at the time of this homework.

2. **[5/20]** In deriving the Dirac equation Dirac made the stipulation that *each component of the spinor should satisfy the Klein-Gordon equation.* As we will see in due time, there are various connections between the two theories (i.e. spectral functions, propagators, etc.). For now we establish the first property by noting:

$$(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi = -(\partial\!\!\!/\partial\!\!\!/ + m^2)\psi = 0$$

   Next we use the fact that a symmetric matrix contracted with an anti-symmetric matrix is zero to simplify our above equation. Recall the Dirac matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, which is symmetric, the commutator $[\gamma^\mu, \gamma^\nu]$ is antisymmetric, while $\partial_\mu\partial_\nu$ is symmetric. With these we can see:

$$\begin{aligned}\partial\!\!\!/\partial\!\!\!/ &= \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu \\ &= \frac{1}{2}\Big[[\gamma^\mu, \gamma^\nu] + \{\gamma^\mu, \gamma^\nu\}\Big]\partial_\mu\partial_\nu \\ &= \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu \\ &= g^{\mu\nu}\partial_\mu\partial_\nu \\ &= \partial_\mu\partial^\mu\end{aligned}$$

   The desired result follows:

$$(\partial_\mu\partial^\mu + m^2)\psi = 0$$

3. **[10/20]** Using properties of the γ-matrices we now derive various identities.

(a) **[4/10]** To begin:

$$
\begin{aligned}
\not{A}\not{B} &= A_\mu \gamma^\mu B_\nu \gamma^\nu \\
&= A_\mu B_\nu \Big(2g^{\mu\nu} - \gamma^\nu \gamma^\mu\Big) \\
&= A_\mu B_\nu \Big(g^{\mu\nu} + \frac{1}{2}\big(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\big) - \gamma^\nu\gamma^\mu\Big) \\
&= A_\mu B_\nu \Big(g^{\mu\nu} + \frac{1}{2}\big[\gamma^\mu, \gamma^\nu\big]\Big) \\
&= A \cdot B - i\sigma^{\mu\nu} A_\mu B_\nu
\end{aligned}
\tag{1}
$$

Note that there is the 4×4 identity matrix implied after the dot product.

(b) **[2/10]** We can use (a) to easily derive the following identity:

$$
\begin{aligned}
\mathrm{Tr}\big[\not{A}\not{B}\big] &= A \cdot B\,\mathrm{Tr}\big[I\big] - i\mathrm{Tr}\big[\sigma^{\mu\nu}\big] A_\mu B_\nu \\
&= 4A \cdot B
\end{aligned}
\tag{2}
$$

Obviously $\mathrm{Tr}\big[\sigma^{\mu\nu}\big] = 0$ as a result of its definition, $\sigma^{\mu\nu} \propto \big[\gamma^\nu, \gamma^\mu\big]$, and the linearity of the trace operator as well as invariance of permutations of matrices under the trace operator.

(c) **[4/10]** Our final identity uses the anticommutation relation for the Clifford algebra:

$$
\begin{aligned}
\gamma^\lambda \gamma^\mu \gamma_\lambda &= \Big(2g^{\lambda\mu} - \gamma^\mu\gamma^\lambda\Big)\gamma_\lambda \\
&= \gamma^\mu \Big(2 - \frac{1}{2}\big\{\gamma^\lambda, \gamma_\lambda\big\}\Big) \\
&= \gamma^\mu \Big(2 - g^\lambda_\lambda\Big) \\
&= -2\gamma^\mu
\end{aligned}
\tag{3}
$$

The last line follows from: $g^\lambda_\lambda = D = 4$ (where D is the dimension of spacetime).

2 Transformation Properties of Dirac Fermion Bilinears

In this problem you will consider again the Dirac Theory and study the transformation properties of its physical observable under Lorentz transformations. Let:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

be a general Lorentz transformation, and $S(\Lambda)$ be the induced transformation for the Dirac spinors $\psi'_a(x)$ (with $a = 1, \ldots, 4$):

$$\psi'_a(x') = S(\Lambda)_{ab}\psi_b(x)$$

From this we can see:

$$\begin{aligned}
\bar{\psi}'(x') &= \psi'(x')^{\dagger}\gamma^0 \\
&= \Big(S(\Lambda)\psi(x)\Big)^{\dagger}\gamma^0 \\
&= \psi^{\dagger}(x)S^{\dagger}(\Lambda)\gamma^0
\end{aligned}$$

We can parametrize these induced transformations in terms of our generators $\sigma^{\mu\nu}$; that is:

$$S(\Lambda) = \exp\big(-i\sigma_{\mu\nu}\omega^{\mu\nu}/2\big)$$

Where $\omega_{\mu\nu}$ is a real constant which is antisymmetric in its indices. *We will be verifying transformation laws of Dirac bilinears, but it helps to begin by proving the following relationship.*

$$S^{\dagger}(\Lambda) = \gamma^0 S^{-1}(\Lambda)\gamma^0$$

Proof: To begin we just take the adjoint of our representation for our induced transformation:

$$\big(S(\Lambda)\big)^{\dagger} = \exp\big(i\sigma^{\dagger}_{\mu\nu}\omega^{\mu\nu}/2\big)$$

Using the definition of $\sigma_{\mu\nu}$ in terms of our γ-matrices we can come up with the desired result. Recall that we need to choose a basis for our gamma matrices; *here we use the chiral basis*:

$$\gamma_0 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}, \gamma_5 = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$$

Taking the adjoint of the first four matrices we can see:

$$\gamma_0^{\dagger} = \gamma_0 = \gamma_0\gamma_0\gamma_0$$

since $\gamma_0^2 = I_4$, and:

$$\gamma_k^{\dagger} = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix} = \gamma_0\gamma_k\gamma_0$$

It follows that:

$$\begin{aligned}
\sigma^{\dagger}_{\mu\nu} &= -\frac{i}{2}\big[\gamma^{\dagger}_{\nu}, \gamma^{\dagger}_{\mu}\big] = \frac{i}{2}\big[\gamma^{\dagger}_{\mu}, \gamma^{\dagger}_{\nu}\big] \\
&= \frac{i}{2}\big[\gamma_0\gamma_\mu\gamma_0, \gamma_0\gamma_\nu\gamma_0\big] \\
&= \frac{i}{2}\gamma_0\big[\gamma_\mu, \gamma_\nu\big]\gamma_0 \\
&= \gamma_0\sigma_{\mu\nu}\gamma_0
\end{aligned}$$

We can now easily prove our result:

$$\begin{aligned}
S^{\dagger}(\Lambda) &= \exp\big(i\sigma^{\dagger}_{\mu\nu}\omega^{\mu\nu}/2\big) \\
&= \sum_k \frac{i^k}{2^k k!}\Big(\gamma_0\sigma_{\mu\nu}\gamma_0\omega^{\mu\nu}\Big)^k \\
&= \sum_k \frac{i^k}{2^k k!}\gamma_0\Big(\sigma_{\mu\nu}\omega^{\mu\nu}\Big)^k\gamma_0 \\
&= \gamma_0\exp\big(i\sigma_{\mu\nu}\omega^{\mu\nu}/2\big)\gamma_0 \\
&= \gamma_0 S^{-1}(\Lambda)\gamma_0 \qquad \textbf{Q.E.D.}
\end{aligned}$$

As a quick application of this result, we can see:

$$\begin{aligned}\bar{\psi}'(x') &= \psi^\dagger(x)S^\dagger(\Lambda)\gamma_0 \\ &= \psi^\dagger(x)\Big(\gamma_0 S^{-1}(\Lambda)\gamma_0\Big)\gamma_0 \\ &= \bar{\psi}(x)S^{-1}(\Lambda)\end{aligned}$$

1. **[3/20]** The transformation properties of the first bilinear follows from our above work immediately:

$$\begin{aligned}\bar{\psi}'(x')\psi'(x') &= \bar{\psi}(x)S^{-1}(\Lambda)S(\Lambda)\psi(x) \\ &= \bar{\psi}(x)\psi(x)\end{aligned} \tag{4}$$

This bilinear is a *Lorentz scalar* since it remains invariant under Lorentz transformations.

2. **[8/20]** Next we consider the following bilinear:

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \bar{\psi}(x)S^{-1}(\Lambda)\gamma_5 S(\Lambda)\psi(x)$$

Let's simplify the middle bit by recalling the definition of γ_5:

$$\gamma_5 = \frac{i}{4!}\epsilon^{\mu\nu\lambda\sigma}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma$$

We'll need the following identity, which is just rewriting the above expression:

$$i\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma = \epsilon_{\mu\nu\lambda\sigma}\gamma_5$$

since,

$$\epsilon_{\mu\nu\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma} = 4!$$

Furthermore, we know from lecture that:

$$S^{-1}(\Lambda)\gamma_\mu S(\Lambda) = \Lambda^\nu_\mu\gamma_\nu$$

Using this property and strategically "inserting" an identity matrix between certain gamma matrices we see:

$$\begin{aligned}S^{-1}(\Lambda)\gamma_5 S(\Lambda) &= \frac{i}{4!}\epsilon^{\mu\nu\lambda\sigma}S^{-1}(\Lambda)\gamma_\mu S(\Lambda)S^{-1}(\Lambda)\gamma_\nu S(\Lambda)S^{-1}(\Lambda)\gamma_\lambda S(\Lambda)S^{-1}(\Lambda)\gamma_\sigma S(\Lambda) \\ &= \frac{i}{4!}\epsilon^{\mu\nu\lambda\sigma}\Lambda^{\bar{\mu}}_\mu\Lambda^{\bar{\nu}}_\nu\Lambda^{\bar{\lambda}}_\lambda\Lambda^{\bar{\sigma}}_\sigma\gamma_{\bar{\mu}}\gamma_{\bar{\nu}}\gamma_{\bar{\lambda}}\gamma_{\bar{\sigma}} \\ &= \left(\frac{1}{4!}\epsilon_{\bar{\mu}\bar{\nu}\bar{\lambda}\bar{\sigma}}\epsilon^{\mu\nu\lambda\sigma}\Lambda^{\bar{\mu}}_\mu\Lambda^{\bar{\nu}}_\nu\Lambda^{\bar{\lambda}}_\lambda\Lambda^{\bar{\sigma}}_\sigma\right)\gamma_5 \\ &= \det\big(\Lambda\big)\gamma_5\end{aligned}$$

Which implies:

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \det\big(\Lambda\big)\bar{\psi}(x)\gamma_5\psi(x) \tag{5}$$

Since $\det\big(\Lambda\big) = \pm 1$ (check it!), this bilinear is a *Lorentz pseudo-scalar*.

3. **[3/20]** Playing a similar game:

$$\begin{aligned}\bar{\psi}'(x')\gamma_\mu\psi'(x') &= \bar{\psi}(x)S^{-1}(\Lambda)\gamma_\mu S(\Lambda)\psi(x) \\ &= \Lambda^\mu_\nu\bar{\psi}(x)\gamma_\nu\psi(x)\end{aligned} \tag{6}$$

we see this bilinear transforms as a *Lorentz vector*.

4. **[3/20]** Next we have:

$$\begin{aligned}\bar{\psi}'(x')\gamma_5\gamma_\mu\psi'(x') &= \bar{\psi}(x)S^{-1}(\Lambda)\gamma_5 S(\Lambda)S^{-1}(\Lambda)\gamma_\mu S(\Lambda)\psi(x) \\ &= \det\big(\Lambda\big)\Lambda^\nu_\mu\bar{\psi}(x)\gamma_5\gamma_\nu\psi(x)\end{aligned} \tag{7}$$

Thus, this bilinear behaves as a *Lorentz pseudo-vector*.

5. **[3/20]** Finally, we have arrived at our final bilinear:

$$\begin{aligned}\bar{\psi}'(x')\sigma_{\mu\nu}\psi'(x') &= \frac{i}{2}\bar{\psi}(x)S^{-1}(\Lambda)[\gamma_\mu,\gamma_\nu]S(\Lambda)\psi(x) \\ &= \Lambda^\alpha_\mu\Lambda^\beta_\nu\bar{\psi}(x)\sigma_{\alpha\beta}\psi(x)\end{aligned} \tag{8}$$

This bilinear transforms as a *Lorentz tensor*!

3 Chiral Symmetry

Once again we will be looking at our friend the Dirac equation:

$$(i\not{\partial} - m)\psi = 0$$

but this time let's consider the chiral representation of the Dirac γ-matrices:

$$\gamma_0 = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$
$$\vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$$
$$\gamma_5 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Thus, γ_0 and γ_5 look different in this representation, but the remaining γ-matrices are the same.

1. **[2/20]** Let's begin by writing the Dirac equation as a set of two equation in terms of the 2-spinors ϕ and χ. Our Dirac spinor takes the form:

$$\psi = \begin{bmatrix} \phi \\ \chi \end{bmatrix}$$

and a straight forward substitution gives:

$$\begin{aligned} (i\not{\partial} - m)\psi &= 0 \\ (i\gamma^0\partial_0 - i\vec{\gamma}\cdot\vec{\nabla} - m)\psi &= 0 \\ \left(i \begin{bmatrix} 0 & -\partial_0 \\ -\partial_0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \cdot \vec{\nabla} - m \right)\psi &= 0 \\ \begin{bmatrix} -m & -i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} \\ -i\partial_0 + i\vec{\sigma}\cdot\vec{\nabla} & -m \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} &= 0 \end{aligned} \tag{9}$$

2. **[10/20]** Notice these two equations are coupled to one another as a result of the mass term, so *if we set $m = 0$, we decouple the equations.* As we'll see, we get an additional (continuous) symmetry along with the usual $U(1)$ symmetry. Proceeding forward we arrive at a set of two equations:

$$(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\chi = 0$$
$$(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\phi = 0$$

We can solve this set of equations[1] *with plane wave solutions*[2]: $\chi = \chi_0 e^{ia_\mu x^\mu}$ and $\phi = \phi_0 e^{ib_\mu x^\mu}$. Let's solve the equation involving χ first by simply substituting in our above ansatz:

$$i(-a_0 + \vec{\sigma}\cdot\vec{\nabla})\chi = 0$$
$$\begin{bmatrix} -a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_0 - a_3 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = 0$$

This gives us a relationship between the components of our spinors:

$$\chi_1 = \frac{a_1 - ia_2}{a_0 - a_3}\chi_2$$

[1] There are many ways to solve this and you will not be penalized for using a different method as long as it is valid.

[2] I expand the exponents out with arbitrary constants a_μ and b_μ first. We will see what they mean physically in a second.

Our parameters a_μ also posses a relation; we just consider a_o as our eigenvalue. A solution of this matrix equation exists if and only if the determinant of our square matrix is zero; thus:

$$\begin{aligned}(a_0 - a_3)(a_0 + a_3) - |a_1 + ia_2|^2 = a_0^2 - a_3^2 - a_1^2 - a_2^2 = 0 \\ \Rightarrow a_0 = \pm|\vec{a}|\end{aligned} \tag{10}$$

If we now identify a_μ with the four momentum then our condition above is the dispersion relation for the energy of a massless particle, which it should be! Thus, we replace $a_\mu \to p_\mu$ in our general solution:

$$\chi = C_1 \begin{bmatrix} 1 \\ \frac{p_1 - ip_2}{\pm p_o - p_3} \end{bmatrix} e^{ip_\mu x^\mu} \tag{11}$$

C_1 is just a normalization constant.

We can repeat the procedure for ϕ: all that is different is the relationship between the components of the bi-spinor. We again get the same dispersion relation, and identify b_μ with the four momenta, p_μ. The final result is:

$$\phi = C_2 \begin{bmatrix} 1 \\ \frac{p_1 - ip_2}{\pm p_o + p_3} \end{bmatrix} e^{ip_\mu x^\mu} \tag{12}$$

Notice that the two solutions:

$$\psi_- = \begin{bmatrix} 0 \\ \chi \end{bmatrix}, \psi_+ = \begin{bmatrix} \phi \\ 0 \end{bmatrix}$$

are eigenstates of γ_5. *Thus, we say* ϕ has a chirality of $+1$ and χ has a chirality of -1.

3. **[8/20]** As previously stated, there is a new continuous symmetry for our massless theory. We are interested in the action of the following *chiral transformation*:

$$\psi' = e^{i\gamma_5\theta}\psi$$

on various states and bilinears.

(a) **[2/10]** First we look at how the two spinors we constructed in the previous part of this problem transform. This is trivial:

$$\begin{aligned}\psi' = \begin{bmatrix} \phi' \\ \chi' \end{bmatrix} &= e^{i\gamma_5\theta} \begin{bmatrix} \phi \\ \chi \end{bmatrix} \\ &= \begin{bmatrix} e^{i\theta}\phi \\ e^{-i\theta}\chi \end{bmatrix}\end{aligned}$$

That is:

$$\begin{aligned}\phi' &= e^{i\theta}\phi \\ \chi' &= e^{-i\theta}\chi\end{aligned} \tag{13}$$

(b) **[2/10]** Now we consider:

$$\begin{aligned}\bar{\psi}' &= \psi'^\dagger\gamma_0 \\ &= \psi^\dagger e^{-i\gamma_5\theta}\gamma_0 \\ &= \psi^\dagger \Big[I\cos\theta - i\gamma_5\sin\theta\Big]\gamma_0 \\ &= \psi^\dagger\gamma_0 \Big[I\cos\theta + i\gamma_5\sin\theta\Big] \\ &= \bar{\psi}e^{i\gamma_5\theta}\end{aligned} \tag{14}$$

Where I used: $\{\gamma_5, \gamma_\mu\} = 0$, $\gamma_5^\dagger = \gamma_5$, and $\gamma_5^2 = I$.

(c) **[2/10]** As a result of our previous analysis, we see:

$$\begin{aligned}\bar{\psi}'\psi' &= \bar{\psi}e^{i\gamma_5\theta}e^{i\gamma_5\theta}\psi = \bar{\psi}e^{2i\gamma_5\theta}\psi \\ &= \cos(2\theta)\bar{\psi}\psi + \sin(2\theta)i\bar{\psi}\gamma_5\psi\end{aligned} \tag{15}$$

The first bilinear is called the *Dirac mass operator* while the second is called the *pseudo-mass operator*. Let's also look at the action of the chiral transformation on the following bilinear:

$$\begin{aligned}\bar{\psi}'\gamma_\mu\psi' &= \bar{\psi}e^{i\gamma_5\theta}\gamma_\mu e^{i\gamma_5\theta}\psi \\ &= \bar{\psi}e^{i\gamma_5\theta}e^{-i\gamma_5\theta}\gamma_\mu\psi \\ &= \bar{\psi}\gamma_\mu\psi\end{aligned} \tag{16}$$

This means the *4-current (Lorentz vector bilinear) transforms invariantly under the chiral transformation*, but the *Lorentz scalar bilinear is not invariant.* Note the $\bar{\psi}\psi$ is the term which accompanies the mass, m, so it may not be a surprise that it doesn't transform invariantly under our chiral transformation.

(d) **[2/10]** We do not expect the Dirac equation to transform invariantly when there is a mass term due to the previous part of this question. Indeed [3]:

$$\begin{aligned}\bar{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' &= \bar{\psi}e^{i\gamma_5\theta}(i\gamma^\mu\partial_\mu - m)e^{i\gamma_5\theta}\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu - me^{2i\gamma_5\theta})\psi \qquad\qquad *\{\gamma_\mu,\gamma_5\} = 0 \\ \Rightarrow 0 &= (i\gamma^\mu\partial_\mu - me^{2i\gamma_5\theta})\psi \\ &= \Big(i\gamma^\mu\partial_\mu - m\big(I\cos(2\theta) + i\gamma_5\sin(2\theta)\big)\Big)\psi\end{aligned} \tag{17}$$

Note that we get a new pseudo mass scalar in our Dirac equation under the action of a chiral transformation!

[3] I use the Dirac Lagrangian, but you can do this directly with the Dirac equation as well

4 The Landau Theory of Phase Transitions as a Classical Field Theory

In this problem we will be working with the Landau-Ginzburg free energy density:

$$\mathcal{E} = \frac{1}{2}\left(\nabla\phi\right)^2 + U[\phi]$$

where,

$$U[\phi] = \frac{m_o^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6$$

We have: $m_o^2 = a(T - T_0)$ and the $\lambda_6 > 0$ term is required for stability when $\lambda_4 < 0$, which we will be considering shortly.

1. **[2/20]** Let's begin by deriving the saddle point equations for this system. Obviously:

   $$(\nabla\phi)^2 \geq 0 \;\; \forall\phi$$

   so *minimizing the gradient term implies we set ϕ equal to a constant.* That is, we are restricting our analysis to the case of a uniform magnetization, which is fairly intuitive. This makes for a laughably easy derivation of the saddle point equations:

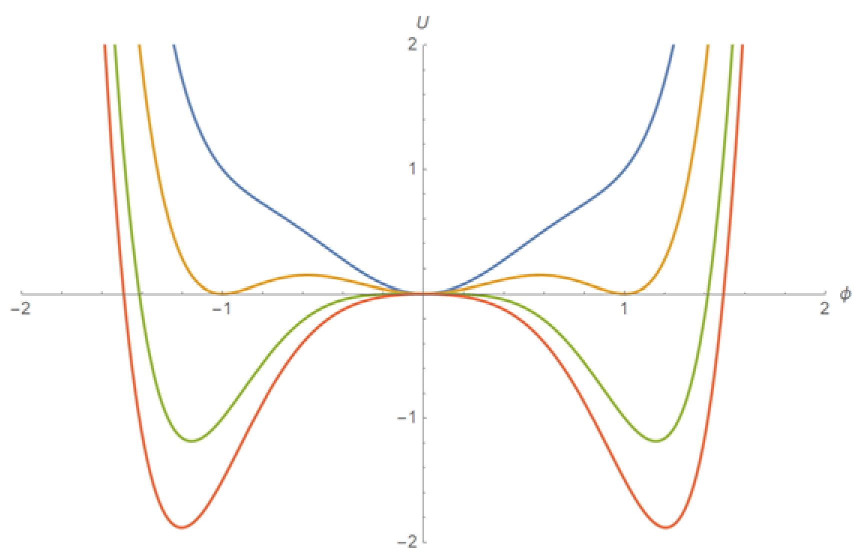
   $$0 = \frac{\delta\mathcal{E}}{\delta\phi}\Big|_{\phi=\bar{\phi}} = m_o^2\bar{\phi} + \frac{\lambda_4}{6}\bar{\phi}^3 + \frac{\lambda_6}{5!}\bar{\phi}^5 \tag{18}$$

   An obvious solution to this equation is $\bar{\phi} = 0$; that is, no magnetization. As we will see, for certain sets of parameters there exists minimums below this value. That is, *there will be a phase transition and so the magnetization is an order parameter.*

2. **[7/20]** Let's suppose $\lambda_4 < 0$, and examine the qualitative behaviour of $U(\phi)$ at various temperatures (see Figure 1).

Potential Energy for $\lambda_4 < 0$

Figure 1: The yellow curve is for $T = T_c > T_0$; notice there are two nonzero possibilities which appeared. Furthermore, these nontrivial magnetization's become more stable as we decrease the temperature.



   We see *non-trivial solutions pop up when we decrease the temperature past a critical value.* Furthermore, the yellow curve in Figure 1 marks the transition point since this is where the nontrivial solutions pop up. We note that we have the two conditions at the transition point:

   i. $U(\bar{\phi}) = 0$

ii. $U'(\bar{\phi}) = 0$

The latter one is of course our saddle point equation.

Let's now search for these nontrivial solutions: $\bar{\phi} \neq 0$. This allows us to divide out extraneous values of $\bar{\phi}$. We solve the following system of equations:

$$\begin{aligned}
0 &= m_o^2 + \frac{\lambda_4}{12}\bar{\phi}^2 + \frac{\lambda_6}{360}\bar{\phi}^4 \\
0 &= m_o^2 + \frac{\lambda_4}{6}\bar{\phi}^2 + \frac{\lambda_6}{5!}\bar{\phi}^4 \\
\Rightarrow 0 &= \lambda_4 + \frac{\lambda_6}{15}\bar{\phi}^2 \\
\Rightarrow \bar{\phi}^2 &= \frac{15|\lambda_4|}{\lambda_6}
\end{aligned}$$

This illustrates a *first order phase transition* since the magnetization jumps from 0 to $\bar{\phi}$ above.

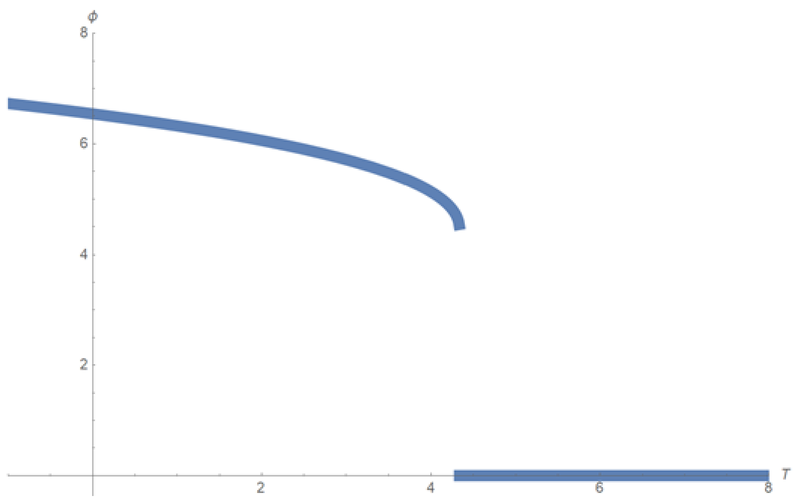
Next we can find the transition temperature by plugging $\bar{\phi}$ into our saddle point equation and solving for m_o^2; we find:

$$T - T_0 = \frac{m_o^2}{a} = \frac{|\lambda_4|^2}{8\lambda_6} \tag{19}$$

Note this tells us that $T_c > T_0$! Figure 2 gives the qualitative behavior of $\bar{\phi}$ as a function of temperature. We just note that the value of the energy in the ordered state is $U\big(\bar{\phi}(T)\big)$ such that $T \leq T_c \Rightarrow \bar{\phi} \neq 0$.

Magnetization for $\lambda_4 < 0$

Figure 2: There is a discontinuous jump at the transition temperature which indicates a first order phase transition. $T_0 = 1$ below, and so we see this jump occurs at a greater temperature.



3. **[7/20]** Now let's consider the case $\lambda_4 > 0$. As we will see, this signifies a continuous phase transition! Indeed, if we try and make the assumption that $\bar{\phi} \neq 0$ at the transition point (same two conditions above), we find:

$$\bar{\phi}^2 = -\frac{15|\lambda_4|}{\lambda_6}$$

This is an imaginary solution, and hence not physical. We conclude that $\bar{\phi} = 0$ at the transition point and we are working with a *continuous phase transition*. We can find the transition temperature in the same way as before (being sure to use the original saddle point equation before dividing by $\bar{\phi}$!); the result is simply:

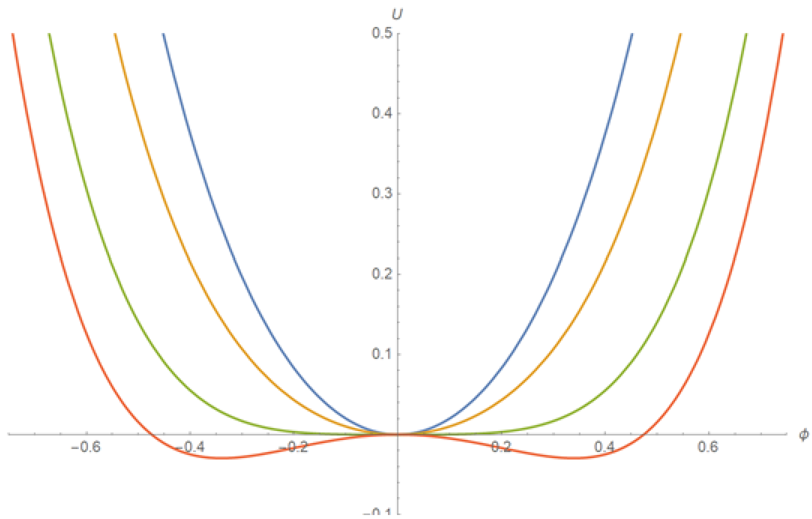
$$T_c - T_0 = \frac{m_o^2}{a} = 0 \tag{20}$$

Thus, the transition temperature occurs at $T_c = T_0$.

Now we look at a plot of the potential energy (see Figure 3). The yellow curve again signifies the transition point, which is at $T = T_0$ this time. Notice this has only one minimum: $\bar{\phi} = 0$. Figure 4 gives the qualitative behavior of $\bar{\phi}$ for $\lambda_4 > 0$ with the same magnitudes for all the parameters as in 4.2. We see it is indeed continuous, and the transition occurs at $T_c = T_0$.

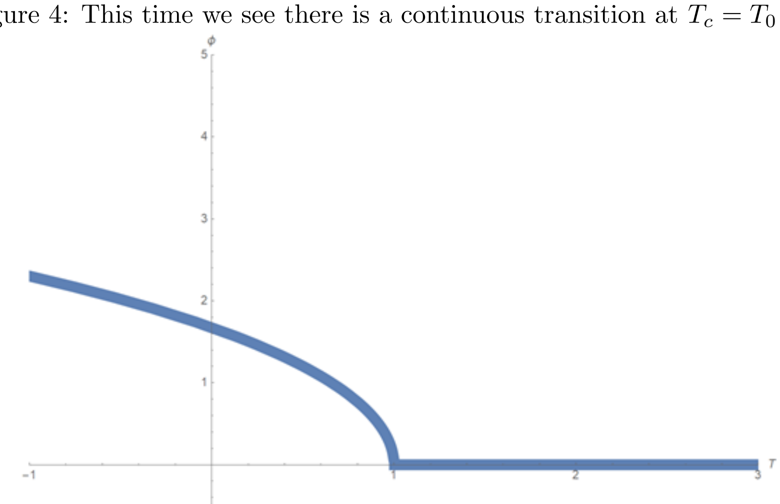
Potential Energy for $\lambda_4 > 0$

Figure 3: The yellow curve again marks the transition point at $T_c = T_0$. Notice there is only one zero here indicating a continuous phase transition.



Magnetization around T_c for $\lambda_4 > 0$

Figure 4: This time we see there is a continuous transition at $T_c = T_0 = 1$.

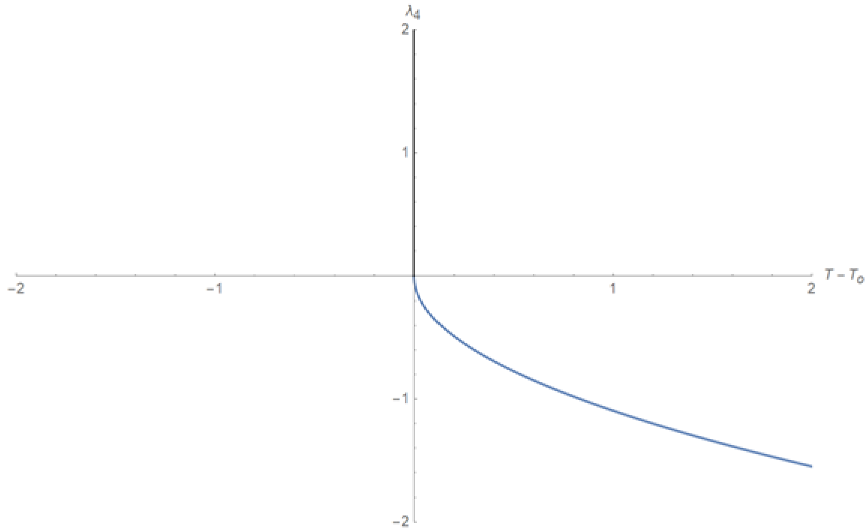


4. **[4/20]** Let's use our previous results to analyze the phase diagram for $\lambda_4(T - T_0)$. *For the upper quadrants*, $\lambda_4 > 0$ we know that *the phase boundary between the ordered and disordered phase is located at* $T - T_0 = 0$. This is just a vertical line!
Now for $\lambda_4 < 0$ we know that *the phase boundary can be found with*:

$$T - T_0 = \frac{|\lambda_4|^2}{8\lambda_6}$$

Inverting this we can find our relation, which is shown in Figure 5 below.

Figure 5: **Phase boundary with λ_4 given as a function of $T - T_0$**



The black line is the phase boundary for our continuous (second order) phase transition while the blue curve is the boundary for the discontinuous (first order) phase transition. The region to the left of the curves is the *ordered phase* while the region to the right represents the *disordered phase* since the temperature will be less than or greater than T_c, respectively. Notice how the critical temperature for the first order phase transition increases with more negative λ_4 as it should! The origin is known as a *tricritical point*: varying λ_4 takes us from a first order phase transition to a second order one!

5 Scalar Electrodynamics

The dynamics of a charged complex scalar field coupled to the electromagnetic field $A_\mu(x)$ is governed by the Lagrangian density:

$$\begin{aligned}\mathcal{L} &= \Big(D_\mu\phi(x)\Big)^*\Big(D^\mu\phi(x)\Big) - m_o^2|\phi(x)|^2 - \frac{\lambda}{2}|\phi(x)|^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\\ &= \Big(D_\mu\phi(x)\Big)^*\Big(D^\mu\phi(x)\Big) - V(\phi(x)) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\end{aligned}$$

where the covariant derivative is defined by:

$$D_\mu = \partial_\mu + ieA_\mu$$

1. **[2/20]** Let's begin by showing the above Lagrangian is invariant under the gauge transformations:

$$\begin{aligned}\phi'(x) &= e^{-ie\Lambda(x)}\phi(x)\\ \phi^{*\prime}(x) &= e^{ie\Lambda(x)}\phi^*(x)\\ A'_\mu(x) &= A_\mu(x) + \Lambda(x)\end{aligned}$$

The potential is obviously invariant under these transforms, so we just check the covariant derivatives and the field tensors. Starting with the covariant derivatives:

$$\begin{aligned}D'_\mu\phi'(x) &= \big(\partial_\mu + ieA'_\mu\big)e^{-ie\Lambda(x)}\phi(x)\\ &= e^{-ie\Lambda(x)}\big(\partial_\mu - ie\partial_\mu\lambda(x) + ieA'_\mu\big)\phi(x)\\ &= e^{-ie\Lambda(x)}\big(\partial_\mu + ieA_\mu\big)\phi(x)\end{aligned}$$

Similarly:

$$\Big(D'_\mu\phi'(x)\Big)^* = e^{ie\Lambda(x)}\big(\partial_\mu - ieA_\mu\big)\phi^*(x) \tag{21}$$

It follows:

$$\Big(D'_\mu\phi'(x)\Big)^*\Big(D'^\mu\phi'(x)\Big) = \Big(D_\mu\phi(x)\Big)^*\Big(D^\mu\phi(x)\Big)$$

The electromagnetic term is straight forward to check as well; indeed, we just check the field tensor itself [4]:

$$\begin{aligned}F'_{\mu\nu}(x) &= \partial_\mu\big(A_\nu(x) + \partial_\mu\Lambda(x)\big) - \partial_\nu\big(A_\mu(x) + \partial_\nu\Lambda(x)\big)\\ &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\\ &= F_{\mu\nu}(x)\end{aligned} \tag{22}$$

The invariance of the Lagrangian follows.

2. **[8/20]** Let's now find our equations of motion. I may have done this in a slightly different manner than yourself, but this method gives some good practice in differentiation of co and contra-variant tensors. Furthermore, I keep the indices on the tensors as general as they can be, but we will neglect any spatial dependence from here on out since this clears up a lot of clutter. We just keep in mind:

$$\frac{\delta A_\mu(x)}{\delta A_\nu(y)} = \delta_{\mu\nu}\delta(x-y)$$

and so on. So, we'll understand there is a Dirac delta function lurking in the background. *Let's begin with*

[4] We are of course assuming $\Lambda(x)$ is second order continuously differentiable.

derivatives in respect to our gauge field:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta A^\mu} &= \frac{\delta}{\delta A^\mu}\Big(\big(D_\sigma\phi\big)^*\big(D^\sigma\phi\big)\Big)\\ &= \left(\frac{\delta\big(D_\sigma\phi\big)^*}{\delta A^\mu}D^\sigma\phi + \big(D_\sigma\phi\big)^*\frac{\delta D^\sigma\phi}{\delta A^\mu}\right)\\ &= \left(g_{\sigma\lambda}\frac{\delta\big(D^\lambda\phi\big)^*}{\delta A^\mu}D^\sigma\phi + \big(D_\sigma\phi\big)^*\frac{\delta D^\sigma\phi}{\delta A^\mu}\right)\\ &= ie\Big(-g_{\sigma\lambda}\delta^{\mu\lambda}\phi^* D^\sigma\phi + \big(D_\sigma\phi\big)^*\phi\,\delta^{\mu\sigma}\Big)\\ &= ie\Big(\big(D_\mu\phi\big)^*\phi - \phi^* D_\mu\phi\Big)\end{aligned}$$

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\partial^\nu A^\mu} &= -\frac{1}{4}\left(\frac{\delta F^{\alpha\beta}}{\delta\partial^\nu A^\mu}F_{\alpha\beta} + F^{\alpha\beta}\frac{\delta F_{\alpha\beta}}{\delta\partial^\nu A^\mu}\right)\\ &= -\frac{1}{4}\left(\frac{\delta F^{\alpha\beta}}{\delta\partial^\nu A^\mu}F_{\alpha\beta} + g_{\alpha\sigma}g_{\beta\gamma}F^{\alpha\beta}\frac{\delta F^{\sigma\gamma}}{\delta\partial^\nu A^\mu}\right)\\ &= -\frac{1}{4}\left(\frac{\delta F^{\alpha\beta}}{\delta\partial^\nu A^\mu}F_{\alpha\beta} + F_{\sigma\gamma}\frac{\delta F^{\sigma\gamma}}{\delta\partial^\nu A^\mu}\right)\end{aligned}$$

Let's evaluate the derivative of the field tensor separately:

$$\begin{aligned}\frac{\delta F^{\alpha\beta}}{\delta\partial^\nu A^\mu} &= \frac{\delta\partial^\alpha A^\beta}{\delta\partial^\nu A^\mu} - \frac{\delta\partial^\beta A^\alpha}{\delta\partial^\nu A^\mu}\\ &= \Big(\delta^{\nu\alpha}\delta^{\mu\beta} - \delta^{\nu\beta}\delta^{\mu\alpha}\Big)\end{aligned}$$

Substituting in we find:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\partial^\nu A^\mu} &= -\frac{1}{4}\Big(\Big(\delta^{\nu\alpha}\delta^{\mu\beta} - \delta^{\nu\beta}\delta^{\mu\alpha}\Big)F_{\alpha\beta} + F_{\sigma\gamma}\Big(\delta^{\nu\sigma}\delta^{\mu\gamma} - \delta^{\nu\gamma}\delta^{\mu\sigma}\Big)\Big)\\ &= -\frac{1}{4}\Big(F_{\nu\mu} - F_{\mu\nu} + F_{\nu\mu} - F_{\mu\nu}\Big)\\ &= F_{\mu\nu}\end{aligned}$$

Going back to the Euler Lagrange equation we see [5]:

$$\partial^\nu\left(\frac{\delta\mathcal{L}}{\delta\partial^\nu A^\mu}\right) = \frac{\delta\mathcal{L}}{\delta A^\mu} \Rightarrow \partial^\nu F_{\nu\mu} = ie\Big(\phi^* D_\mu\phi - \big(D_\mu\phi\big)^*\phi\Big) := j_\mu \tag{23}$$

Now let's work out the derivatives in respect to our complex scalar field:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\phi} &= \big(D_\mu\phi\big)^*\frac{\delta\big(D^\mu\phi\big)}{\delta\phi} - \frac{\delta V(\phi)}{\delta\phi}\\ &= ieA^\mu\big(D_\mu\phi\big)^* - m_o^2\phi^* - \lambda|\phi|^2\phi^*\end{aligned}$$

Similarly:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\partial^\nu\phi} &= \big(D_\mu\phi\big)^*\frac{\delta\big(D^\mu\phi\big)}{\delta\partial^\nu\phi}\\ &= \big(D_\mu\phi\big)^*\delta^{\mu\nu}\end{aligned}$$

[5] I used the anti-symmetry of the electromagnetic field tensor to cast the final equation in its above form. This should be reminiscent of Maxwell's equations in tensor form.

Throwing everything together we find:

$$
\begin{aligned}
\partial^\nu\left(\frac{\delta\mathcal{L}}{\delta\partial^\nu\phi}\right) &= \frac{\delta\mathcal{L}}{\delta\phi} \\
\Rightarrow \partial^\nu\Big(D_\nu\phi\Big)^* &= ieA^\nu\Big(D_\nu\phi\Big)^* - m_o^2\phi^* - \lambda|\phi|^2\phi^* \\
\Rightarrow \Big(D_\nu^* D^{*\nu} + m_o^2\Big)\phi^* &= -\lambda|\phi|^2\phi^*
\end{aligned} \tag{24}
$$

This looks like a Klein-Gordon equation with a source term! The final EOM we get from variations in respect to ϕ^* gives the complex conjugate of the above expression.

3. **[2/20]** We now find the Hamiltonian density for this system. We begin by listing the momentums that are conjugate to our fields:

$$
\begin{aligned}
\Pi &= \frac{\delta\mathcal{L}}{\delta\partial^0\phi} = \Big(D_0\phi\Big)^* \\
\Pi^* &= \frac{\delta\mathcal{L}}{\delta\partial^0\phi^*} = D_0\phi \\
\Pi_\mu &= \frac{\delta\mathcal{L}}{\delta\partial^0 A^\mu} = F_{\mu 0} = \begin{bmatrix} 0 \\ -E_i \end{bmatrix}
\end{aligned} \tag{25}
$$

Applying a Legendre transformation we see:

$$
\begin{aligned}
\mathcal{H} &= \Pi\partial^0\phi + \Pi^*\partial^0\phi^* + \Pi_\mu\partial^0A^\mu - \mathcal{L} \\
&= \Pi\big(\Pi^* - ieA^0\phi\big) + \Pi^*\big(\Pi + ieA^0\phi^*\big) + F_{0\mu}\partial^0A^\mu - \Pi\,\Pi^* + |D_i\phi|^2 + V(\phi) + \frac{1}{2}\big(|\vec{B}|^2 - |\vec{E}|^2\big) \\
&= \Pi\big(\Pi^* - ieA^0\phi\big) + \Pi^*\big(\Pi + ieA^0\phi^*\big) - E_i\partial^0A^i + \Big(E_i\partial^iA^0 - E_i\partial^iA^0\Big) - \Pi\,\Pi^* + |D_i\phi|^2 + V(\phi) + \frac{1}{2}\big(|\vec{B}|^2 - |\vec{E}|^2\big) \\
&= \Pi\,\Pi^* + |D_i\phi|^2 + \frac{1}{2}\big(|\vec{E}|^2 + |\vec{B}|^2\big) + V(\phi) - A^0\Big(\partial^iE_i - ie\big(\Pi^*\phi^* - \Pi\phi\big)\Big)
\end{aligned} \tag{26}
$$

In the last step I integrated by parts on the term $E_i\partial^iA^0$ so that the scalar potential becomes a Lagrange multiplier which enforces our Gauss' Law constraint. Indeed, when we learn about Noether's theorem, we will see that there are conserved charges which accompany our conserved currents j^μ. In particular, the relationship is:

$$
Q = e\int d^3x j^0
$$

Hence, ej^0 is a charge density. Variations of our scalar potential, A^0, leads us to the condition:

$$
\partial^iE_i = ie\big(\Pi^*\phi^* - \Pi\phi\big) \Rightarrow \vec{\nabla}\cdot\vec{E} = ej^0
$$

4. **[4/20]** We can always rewrite a complex function in terms of a magnitude and a phase:

$$
\phi(x) = \rho(x)e^{i\theta(x)}
$$

With this we can easily cast our Lagrangian into it's polar form; we just note:

$$
D_\mu\phi = e^{i\theta}\Big(\partial_\mu\rho + i\rho(\partial_\mu + eA_\mu)\Big)
$$

Performing the requisite substitutions we get:

$$
\mathcal{L} = (\partial_\mu\rho)^2 + \rho^2(\partial_\mu\theta + eA_\mu)^2 - V(\rho) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{27}
$$

From this we derive our EOM for ρ and θ respectively:

$$
\begin{aligned}
0 &= \partial_\mu\partial^\mu\rho - \rho(\partial_\mu\theta + eA_\mu)^2 + m_o^2\rho + \lambda\rho^3 \\
0 &= \partial^\mu\Big(\rho^2\big(\partial_\mu\theta + eA_\mu\big)\Big)
\end{aligned} \tag{28}
$$

The EOM for θ enforces the conservation of the gauge current, and so we can eliminate it with a gauge transform. We choose the *London gauge*: $\theta = 0$. The gauge fixed Lagrangian is then [6]:

$$\mathcal{L} = (\partial_\mu \rho)^2 + \rho^2 e^2 A'_\mu A'^\mu - V(\rho) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{29}$$

I'll drop the primes on the vector potential from here on out. Furthermore, we can see the gauge field has acquired a mass! This is the famous Higgs mechanism.

5. **[4/20]** For this final question we take $m_o^2 < 0$, so then we can see, from our EOM, that we have the following solutions for ρ constant:

$$\rho = 0$$

$$\rho = \bar{\rho} = \pm\sqrt{\frac{|m_o^2|}{\lambda}}$$

Note that I neglected the gauge field, which we need to do if ρ is constant since A_μ is our only remaining degree of freedom. After freezing the ρ-field at $\bar{\rho}$, our new effective Lagrangian is:

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{|m_o^2| e^2}{\lambda} A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= \bar{\rho}^2 e^2 A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \end{aligned} \tag{30}$$

I dropped all constant terms from the above Lagrangian.

Now we can make it more explicit that the coefficient of the term quadratic in A_μ can be interpreted as the photon mass. We can easily find the EOM with this remaining degree of freedom (we already did the hard part in 5.2):

$$2\bar{\rho}^2 e^2 A_\mu + \partial^\nu F_{\nu\mu} = 0 \tag{31}$$

Quickly notice that if we contract both sides of the above equation with ∂^μ, then:

$$\partial^\mu A_\mu = 0$$

This implies:

$$\partial^\nu F_{\nu\mu} = \partial^\nu \partial_\nu A_\mu$$

Thus:

$$\left(\partial^2 + 2\bar{\rho}^2 e^2\right) A_\mu = 0 \tag{32}$$

This is a Klein-Gordon equation for the vector potential, so we interpret $\sqrt{2\bar{\rho}^2 e^2} = \sqrt{\frac{2|m_o^2|}{\lambda}} e$ as the mass of the photon.

[6]Choosing the gauge $\theta = 0$ is equivalent to the gauge transformation $\phi'(x) = e^{-i\theta}\phi(x) \Rightarrow A'_\mu = A_\mu(x) + \frac{i}{e}\partial_\mu \theta$.