Homework 2 Solutions

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Total Points: 100

We consider the scalar electrodynamics field theory given by the Lagrangian

$$\mathcal{L} = |D_\mu \phi|^2 - m_0^2 |\phi|^2 - \frac{1}{2}\lambda \left(|\phi|^2\right)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{0.1}$$

with the covariant derivative defined as

$$D_\mu \doteq \partial_\mu + ieA_\mu. \tag{0.2}$$

Note that this definition differs from the one used in the lecture notes by the sign of the coupling constant.

1. [12 Points] An arbitrary variation of the Lagrangian (with no change in the coordinates) can be written as

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\partial_\mu\phi^* + \frac{\delta\mathcal{L}}{\delta A_\nu}\delta A_\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta\partial_\mu A_\nu. \tag{1.1}$$

We now assume that all fields obey the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} = 0, \tag{1.2a}$$

$$\frac{\delta\mathcal{L}}{\delta\phi^*} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} = 0, \tag{1.2b}$$

$$\frac{\delta\mathcal{L}}{\delta A_\nu} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} = 0, \tag{1.2c}$$

which imply that

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi\right] = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi = \frac{\delta\mathcal{L}}{\delta\phi} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi, \tag{1.3a}$$

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\phi^*\right] = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\partial_\mu\phi^* = \frac{\delta\mathcal{L}}{\delta\phi^*} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\partial_\mu\phi^*, \tag{1.3b}$$

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta A_\nu\right] = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta A_\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta\partial_\mu A_\nu = \frac{\delta\mathcal{L}}{\delta A_\nu} + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta\partial_\mu A_\nu, \tag{1.3c}$$

where we have expanded using the product rule, and then simplified using Eqs. (1.2). Therefore, the variation of the Lagrangian simplifies to

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta A_\nu\right]. \tag{1.4}$$

In the case of the specific global transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{i\theta}\phi(x) \simeq (1 + i\theta)\phi(x) \tag{1.5a}$$

$$\phi^*(x) \longrightarrow {\phi'}^*(x) = e^{-i\theta}\phi^*(x) \simeq (1 - i\theta)\phi^*(x) \tag{1.5b}$$

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x), \tag{1.5c}$$

where $\theta \in \mathbb{R}$ is a constant, we have the field variations

$$\delta\phi = i\theta\phi, \qquad \delta\phi^* = -i\theta\phi^*, \qquad \delta A_\mu = 0. \tag{1.6}$$

Then, we see that the variation (1.4) is

$$\delta\mathcal{L} = \partial_\mu\Big(i\left[(D^\mu\phi)^*\phi - (D^\mu\phi)\phi^*\right]\Big)\theta. \tag{1.7}$$

Since θ is arbitrary, we see that the variation of the Lagrangian can only vanish $\delta\mathcal{L} = 0$, if the quantity

$$j^\mu = i\left[(D^\mu\phi)^*\phi - \phi^* D^\mu\phi\right] = i(\phi\,\partial^\mu\phi^* - \phi^*\,\partial^\mu\phi) + 2eA^\mu|\phi|^2, \tag{1.8}$$

is a locally conserved current. That is, it satisfies the continuity equation $\partial_\mu j^\mu = 0$.

2. [10 Points] Define the quantity

$$\mathcal{Q} = \int \mathrm{d}^3\boldsymbol{x}\, j^0(\boldsymbol{x}, t), \tag{2.1}$$

where the integral is over all of space (not time), and j^0 is the temporal component of the conserved current j^μ (1.8), which we assume vanishes sufficiently rapidly as $|\mathbf{x}| \to \infty$. Since j^μ satisfies the continuity equation

$$\partial_\mu j^\mu(x) = \partial_t j^0(\boldsymbol{x}, t) + \boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x}, t) = 0, \tag{2.2}$$

we find that

$$\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}t} = \int \mathrm{d}^3\boldsymbol{x}\,\partial_t j^0(\boldsymbol{x}, t) = -\int \mathrm{d}^3\boldsymbol{x}\,\boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x}, t). \tag{2.3}$$

Stokes' theorem then tells us that

$$\int \mathrm{d}^3\boldsymbol{x}\,\boldsymbol{\nabla}\cdot\boldsymbol{j}(\boldsymbol{x}, t) = \oint_{\partial\mathbb{R}^3} \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{j}(\boldsymbol{x}, t) = 0, \tag{2.4}$$

where $\partial\mathbb{R}^3$ is the "boundary" of space at $|\boldsymbol{x}| \to \infty$, and the last equality comes from our assumption that j^μ vanishes at ∞. Therefore, $\mathcal{Q}$ is a constant of motion, that is $\mathrm{d}\mathcal{Q}/\mathrm{d}t = 0$. For the scalar ED theory, we have

$$\mathcal{Q} = \int \mathrm{d}^3\boldsymbol{x}\left[i\Big(\phi(\boldsymbol{x}, t)\partial_t\phi^*(\boldsymbol{x}, t) - \phi^*(\boldsymbol{x}, t)\partial_t\phi(\boldsymbol{x}, t)\Big) + 2eA^0(\boldsymbol{x}, t)|\phi(\boldsymbol{x}, t)|^2\right]. \tag{2.5}$$

3. [10 Points] We now consider the local gauge transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{i\theta(x)}\phi(x), \tag{3.1a}$$

$$\phi^*(x) \longrightarrow \phi'^*(x) = e^{-i\theta(x)}\phi^*(x), \tag{3.1b}$$

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x), \tag{3.1c}$$

where $\Lambda(x)$ is a sufficiently smooth function. First, note that $|\phi|^2$ is manifestly gauge invariant. Then, observe that

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu \partial_\nu A'_\mu \\ &= \partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda) \\ &= F_{\mu\nu} + \partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda \\ &= F_{\mu\nu}, \end{aligned} \tag{3.2}$$

so the field strength tensor is also gauge invariant (assuming the function Λ is sufficiently differentiable). Finally, we consider the derivative term. For it to be gauge invariant, we need

$$|D'_\mu \phi'(x)|^2 = |D_\mu \phi(x)|^2 \qquad \Longrightarrow \qquad D'_\mu \phi'(x) = e^{i\theta(x)} D_\mu \phi(x). \tag{3.3}$$

Expanding this out, we find

$$\begin{aligned} D'_\mu \phi' &= (\partial_\mu + ieA'_\mu)\phi' \\ &= (\partial_\mu + ieA_\mu + ie\partial_\mu \Lambda)e^{i\theta}\phi \\ &= e^{i\theta}\left[\partial_\mu + ieA_\mu + i(\partial_\mu\theta + e\partial_\mu\Lambda)\right]\phi \\ &= e^{i\theta} D_\mu \phi + ie^{i\theta}(\partial_\mu\theta + e\partial_\mu\Lambda)\phi, \end{aligned} \tag{3.4}$$

which implies the relation

$$\partial_\mu \Lambda(x) = -\frac{1}{e}\partial_\mu \theta(x). \tag{3.5}$$

4. **[12 Points]** For this local gauge transformation, the infinitesimal variations of the fields are

$$\delta\phi(x) = i\theta(x)\phi(x), \qquad \delta\phi^*(x) = -i\theta(x)\phi^*(x), \qquad \delta A_\mu(x) = -\frac{1}{e}\partial_\mu\theta(x). \tag{4.1}$$

We note that it is easy to check that the variation and partial derivatives commute. That is, for example,

$$\delta\partial_\mu\phi \simeq \partial_\mu\phi' - \partial_\mu\phi = \partial_\mu(\phi' - \phi) \simeq \partial_\mu\delta\phi. \tag{4.2}$$

We then re-write the general variation of the Lagrangian (1.4) as

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\phi^*\right] + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta\partial_\mu A_\nu + \frac{\delta\mathcal{L}}{\delta A_\nu}\delta A_\nu, \tag{4.3}$$

so that specifically for scalar ED,

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu\Big(i\left[(D^\mu\phi)^*\phi - (D^\mu\phi)\phi^*\right]\theta\Big) - \frac{1}{e}F^{\mu\nu}\partial_\mu\partial_\nu\theta + \frac{1}{e}\frac{\delta\mathcal{L}}{\delta A_\nu}\partial_\nu\theta \\ &= \partial_\mu(j^\mu\theta) + \frac{1}{e}F^{\mu\nu}\partial_\mu\partial_\nu\theta - \frac{1}{e}\frac{\delta\mathcal{L}}{\delta A_\nu}\partial_\nu\theta, \end{aligned} \tag{4.4}$$

where j^μ is the locally conserved "matter current" from the previous sections. For a sufficiently smooth function $\theta(x)$, $\partial_\mu\partial_\nu\theta(x)$ is a symmetric tensor, so its contraction with the antisymmetric tensor $F^{\mu\nu}$ must vanish. Also, $\partial_\mu j^\mu = 0$, so

$$\delta\mathcal{L} = (\partial_\mu\theta)\left[j^\mu - \frac{1}{e}\frac{\delta\mathcal{L}}{\delta A_\mu}\right]. \tag{4.5}$$

Since the Lagrangian is invariant under the gauge transformation, this term must also vanish, and since $\theta(x)$ is arbitrary, this implies that

$$\frac{\delta\mathcal{L}}{\delta A_\mu(x)} = ej^\mu(x). \tag{4.6}$$

This relation implies the existence of a gauge current

$$J^\mu(x) \doteq -\frac{\delta \mathcal{L}}{\delta A_\mu(x)}, \tag{4.7}$$

which is necessarily locally conserved, since it is proportional to the locally conserved matter current current j^μ:

$$J^\mu = -ej^\mu = ie\left[\phi^*(D^\mu\phi) - (D^\mu\phi)^*\phi\right] = ie(\phi^*\,\partial^\mu\phi - \phi\,\partial^\mu\phi^*) - 2e^2A^\mu|\phi|^2. \tag{4.8}$$

The choice of minus sign in Eq. (4.7) is a matter of convention, chosen here so that the equations of motion for A_μ exactly match the Maxwell equations.

To understand the difference between j^μ and J^μ, we observe the following:

- Noether's theorem connects global symmetries and local conservation laws. In particular, the **global** $U(1)$ symmetry of the Lagrangian (0.1) is a sufficient condition for the existence of the locally conserved matter current j^μ; conservation of j^μ does not require gauge invariance, and there is still a conserved current in the absence of the gauge field A_μ.
- The Lagrangian (0.1) can be written schematically as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu J^\mu + \text{terms involving } \phi. \tag{4.9}$$

  (Note that this is not strictly correct for the scalar field theory, since J^μ depends explicitly on A^μ, unlike in the Dirac theory). As discussed in Sec. 2.6 of the lecture notes/textbook, conservation of the current J^μ is a *requirement* of the **local** $U(1)$ gauge invariance of electromagnetism. That is, attempting to couple a gauge field to matter in any other way would violate gauge invariance, which is a core *desideratum* of the theory. Therefore, the current J^μ does not exist without a notion of local $U(1)$ symmetry.

By inspection of (4.8), we immediately identify that the constant of motion associated with J^μ is simply

$$\mathcal{Q}_{\text{gauge}} = -e\mathcal{Q} = -e\int \mathrm{d}^3\boldsymbol{x}\, j^0. \tag{4.10}$$

where $\mathcal{Q}$ is given by (2.5). Since J^μ is the gauge/electrical current [from the definition Eq. (4.7)], and $\mathcal{Q}_{\text{gauge}}$ is a constant of motion proportional to the coupling constant e, we can identify $\mathcal{Q}_{\text{gauge}}$ as the total electrical charge of the system. Therefore, $U(1)$ symmetry enforces conservation of electrical charge.

5. **[12 Points]** The energy-momentum tensor is defined by a suitable generalisation of Eq. (3.161) of the lecture notes/textbook

$$\tilde{T}^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\nu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\partial^\nu\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\lambda}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L} \tag{5.1a}$$

$$= (D^\mu\phi)^*\partial^\nu\phi + (D^\mu\phi)\partial^\nu\phi^* - F^{\mu\lambda}\partial^\nu A_\lambda - g^{\mu\nu}\mathcal{L}. \tag{5.1b}$$

We can add any total divergence to this expression without altering its physical content:

$$T^{\mu\nu} \doteq \tilde{T}^{\mu\nu} + \partial_\lambda(F^{\mu\lambda}A^\nu) = \tilde{T}^{\mu\nu} + A^\nu\partial_\lambda F^{\mu\lambda} + F^{\mu\lambda}\partial_\lambda A^\nu. \tag{5.2}$$

Then, consider the Euler-Lagrange equations

$$-J^\mu = \frac{\delta \mathcal{L}}{\delta A_\mu(x)} = \partial_\nu \frac{\delta \mathcal{L}}{\delta \partial_\nu A_\mu} = \partial_\nu F^{\mu\nu}, \tag{5.3}$$

where J^μ is the gauge current from the previous part. We can then use this to re-write the energy-momentum tensor obtained from Eq. (5.2) in a manifestly gauge invariant and symmetric form:

$$\begin{aligned}
T^{\mu\nu} &= (D^\mu\phi)^*\partial^\nu\phi + (D^\mu\phi)\partial^\nu\phi^* - F^{\mu\lambda}(\partial^\nu A_\lambda - \partial_\lambda A^\nu) - A^\nu J^\mu - g^{\mu\nu}\mathcal{L} \\
&= (D^\mu\phi)^*\partial^\nu\phi + (D^\mu\phi)\partial^\nu\phi^* - F^{\mu\lambda}F^\nu{}_\lambda + ieA^\nu[(D^\mu\phi)^*\phi - \phi^* D^\mu\phi] - g^{\mu\nu}\mathcal{L} \\
&= (D^\mu\phi)^*(\partial^\nu\phi + ieA^\nu\phi) + (D^\mu\phi)(\partial^\nu\phi^* - ieA^\nu\phi^*) - F^{\mu\lambda}F^\nu{}_\lambda - g^{\mu\nu}\mathcal{L} \\
&= (D^\mu\phi)^*(D^\nu\phi) + (D^\mu\phi)(D^\nu\phi)^* - g^{\mu\nu}\left[|D_\sigma\phi|^2 - m_0^2|\phi|^2 - \frac{1}{2}\lambda\left(|\phi|^2\right)^2\right] \\
&\qquad + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\sigma}F^\nu{}_\sigma\,.
\end{aligned} \tag{5.4}$$

On the last line, we identify the contribution to the energy-momentum from the electromagnetic field

$$\begin{aligned}
T^{\mu\nu}(A) &= \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\lambda}\partial^\nu A_\lambda + F^{\mu\lambda}\partial_\lambda A^\nu - g^{\mu\nu}\mathcal{L}_{\text{Maxwell}}(A) \\
&= \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\sigma}F^\nu{}_\sigma\,.
\end{aligned} \tag{5.5}$$

Then, observe that the energy-momentum tensor for the decoupled scalar field is

$$\begin{aligned}
T^{\mu\nu}(\phi) &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\nu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\partial^\nu\phi^* - g^{\mu\nu}\mathcal{L}(\phi,\phi^*) \\
&= (\partial^\mu\phi^*)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial^\nu\phi^*) - g^{\mu\nu}\left[|\partial_\sigma\phi|^2 - m_0^2|\phi|^2 - \frac{1}{2}\lambda\left(|\phi|^2\right)^2\right].
\end{aligned} \tag{5.6}$$

This expression can be made gauge invariant by replacing all the derivatives by covariant derivatives. That is, modifying it by the minimal coupling procedure:

$$T^{\mu\nu}(\phi, A) = (D^\mu\phi)^*(D^\nu\phi) + (D^\mu\phi)(D^\nu\phi)^* - g^{\mu\nu}\left[|D_\sigma\phi|^2 - m_0^2|\phi|^2 - \frac{1}{2}\lambda\left(|\phi|^2\right)^2\right], \tag{5.7}$$

which is simply the scalar field contribution on the first line of Eq. (5.4). Therefore, we have shown that the energy-momentum tensor of scalar electrodynamics has the form

$$T^{\mu\nu} = T^{\mu\nu}(A) + T^{\mu\nu}(\phi, A). \tag{5.8}$$

6. **[10 Points]** The Hamiltonian density is simply the temporal component of the energy-momentum tensor (5.4) $\mathcal{H} = T^{00}$, written as a function of the canonical momenta:

$$\Pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\phi(x)} = (D^0\phi(x))^*, \qquad \Pi^*(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\phi^*(x)} = D^0\phi(x). \tag{6.1}$$

We also note that $F^{\alpha\beta}F_{\alpha\beta} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$, $F_{0i} = -F^{0i} = E^i$, $A_\mu = (A^0, -\boldsymbol{A})$, and $\partial^\mu = (\partial_0, -\boldsymbol{\nabla})$. Therefore,

$$\begin{aligned}
\mathcal{H} = |\Pi|^2 + |\boldsymbol{\nabla}\phi|^2 + m_0^2|\phi|^2 + \frac{1}{2}\lambda\left(|\phi|^2\right)^2 + \frac{1}{2}\Big(\boldsymbol{E}^2 + \boldsymbol{B}^2\Big) \\
+ ie\boldsymbol{A}\cdot\Big(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*\Big) + (e\boldsymbol{A})^2|\phi|^2.
\end{aligned} \tag{6.2}$$

The different contributions are as follows:

- $|\Pi|^2$ is the canonical momentum squared, and so is related to the kinetic energy of the scalar field.
- $|\boldsymbol{\nabla}\phi|^2$ is the "tension" stored in the scalar field from "stretching" it, similarly to a spring.
- $U(\phi) = m_0^2|\phi|^2 + (\lambda/2)(|\phi|^2)^2$ is the potential energy of the scalar field.
- $(\boldsymbol{E}^2 + \boldsymbol{B}^2)/2$ is the usual energy density of the electromagnetic field.
- The terms on the second line represent the energy associated with the interaction between charges and electromagnetic fields. Apart from a factor of 2 in front of the term quadratic in $\boldsymbol{A}$, these have the form $\boldsymbol{J}\cdot\boldsymbol{A}$. This is not surprising, since classically, we have that the energy of a current density $\boldsymbol{j}_{\text{elec}}$ passing through a *static* magnetic field $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, is

$$U = \frac{1}{2}\int \mathrm{d}V\, \boldsymbol{j}_{\text{elec}}\cdot\boldsymbol{A}. \tag{6.3}$$

- This expression for the Hamiltonian may differ from your answer in Homework 1 by the terms

$$\boldsymbol{E}\cdot\boldsymbol{\nabla}A_0 + ieA_0(\phi^*\Pi^* - \phi\Pi). \tag{6.4}$$

  However, we should understand that the Hamiltonian is an energy *density*, and appears underneath an integral over all space. Therefore, integrating the first term by parts yields

$$-A_0\big[\boldsymbol{\nabla}\cdot\boldsymbol{E} - ie(\phi^*\Pi^* - \phi\Pi)\big], \tag{6.5}$$

  which must vanish, since we recognise the term in brackets as the conservation law for the temporal component of the gauge current $\partial_\mu F^{0\mu} = -J^0$, that is, Gauss' law. The boundary term in the integration by parts is exactly the contribution which we cancelled when we added a total divergence to the energy-momentum tensor in the previous part. Incidentally, this implies that the component A_0 of the gauge field can be interpreted as a Lagrange multiplier which enforces Gauss' law as a constraint.

Similarly, the linear momentum density $\mathcal{P}^i = T^{i0}$ can be read off from Eq. (5.4). Using the fact that $F^{i\sigma}F^0{}_\sigma = [\boldsymbol{B}\times\boldsymbol{E}]_i$, we find that

$$\boldsymbol{\mathcal{P}} = \Pi(-\boldsymbol{\nabla} + ie\boldsymbol{A})\phi + \Pi^*[(-\boldsymbol{\nabla} + ie\boldsymbol{A})\phi]^* + \boldsymbol{E}\times\boldsymbol{B} \tag{6.6}$$

where the contributions are as follows:

- The first two terms are the linear momentum density of the complex scalar field (and its complex conjugate). One can see this by setting $A_\mu = 0$ and $\phi \propto \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$, in which case $\boldsymbol{\mathcal{P}} \propto \boldsymbol{k}$.
- $\boldsymbol{E}\times\boldsymbol{B}$ is the usual Poynting vector; the momentum density of the EM field.

7. [**12 Points**] Accounting for a variation of the coordinate system δx^μ leads to a modification of the variation of the Lagrangian from part 1:

$$\delta\mathcal{L} = \partial_\mu\left[\mathcal{L}\delta x^\mu + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta A_\nu\right]. \tag{7.1}$$

If we consider a Lorentz transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$, where $\omega_{\mu\nu}$ is infinitesimal and antisymmetric, then $\delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu}$ and the Jacobian determinant is equal to 1. Therefore, all variation in the Action comes from the variation of the Lagrangian.

Now, a scalar field is, by definition, invariant under Lorentz transformations. Therefore, the total variation is

$$\begin{aligned} 0 = \delta_T \phi &\doteq \delta\phi + \delta x^{\mu}\partial_{\mu}\phi = \delta\phi + \omega^{\mu\nu}x_{\nu}\partial_{\mu}\phi, \\ \implies \delta\phi &= -\omega^{\mu\nu}x_{\nu}\partial_{\mu}\phi, \end{aligned} \tag{7.2}$$

where $\delta\phi$ is the variation of the field in the absence of coordinate transformations. On the other hand, the vector potential A_{μ} transforms like a vector under Lorentz transformations:

$$\begin{aligned} A'^{\mu}(x') &= \Lambda^{\mu}{}_{\nu}A^{\nu}(x), \\ \implies \omega^{\mu}{}_{\nu}A^{\nu} &= \delta_T A^{\mu} \doteq \delta A^{\mu} + \delta x^{\lambda}\partial_{\lambda}A^{\mu}, \\ \implies \delta A_{\mu} &= \omega_{\mu}{}^{\nu}A_{\nu} - \omega^{\lambda\sigma}x_{\sigma}\partial_{\lambda}A_{\mu}. \end{aligned} \tag{7.3}$$

Therefore, the variation of the Lagrangian becomes

$$\begin{aligned} \delta\mathcal{L} &= \partial_{\mu}\left[\mathcal{L}\omega^{\mu\sigma}x_{\sigma} - \Big((D^{\mu}\phi)^*\partial_{\lambda}\phi + (D^{\mu}\phi)\partial_{\lambda}\phi^*\Big)\omega^{\lambda\sigma}x_{\sigma} - F^{\mu\nu}\Big(\omega_{\nu\sigma}A^{\sigma} - \omega^{\lambda\sigma}x_{\sigma}\partial_{\lambda}A_{\nu}\Big)\right] \\ &= \partial_{\mu}\left[\Big(\mathcal{L}g^{\mu}{}_{\lambda} - (D^{\mu}\phi)^*\partial_{\lambda}\phi - (D^{\mu}\phi)\partial_{\lambda}\phi^* + F^{\mu\nu}\partial_{\lambda}A_{\nu}\Big)x_{\sigma} - F^{\mu\nu}g_{\nu\lambda}A_{\sigma}\right]\omega^{\lambda\sigma} \\ &= \partial_{\mu}\left[\Big(\mathcal{L}g^{\mu\lambda} - (D^{\mu}\phi)^*\partial^{\lambda}\phi - (D^{\mu}\phi)\partial^{\lambda}\phi^* + F^{\mu\nu}\partial^{\lambda}A_{\nu}\Big)x^{\sigma} - F^{\mu\nu}g_{\nu}{}^{\lambda}A^{\sigma}\right]\omega_{\lambda\sigma} \\ &= -\partial_{\mu}\Big(\tilde{T}^{\mu\lambda}x^{\sigma} + F^{\mu\lambda}A^{\sigma}\Big)\omega_{\lambda\sigma}, \end{aligned} \tag{7.4}$$

where $\tilde{T}^{\mu\nu}$ is the non-symmetrised energy-momentum tensor. Since $\omega_{\mu\nu}$ is arbitrary, for the variation to be consistent with the Lorentz invariance of the Lagrangian, we must have

$$\partial_{\mu}\Big(\tilde{T}^{\mu\nu}x^{\sigma} + F^{\mu\lambda}A^{\sigma}\Big) = 0, \tag{7.5}$$

and hence, the terms in the parentheses define a conserved tensor. Then, using the fact that $\omega_{\mu\nu}$ is an antisymmetric tensor, we can re-write this expression as

$$\Big(\tilde{T}^{\mu\nu}x^{\sigma} + F^{\mu\lambda}A^{\sigma}\Big)\omega_{\lambda\sigma} \doteq \frac{1}{2}M^{\mu\lambda\sigma}\omega_{\lambda\sigma}, \tag{7.6}$$

where

$$M^{\mu\nu\lambda} = \Big(\tilde{T}^{\mu\nu}x^{\lambda} - \tilde{T}^{\mu\lambda}x^{\nu}\Big) + \Big(F^{\mu\nu}A^{\lambda} - F^{\mu\lambda}A^{\nu}\Big), \tag{7.7}$$

is the locally conserved tensor ($\partial_{\mu}M^{\mu\nu\lambda} = 0$).

We can define a pseudo-vector from the spatial components of the above tensor:

$$\mathcal{J}_i = -\frac{1}{2}\varepsilon_{ijk}M^{0jk} = -\varepsilon_{ijk}\left(\tilde{T}^{0j}x^k + F^{0j}A^k\right). \tag{7.8}$$

Since $\tilde{T}^{\mu\nu}$ is the non-symmetrised tensor, $\tilde{T}^{0j}$ is not the same linear momentum density defined above. Instead,

$$\begin{aligned} \tilde{T}^{0j} &= (D^0\phi)^*\partial^j\phi + (D^0\phi)\partial^j\phi^* - F^{0\lambda}\partial^j A_{\lambda} \\ &\doteq \mathcal{P}^j_{\phi} - \boldsymbol{E}\cdot(\partial^j\boldsymbol{A}), \end{aligned} \tag{7.9}$$

Substituting this into Eq. (7.8) then yields

$$\boldsymbol{\mathcal{J}} = \boldsymbol{x} \times \boldsymbol{\mathcal{P}}_\phi + \boldsymbol{E} \cdot (\boldsymbol{x} \times \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{E} \times \boldsymbol{A}. \tag{7.10}$$

Since the first two terms originated from $\tilde{T}^{0j}$, we can identify them as the orbital angular momentum of the scalar and electromagnetic fields, respectively. By examining Eq. (7.4), we see that the third term only arose due to the transformation property of the vector field A^μ under Lorentz transformations. Therefore, we can identify $\boldsymbol{E} \times \boldsymbol{A}$ with the intrinsic or spin angular momentum of the electromagnetic field.

Finally, if we define

$$M^{\mu\nu\lambda} \doteq \left(\tilde{T}^{\mu\nu}x^\lambda - \tilde{T}^{\mu\lambda}x^\nu\right) + S^{\mu\nu\lambda}, \tag{7.11}$$

where $S^{\mu\nu\lambda}$ represents the intrinsic AM contributions, then the conservation laws

$$\partial_\mu M^{\mu\nu\lambda} = 0, \qquad \partial_\mu \tilde{T}^{\mu\nu} = 0, \tag{7.12}$$

imply that

$$\partial_\mu S^{\mu\nu\lambda} = \partial_\mu\left(\tilde{T}^{\mu\nu}x^\lambda - \tilde{T}^{\mu\lambda}x^\nu\right) = \tilde{T}^{\lambda\nu} - \tilde{T}^{\nu\lambda}. \tag{7.13}$$

Therefore, the asymmetry of the canonical tensor is directly related to the intrinsic angular momentum. This is the only explicit restriction on the components of $\tilde{T}^{\mu\nu}$ from the conservation of $M^{\mu\nu\lambda}$. However, if we re-define $M^{\mu\nu\lambda}$ by adding a total divergence

$$\begin{aligned} M^{\mu\nu\lambda} &\longrightarrow M^{\mu\nu\lambda} + \partial_\alpha\left(F^{\mu\alpha}A^\nu x^\lambda - F^{\mu\alpha}A^\lambda x^\nu\right) \\ &= T^{\mu\nu}x^\lambda - T^{\mu\lambda}x^\nu, \end{aligned} \tag{7.14}$$

where $T^{\mu\nu}$ is the "improved" energy-momentum tensor we worked with in part 5, then the conservation laws instead imply that

$$T^{\lambda\nu} - T^{\nu\lambda} = 0, \tag{7.15}$$

as expected, since the expression for $T^{\mu\nu}$ found in part 5 was manifestly symmetric. Note that if we had used this modified tensor to define $\mathcal{J}_i$, then the total angular momentum density would instead be

$$\boldsymbol{\mathcal{J}} \longrightarrow \boldsymbol{x} \times \boldsymbol{\mathcal{P}}, \tag{7.16}$$

where $\boldsymbol{\mathcal{P}}$ is the linear momentum density calculated in part 6. In this form, we cannot identify the orbital and intrinsic contributions intuitively.

8. [**10 Points**] We now define the complex scalar field in the polar representation $\phi(x) = \rho(x)e^{i\omega(x)}$. In this case, the Lagrangian in terms of $\rho(x)$ and $\omega(x)$ is

$$\mathcal{L} = (\partial_\mu\rho)^2 + \rho^2(\partial_\mu\omega + eA_\mu)^2 - m_0^2\rho^2 - \frac{1}{2}\lambda\rho^4 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{8.1}$$

When $m_0^2 < 0$, minimising the potential energy

$$U(\rho) = -|m_0^2|\rho^2 + \frac{1}{2}\lambda\rho^4, \tag{8.2}$$

gives $\rho_0 = \sqrt{|m_0^2|/\lambda}$. From hereon we work in the unitary gauge where $\omega(x) = 0$. Then, freezing $\rho(x) = \rho_0$, the covariant derivative becomes

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\rho_0 = ie\rho_0 A_\mu, \tag{8.3}$$

since $\partial_\mu \phi = 0$. Therefore, we find the following:

(a) The gauge current (4.8) simplifies to

$$J^\mu = -2e^2\rho_0^2 A^\mu. \tag{8.4}$$

(b) Noting that the canonical momentum $\Pi(x) = (D^0\phi(x))^*$, the total energy follows from (6.2)

$$H = \int \mathrm{d}^3\boldsymbol{x}\, \mathcal{H}(x) = \int \mathrm{d}^3\boldsymbol{x} \left[e^2\rho_0^2(A_0^2 + \boldsymbol{A}^2) + \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) \right], \tag{8.5}$$

and we have dropped the constant term $-|m_0^2|^2/2\lambda$ in the Hamiltonian density, since it gives a formally infinite contribution to the total energy.

(c) Similarly, the total linear momentum follows from (6.6)

$$P^j = \int \mathrm{d}^3\boldsymbol{x}\, \mathcal{P}^j(x) = \int \mathrm{d}^3\boldsymbol{x} \Big(2e^2\rho_0^2 A^0 A^j + [\boldsymbol{E}\times\boldsymbol{B}]_j \Big). \tag{8.6}$$

9. [12 Points] We now consider the analytic continuation of the scalar EM theory in $D = d+1$ spacetime dimensions to D Euclidean dimensions. First, we define the imaginary time coordinate τ:

$$t = -i\tau, \qquad \Longrightarrow \qquad \partial_t = i\partial_\tau. \tag{9.1}$$

Then, we need to make sure that the Euclidean theory will still be gauge invariant after the analytic continuation. The gauge transformation

$$A^\mu \longrightarrow A'^\mu = A^\mu - \frac{1}{e}\partial^\mu\theta(x) \qquad \Longrightarrow \qquad \begin{cases} A^0 \longrightarrow A'^0 = A^0 - \dfrac{1}{e}\partial_t\theta(x), \\ \boldsymbol{A}' \longrightarrow \boldsymbol{A}' = \boldsymbol{A} + \dfrac{1}{e}\boldsymbol{\nabla}\theta(x), \end{cases} \tag{9.2}$$

implies that the Wick rotated vector potential $\tilde{A}$ should have an imaginary time component which transforms like the spatial components:

$$\tilde{A}_\tau \longrightarrow \tilde{A}'_\tau = \tilde{A}_\tau + \frac{1}{e}\partial_\tau\theta(x), \tag{9.3}$$

and hence,

$$A^0 = -i\tilde{A}_\tau. \tag{9.4}$$

From hereon, we drop the tilde on the Wick rotated vector potential. Thus, the covariant derivative is

$$D^0 = \partial_t + ieA^0 = i\partial_\tau + eA_\tau = i(\partial_\tau - ieA_\tau) \doteq iD_\tau. \tag{9.5}$$

Therefore,

$$\begin{aligned}(D_\mu\phi)^*(D^\mu\phi) &= -[(D_\tau\phi)^*(D_\tau\phi) + (D^j\phi)^*(D^j\phi)] \\ &= -\big[(\partial_\tau\phi^*)(\partial_\tau\phi) + ieA_\tau(\phi^*\,\partial_\tau\phi - \phi\,\partial_\tau\phi^*) + e^2A_\tau^2|\phi|^2 \\ &\quad + (\boldsymbol{\nabla}\phi^*)\cdot(\boldsymbol{\nabla}\phi) + ie\boldsymbol{A}\cdot(\phi^*\,\boldsymbol{\nabla}\phi - \phi\,\boldsymbol{\nabla}\phi^*) + (e\boldsymbol{A})^2|\phi|^2\big]\end{aligned} \tag{9.6}$$

We then consider the Maxwell terms. First,

$$\begin{aligned} F^{0j}F_{0j} &= g_{jk}(\partial^0 A^j - \partial^j A^0)(\partial^0 A^k - \partial^k A^0) \\ &= -\delta_{jk}(\partial_t A^j + \partial_j A^0)(\partial_t A^k + \partial_k A^0) \\ &= -\delta_{jk}(i\partial_\tau A^j - i\partial_j A_\tau)(i\partial_\tau A^k - i\partial_k A_\tau) \\ &= \delta_{jk}(\partial_\tau A^j - \partial_j A_\tau)(\partial_\tau A^k - \partial_k A_\tau) \\ &\doteq F_{\tau j}F_{\tau j}, \end{aligned} \tag{9.7}$$

with implied summation over all repeated indices regardless of position, and where we have used the fact that the Minkowski metric is diag$(+,-,-,-)$ to make the replacement $g^{jk} = -\delta_{jk}$. Similarly,

$$\begin{aligned} F^{ij}F_{ij} &= g^{ik}g^{j\ell}F_{ij}F_{k\ell} \\ &= (-\delta_{ik})(-\delta_{j\ell})F_{ij}F_{k\ell} \\ &= F_{ij}F_{ij}. \end{aligned} \tag{9.8}$$

Therefore, since $F^{\mu\nu}$ is antisymmetric,

$$F^2 = F^{\mu\nu}F_{\mu\nu} = 2F_{\tau j}F_{\tau j} + F_{ij}F_{ij}, \tag{9.9}$$

is positive definite, as required in a Euclidean theory.

Finally, the Euclidean energy functional E is defined so that

$$iS(\phi, A) = i\int \mathrm{d}t\mathrm{d}^d\boldsymbol{x}\,\mathcal{L}(\phi, A) = -\int \mathrm{d}^D x\,\mathcal{E}(\phi, A) = -E(\phi, A), \tag{9.10}$$

where $\mathrm{d}^D x = \mathrm{d}\tau\mathrm{d}^d\boldsymbol{x}$ and $\mathcal{E}$ is the energy density. Therefore, putting all of the above together, we find

$$\begin{aligned} E(\phi, A) = \int \mathrm{d}^D x \Big[&|\partial_\tau\phi|^2 + |\boldsymbol{\nabla}\phi|^2 + m_0^2|\phi|^2 + \frac{1}{2}\lambda\left(|\phi|^2\right)^2 + \frac{1}{4}F^2 \\ &+ ieA_\tau(\phi^*\,\partial_\tau\phi - \phi\,\partial_\tau\phi^*) + ie\boldsymbol{A}\cdot(\phi^*\,\boldsymbol{\nabla}\phi - \phi\,\boldsymbol{\nabla}\phi^*) + e^2(A_\tau^2 + \boldsymbol{A}^2)|\phi|^2 \Big]. \end{aligned} \tag{9.11}$$

We can also write this in a simpler way by using the D dimensional Euclidean (positive) metric signature:

$$\mathcal{E}(\phi, A) = \int \mathrm{d}^D x\left[|\partial\phi - ieA\phi|^2 + m_0^2|\phi|^2 + \frac{1}{2}\lambda\left(|\phi|^2\right)^2 + \frac{1}{4}F^2\right], \tag{9.12}$$

where $\partial = (\partial_\tau, \boldsymbol{\nabla})$ and $A = (A_\tau, \boldsymbol{A})$, which represents a classical theory in D spatial dimensions.

In three dimensions the 2-form F is (Hodge) dual to a pseudo-vector

$$F_{ij} = \varepsilon_{ijk}B_k, \tag{9.13}$$

which we can identify with the magnetic field. In this case, Eq. (9.12) is exactly the Ginzburg-Landau (free) energy of a superconductor. Therefore, we can understand the physical meaning of each term by analogy: Consider a superconductor in an applied (external) magnetic field, then

- The first term has two contributions:
  - $|\partial\phi|^2$ is the energy cost of gradients in the superconducting condensate.
  - The gauge field terms in the covariant derivative represent the gauge-covariant coupling of the superconducting condensate to the applied magnetic field. In particular,

$$\frac{\delta \mathcal{E}}{\delta A(x)} = ie(\phi^* \, \partial\phi - \phi \, \partial\phi^*) + 2e^2 A|\phi|^2, \tag{9.14}$$

    is the density of supercurrent, which couples to the magnetic field.
- $U(\phi) = m_0^2|\phi|^2 + \lambda\left(|\phi|^2\right)^2/2$ is the potential energy, where $m_0^2 \propto (T - T_c)$ determines the critical temperature, and minimising the potential leads to a finite superconducting condensate density $|\phi|^2$ when $T < T_c$.
- $F^2/4 = \boldsymbol{B}^2/2$ is the energy density of the applied magnetic field.