Physics 582, Fall Semester 2021
Professor Eduardo Fradkin

Problem Set No. 2:
Symmetries and Conservation Laws
Due Date: Friday September 24, 2021,
9:00 pm US Central Time

Here you will look again at problem 5 of Problem Set 1 in which you studied some of the properties of the dynamics of a *charged* (complex) scalar field $\phi(x)$ coupled to the electromagnetic field $A_\mu(x)$. Recall that the Lagrangian density $\mathcal{L}$ for this system is

$$\mathcal{L} = \left(D_\mu \phi(x)\right)^* \left(D^\mu \phi(x)\right) - m_0^2 |\phi(x)|^2 - \frac{\lambda}{2}\left(|\phi(x)|^2\right)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{1}$$

where D_μ is the *covariant derivative*

$$D_\mu \equiv \partial_\mu + ieA_\mu \tag{2}$$

e is the electric charge and * denotes complex conjugation. In this problem set you will determine several important properties of this field theory at the classical level.

1. Derive an expression for the *locally conserved current* $j_\mu(x)$, associated with the *global* symmetry

$$\begin{aligned} \phi(x) \to \phi'(x) =& e^{i\theta}\phi(x) \\ \phi^*(x) \to \phi'^*(x) =& e^{-i\theta}\phi^*(x) \\ A_\mu(x) \to A'_\mu(x) =& A_\mu(x) \end{aligned} \tag{3}$$

   in terms of the fields of the theory.

2. Show that the conservation of the current j_μ implies the existence of a *constant of motion*. Find an explicit form for this constant of motion.

3. Consider now the case of the *local* (*gauge*) transformation

$$\begin{aligned} \phi(x) \to \phi'(x) =& e^{i\theta(x)}\phi(x) \\ \phi^*(x) \to \phi'^*(x) =& e^{-i\theta(x)}\phi^*(x) \\ A_\mu(x) \to A'_\mu(x) =& A_\mu(x) + \partial_\mu \Lambda(x) \end{aligned} \tag{4}$$

   where $\theta(x)$ and $\Lambda(x)$ are two functions. What should be the relation between $\theta(x)$ and $\Lambda(x)$ for this transformation to be a symmetry of the Lagrangian of the system?

4. Show that, if the system has the local symmetry of the previous section, there is a locally conserved *gauge current* $J_\mu(x)$. Find an explicit expression for J_μ and discuss in which way it is different from the current j_μ of item 1). Find an explicit expression for the associated constant of motion and discuss its physical meaning.

5. Find the Energy-Momentum $T^{\mu\nu}$ tensor for this system. Show that it can be written as the sum of two terms

$$T^{\mu\nu} = T^{\mu\nu}(A) + T^{\mu\nu}(\phi, A) \tag{5}$$

where $T^{\mu\nu}(A)$ is the energy-momentum tensor for the free electromagnetic field and $T^{\mu\nu}(\phi, A)$ is the tensor which results by modifying the energy-momentum tensor for the decoupled complex scalar field ϕ by the *minimal coupling* procedure.

6. Find explicit expressions for the Hamiltonian density $\mathcal{H}(x)$ and the linear momentum density $\mathcal{P}(x)$ for this system. Give a physical interpretation for all of the terms that you found for each quantity.

7. Consider now the case of an infinitesimal Lorentz transformation

$$x_\mu \to x'_\mu + \omega_{\mu\nu} x^\nu \tag{6}$$

where $\omega_{\mu\nu}$ infinitesimal and antisymmetric. Show that the invariance of the Lagrangian of this system under these Lorentz transformation leads to the existence of a conserved tensor $M_{\mu\nu\lambda}$. Find the explicit form of this tensor. Give a physical interpretation for its *spacial* components. Does the conservation of this tensor impose any restriction on the properties of the energy-momentum tensor $T^{\mu\nu}$? Explain.
Warning: Be very careful in how you treat the fields. Recall that *not all* of the fields are scalars!

8. In this section yo will consider again the same system but in a *polar* representation for the scalar field ϕ, i.e.

$$\phi(x) = \rho(x)\, e^{i\omega(x)} \tag{7}$$

In problem set 1, problem 2, you showed that for $m_0^2 < 0$ the lowest energy states of the system can be well approximated by freezing the amplitude mode ρ to a constant value ρ_0 which you obtained by an energy minimization argument. In this section you are asked to find the form of

   (a) the conserved gauge current J_μ,
   (b) the total energy H,
   (c) the total linear momentum $\boldsymbol{P}$ in this limit.

9. Consider now the analytic continuation to imaginary time of this theory. Find the energy functional of the equivalent system in classical statistical mechanics. Give a physical interpretation for each of the terms of this energy functional. If D is the dimensionality of *space-time* for the original system, what is the dimensionality of *space* for the equivalent classical problem?.

   Warning : Be very careful in the way you perform the analytic continuation of the components of the vector potential.