582 Homework 3 Solutions

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1 Spin waves in a quantum Heisenberg antiferromagnet

1. First let's recall the Heisenberg equation of motion for an operator $\hat{A}$:

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A}, \hat{H}] + \partial_t \hat{A}$$

In our case we have operators which do not depend explicitly on time; thus, the above equation can be written as:

$$i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}]$$

The Hamiltonian for the Heisenberg antiferromagnet is:

$$\hat{H} = J \sum_{j=-N/2+1}^{N/2} \hat{S}_k(j) \cdot \hat{S}_k(j+1)$$

We are interested in the case that $\hat{A}$ is a spin-operator. Since $\hat{S}_k(j)$ belongs to the spin-S representation irrespective of the parity of j, we do not have to consider each sublattice individually. Furthermore, we can simplify our calculation if we rewrite our Hamiltonian using ladder operators:

$$\hat{H} = J \sum_j \Big[\hat{S}_3(j)\hat{S}_3(j+1) + \frac{1}{2}\Big(\hat{S}^+(j)\hat{S}^-(j+1) + \hat{S}^-(j)\hat{S}^+(j+1)\Big)\Big]$$

It is easy to show for any pair of sites, j and j':

$$[\hat{S}_3(j), \hat{S}^\pm(j')] = \pm\hbar\hat{S}^\pm(j)\delta_{jj'}$$
$$[\hat{S}^\pm(j), \hat{S}^\mp(j')] = \pm 2\hbar\hat{S}_3(j)\delta_{jj'}$$

Using these we can find our EoM!

(a) This is not hard but organization is important in what follows[1]:

$$\begin{aligned}
\frac{d\hat{S}_3(j)}{dt} &= \frac{J}{2i\hbar}\sum_{j'}\Big([\hat{S}_3(j), \hat{S}^+(j')\hat{S}^-(j'+1)] + [\hat{S}_3(j), \hat{S}^-(j')\hat{S}^+(j'+1)]\Big) \\
&= \frac{J}{2i\hbar}\sum_{j'}\Big([\hat{S}_3(j), \hat{S}^+(j')]\hat{S}^-(j'+1) + \hat{S}^+(j')[\hat{S}_3(j), \hat{S}^-(j'+1)] \\
&\qquad + [\hat{S}_3(j), \hat{S}^-(j')]\hat{S}^+(j'+1) + \hat{S}^-(j')[\hat{S}_3(j), \hat{S}^+(j'+1)]\Big) \\
&= \frac{J}{2i}\sum_{j'}\Big(\big(\hat{S}^+(j)\hat{S}^-(j'+1) - \hat{S}^-(j)\hat{S}^+(j'+1)\big)\delta_{jj'} + \big(\hat{S}^-(j')\hat{S}^+(j) - \hat{S}^+(j')\hat{S}^-(j)\big)\delta_{j,j'+1}\Big) \\
&= \frac{J}{2i}\Big(\big(\hat{S}^-(j+1) + \hat{S}^-(j-1)\big)\hat{S}^+(j) - \big(\hat{S}^+(j+1) + \hat{S}^+(j-1)\big)\hat{S}^-(j)\Big)
\end{aligned} \tag{1}$$

Note: I used the fact spin operators on distinct lattice sites commute to get the EoM in the above form.

[1] I drop commutators involving the same k's in $S_k(j)$ since these obviously commute.

(b) Proceeding in an analogous manner to (a):

$$\frac{d\hat{S}^{\pm}(j)}{dt} = \pm\frac{J}{2i}\Big[\Big(\hat{S}^{\pm}(j+1)+\hat{S}^{\pm}(j-1)\Big)\hat{S}_3(j) - 2\Big(\hat{S}_3(j+1)+\hat{S}_3(j-1)\Big)\hat{S}^{\pm}(j)\Big] \tag{2}$$

So we have a system of first order differential equations for operators which depend on time. Since our equations contain products of operators which also depend on time, ***we can not express these differential equation as a differential operator which acts linearly; that is, they are not linear.***

2. *I probably did this problem different than the way you decided to do it. Letting the provided operators speak for themselves is all you needed to do. Here we motivate the alternative form handed to you since it is fairly easy.*

We recall how the raising and lowering operators act on our $2S+1$ degenerate multiplet of states:

$$S^{\pm}|S, M(j)\rangle = \sqrt{(s\mp m)(s\pm m+1)}|S, M(j)\pm 1\rangle$$

Here I use lower case s's and m's without indices in the radical to save on clutter. We will do this when convenient.

In this problem we are interested in how our system deviates from perfect antiferromagentism, so we consider the following operator used to label our eigenstates in terms of its eigenvalues:

$$\hat{n}(j) = S + (-1)^{j-1}\hat{S}_3(j)$$

The staggered nature of the $\hat{n}(j)$'s reflect the fact we are working with an antiferromagnet. We also have the corresponding quantum numbers:

$$M(j) = S + (-1)^{j-1}n(j)$$

Suppressing the j dependence on our eigenvalues, $m = s + (-1)^{j-1}n$, we will substitute this into our above equation. Starting with even j:

$$\begin{aligned} S^{+}|s,m\rangle &= \sqrt{(s-m)(s+m+1)}|s,m+1\rangle \\ &= \sqrt{2sn\big(1-\frac{n-1}{2s}\big)}|s,m+1\rangle \\ \Rightarrow S^{+}|n\rangle &\equiv \sqrt{2sn\big(1-\frac{n-1}{2s}\big)}|n-1\rangle \end{aligned} \tag{3}$$

$$\begin{aligned} S^{-}|s,m\rangle &= \sqrt{(s+m)(s-m+1)}|s,m-1\rangle \\ &= \sqrt{2s\big(n+1)(1-\frac{n}{2s})n\big)}|s,m-1\rangle \\ \Rightarrow S^{-}|n\rangle &\equiv \sqrt{2s\big(n+1)(1-\frac{n}{2s})n\big)}|n+1\rangle \end{aligned} \tag{4}$$

Notice the lowering ladder operator increases n here and vice-versa for the raising operator. This makes sense since even sites where elected to be spin up and so the spin deviation for these states are increased when we decrease M. This is what lead me to label the eigenstates with $n\pm 1$ in the final lines of (3) and (4) after the ladder operators act on the states.

Now for odd j:

$$\begin{aligned} S^+|s,m\rangle &= \sqrt{(s-m)(s+m+1)}|s,m+1\rangle \\ &= \sqrt{2s\big(n+1)(1-\frac{n}{2s})n\big)}|s,m+1\rangle \\ \Rightarrow S^+\,|n\rangle &\equiv \sqrt{2s\big(n+1)(1-\frac{n}{2s})n\big)}|n+1\rangle \end{aligned} \tag{5}$$

Now the raising ladder operator sends $n \to n+1$ here. Similarly (and again counter to what we saw before):

$$\begin{aligned} S^-|s,m\rangle &= \sqrt{(s+m)(s-m+1)}|s,m-1\rangle \\ &= \sqrt{2sn\big(1-\frac{n-1}{2s}\big)}|s,m-1\rangle \\ S^-\,|n\rangle &\equiv \sqrt{2sn\big(1-\frac{n-1}{2s}\big)}|n-1\rangle \end{aligned} \tag{6}$$

The action of the ladder operators on our quantum numbers and eigenstates are reminiscent of the Harmonic oscillator, which inspires us to introduce the following creation and annihilation operators. First, for even sites[2]:

$$\begin{aligned} \hat{a^\dagger}|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned}$$

and for odd sites,

$$\begin{aligned} \hat{b^\dagger}|n\rangle &= \sqrt{n+1}|n+1\rangle \\ \hat{b}|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned}$$

Clearly, $[\hat{a}, \hat{b}^\dagger] = 0$. All the remaining commutation relations we suspect from the harmonic oscillator are immediately satisfied from the above equations and operators on different sites commute.

Using the above definitions we express our ladder operators in the following form (even sites):

$$\begin{aligned} S^+|n\rangle &= \sqrt{2sn\big(1-\frac{n-1}{2s}\big)}|n-1\rangle \\ &= \sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)}\sqrt{n}|n-1\rangle \\ &= \sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)}\ \hat{a}|n\rangle \\ \Rightarrow S^+ &= \sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)}\ \hat{a} \end{aligned} \tag{7}$$

This is the relation we are after!

We can easily justify the second line above. Indeed, it follows if we can expand a function, f, as a Taylor series. It is easy to prove with induction for a Hermitian operator, $\hat{A}$, with it's corresponding eigenstates, $|\alpha\rangle$: $\hat{A}^k|\alpha\rangle = \alpha^k|\alpha\rangle$. We use this to send $n-1 \to \hat{n}$ since the eigenstate is currently $|n-1\rangle$, and the square-root is analytic. We then use the fact n is a real number to permute it with

[2] Reminder: these do have indices and we will reinstate them momentarily.

$\hat{n}$, then we can use the provided properties for the annihilation operator.

We can find S^- in a similar fashion:

$$\begin{aligned} S^-|n\rangle &= \sqrt{2s(n+1)\big(1-\frac{n}{2s}\big)}|n+1\rangle \\ &= \sqrt{2s\big(1-\frac{n}{2s}\big)}\hat{a}^\dagger|n\rangle \\ &= \hat{a}^\dagger\sqrt{2s\big(1-\frac{n}{2s}\big)}|n\rangle \\ &= \hat{a}^\dagger\sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)}|n\rangle \\ &\Rightarrow S^- = \hat{a}^\dagger\sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)} \end{aligned} \tag{8}$$

We can rinse and repeat the above procedure, but instead, we recall the odd sites act counter to the even, so we just swap the roles of $S^+ \leftrightarrow S^-$. For completeness, for odd sites:

$$\begin{aligned} S^+ &= \hat{b}^\dagger\sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)} \\ S^- &= \sqrt{2s\big(1-\frac{\hat{n}}{2s}\big)}\hat{b} \end{aligned} \tag{9}$$

This illustrates we can rewrite our ladder operators with the spin excess operator in place of the run-of-the-mill spin operators!

3. We now rewrite the Hamiltonian in terms of these operators. First recall:

$$\begin{aligned} \hat{H} &= J\sum_{j=-N/2+1}^{N/2}\Big[\hat{S}_3(j)\hat{S}_3(j+1)+\hat{S}_1(j)\hat{S}_1(j+1)+\hat{S}_2(j)\hat{S}_2(j+1)\Big] \\ &= J\sum_{j=-N/2+1}^{N/2}\Big[\hat{S}_3(j)\hat{S}_3(j+1)+\frac{1}{2}\Big(\hat{S}^+(j)\hat{S}^-(j+1)+\hat{S}^-(j)\hat{S}^+(j+1)\Big)\Big] \end{aligned}$$

Caution, these operators are site dependent; i.e. even or odd, so we need to be careful here! First, let's note:

$$\begin{aligned} \hat{S}_3(j)\hat{S}_3(j+1) &= \Bigg[(-1)^{j-1}(\hat{n}(j)-S)\Bigg]\Bigg[(-1)^{(j+1)-1}(\hat{n}(j+1)-S)\Bigg] \\ &= -(\hat{n}(j)-S)(\hat{n}(j+1)-S) \\ &= -S^2+S\big(\hat{n}(j)+\hat{n}(j+1)\big)-\hat{n}(j)\hat{n}(j+1) \end{aligned}$$

Thus, this term is independent of the parity of j. We can easily see:

$$J\sum_{j=-N/2+1}^{N/2}\hat{S}_3(j)\hat{S}_3(j+1) = -NJS^2 + J\sum_{j=-N/2+1}^{N/2}\Bigg(S\big(\hat{n}(j)+\hat{n}(j+1)\big)-\hat{n}(j)\hat{n}(j+1)\Bigg)$$

The ladder operators are where we should be careful, so let's start by considering even sites alone. **First, we define $\sqrt{1-\frac{\hat{n}(j)}{2S}} = \hat{f}(j)$** to reduce clutter. This results in:

$$J\sum_{j\ even}\frac{1}{2}\Big(\hat{S}^{+}(j)\hat{S}^{-}(j+1)+\hat{S}^{-}(j)\hat{S}^{+}(j+1)\Big)$$
$$= SJ\sum_{j\ even}\Big(\hat{f}(j)\hat{f}(j+1)\hat{a}(j)\hat{b}(j+1)+\hat{a}^{\dagger}(j)\hat{b}^{\dagger}(j+1)\hat{f}(j)\hat{f}(j+1)\Big)$$

All that is different when summing the odd sites is the $j+1$ terms correspond to $\hat{a}$ and the j terms to $\hat{b}$. Performing the substitutions[3]:

$$\begin{aligned}\hat{H} = -JNS^2 + J\sum_{j}\Big(S\big(\hat{n}(j)+\hat{n}(j+1)\big)-\hat{n}(j)\hat{n}(j+1)\Big)\\ + SJ\sum_{j\ even}\Big(\hat{f}(j)\hat{f}(j+1)\hat{a}(j)\hat{b}(j+1)+\hat{a}^{\dagger}(j)\hat{b}^{\dagger}(j+1)\hat{f}(j)\hat{f}(j+1)\Big)\\ + SJ\sum_{j\ odd}\Big(\hat{f}(j)\hat{f}(j+1)\hat{a}(j+1)\hat{b}(j)+\hat{a}^{\dagger}(j+1)\hat{b}^{\dagger}(j)\hat{f}(j)\hat{f}(j+1)\Big)\end{aligned} \tag{10}$$

4. Now that we have an expression for our Hamiltonian in terms of our creation and annihilation operators we take the limit $S\to\infty$.

   Looking at our above sum the leading order term is clearly S^2, so let's consider terms linear in S and neglect those of lower order (i.e. S^0, S^{-1}, ...). Following this line of reasoning the summation resulting from the $\hat{S}_3$'s is trivial: drop $\hat{n}(j)\hat{n}(j+1)$. Notice the order of our creation and annihilation operators has been reduced for this part of the Hamiltonian!

   The portion containing the ladder operators takes a bit more work. S shows up as an overall multiplicative factor and also inside our function $\hat{f}(j)$. Thus, we are interested in:

$$\begin{aligned}S\hat{f}(j)\hat{f}(j+1) &= S\sqrt{1-\frac{\hat{n}(j)}{2S}}\sqrt{1-\frac{\hat{n}(j+1)}{2S}}\\ &= S\Big[1-\frac{\hat{n}(j)}{4S}+\mathcal{O}\big(\frac{1}{S^2}\big)\Big]\Big[1-\frac{\hat{n}(j+1)}{4S}+\mathcal{O}\big(\frac{1}{S^2}\big)\Big]\\ &= S-\frac{1}{4}\Big[\hat{n}(j)+\hat{n}(j+1)\Big]+\mathcal{O}\big(\frac{1}{S}\big)\end{aligned}$$

   So, within our approximation we simply replace these products with S! Thus, within the so called *spin-wave approximation* we find:

$$\begin{aligned}\hat{H} = -JNS^2 + SJ\sum_{j}\Big[\hat{n}(j)+\hat{n}(j+1)\Big] + SJ\sum_{j\ even}\Big[\hat{a}(j)\hat{b}(j+1)+\hat{a}^{\dagger}(j)\hat{b}^{\dagger}(j+1)\Big]\\ + SJ\sum_{j\ odd}\Big[\hat{a}(j+1)\hat{b}(j)+\hat{a}^{\dagger}(j+1)\hat{b}^{\dagger}(j)\Big]+\mathcal{O}\Big(S^0\Big)\end{aligned} \tag{11}$$

   This is clearly ***quadratic in our creation and annihilation operators.***

[3] Note: $\hat{n}(j)=\hat{a}^{\dagger}(j)\hat{a}(j)$ for j even and $\hat{n}(j)=\hat{b}^{\dagger}(j)\hat{b}(j)$ for j odd.

5. Let's formally apply the $S \to \infty$ limit to our equations of motion. Since we have rewritten our Hamiltonian in terms of our creation and annihilation operators, we will be looking at the EoM for said operators. Thus, we are about to see that the dynamics of these operators are also simplified in the spin-wave approximation since they are linear. We just calculate the following commutator:

$$\begin{aligned}
[\hat{a}^\dagger(\ell), \hat{H}] =& SJ\Bigg[\sum_{jeven}\Big[\hat{a}^\dagger(j)[\hat{a}^\dagger(\ell), \hat{a}(j)] + [\hat{a}^\dagger(\ell), \hat{a}(j)]\hat{b}(j+1)\Big] \\
&+ \sum_{jodd}\Big[\hat{a}^\dagger(j+1)[\hat{a}^\dagger(\ell), \hat{a}(j+1)] + [\hat{a}^\dagger(\ell), \hat{a}(j+1)]\hat{b}(j)\Big]\Bigg] + \mathcal{O}\Big(S^0\Big) \\
&= -SJ\Bigg[2\hat{a}^\dagger(\ell) + \hat{b}(k+1) + \hat{b}(\ell-1)\Bigg] + \mathcal{O}\Big(S^0\Big)
\end{aligned}$$

where ℓ is an arbitrary lattice site. From this result we can see:

$$i\hbar\frac{d\hat{a}^\dagger(\ell)}{dt} = SJ\Bigg[2\hat{a}^\dagger(\ell) + \hat{b}(\ell+1) + \hat{b}(\ell-1)\Bigg] + \mathcal{O}\Big(S^0\Big) \tag{12}$$

We are retaining the same approximation as 1.3, so we are dropping terms of order $\left(\boldsymbol{S^{-1}}\right)^{\boldsymbol{0}}$. *To leading order in S we have a linear equation of motion.* The time evolution of the annihilation operator can be found from the Hermitian conjugate of this above equation:

$$-i\hbar\frac{d\hat{a}(\ell)}{dt} = SJ\Bigg[2\hat{a}(\ell) + \hat{b}^\dagger(\ell+1) + \hat{b}^\dagger(k-1)\Bigg] + \mathcal{O}\Big(S^0\Big) \tag{13}$$

We can rinse and repeat for $\hat{b}(\ell)$, but this is the same as $\hat{b}^\dagger(l) \to \hat{a}^\dagger(\ell)$ and $\hat{a}(\ell) \to \hat{b}(\ell)$ in the above expressions. ***Thus, all the EoM for the relevant operators are linear (up to order S).***

6. Let's now rewrite our creation and annihilation operators using the following Fourier transforms:

$$\hat{a}(q) = \sum_{j\ even} e^{-iqj}\hat{a}(j), \qquad \hat{b}(q) = \sum_{j\ odd} e^{iqj}\hat{b}(j) \tag{14}$$

Notice that I use a different normalization than the one used in the problem set. To be more specific, we write $j_{\text{even}} = 2r$ and $j_{\text{odd}} = 2r+1$, where the integer r takes the values $r = -\frac{N}{4}+1, \ldots, \frac{N}{4}$, where we assume that there is an even number of even sites (and the same for the odd sites), and hence that N is a multiple of 4.

Periodic boundary conditions require that

$$a(2r) = a(2r+N), \qquad \text{and} \qquad b(2r+1) = b(2r+1+N)$$

which imply that

$$e^{iqN} = 1$$

Hence, the momentum q is restricted to the values

$$q_m = \frac{2\pi}{N}m$$

where $m = -\frac{N}{4}+1, \ldots, \frac{N}{4}$. Hence, the allowed momenta q_m take the values

$$(-\frac{N}{4}+1)\frac{2\pi}{N} \le q_m \le \frac{N}{4}\frac{2\pi}{N}$$

Or, what is the same

$$-\frac{\pi}{2}+\frac{2\pi}{N}\leq q_m\leq\frac{\pi}{2}$$

The allowed values of the momenta have a spacing $\Delta q=\frac{2\pi}{N}$. We are interested in chains with periodic boundary conditions *in the thermodynamic limit* $N\to\infty$. In this limit the spacing of the momenta $\Delta q\to 0$ and the allowed values of the momenta q fill densely the real interval $-\frac{\pi}{2}<q\leq\frac{\pi}{2}$, known as the first Brillouin zone.

The inverse of the Fourier transforms of Eq.(14) are

$$a(2r)=\frac{2}{N}\sum_{m=-N/4+1}^{N/4}a(q_m)e^{iq_m 2r},\qquad b(2r+1)=\frac{2}{N}\sum_{m=-N/4+1}^{N/4}b(q_m)e^{-iq_m(2r+1)} \tag{15}$$

We now recognize that the sums shown in Eq.(15) have the form of Riemann sums which in the thermodynamic limit $N\to\infty$ converge to the integrals

$$a(2r)=\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}\;e^{iq2r}a(q),\qquad b(2r+1)=\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}\;e^{-iq(2r+1)}b(q) \tag{16}$$

where we used that the spacing is $\Delta q=2\pi/N$.

We will now check the commutation relations of the operators $a(q)$ with its Hermitian conjugate:

$$[a(q),a^\dagger(q')]=\sum_{r,r'=-N/4+1}^{N/4}[a(2r),a^\dagger(2r')]e^{iq2r-iq'2r'} \tag{17}$$

$$=\sum_{r=-N/4+1}^{N/4}e^{-i(q-q')2r} \tag{18}$$

$$\equiv\pi\delta_P(q-q') \tag{19}$$

and similarly we obtain

$$[b(q),b^\dagger(q')=\pi\delta_P(q-q')$$

and all other commutators vanish. Here we introduced the periodic delta-function

$$\delta_P(q)\equiv\sum_{m=-\infty}^{\infty}\delta(q+\pi m)$$

In particular, we find the delta function at zero momentum is

$$\delta_P(0)=\frac{N}{2\pi}$$

We will now proceed to express the operators that appear in the effective Hamiltonian of Eq.(11) in terms of the operators $a(k)$ and $b(k)$ (and of their Hermitian conjugates). We will work directly in the thermodynamic limit. We begin with the sum of the local densities $n(j)$ on the even sublattice:

$$\begin{aligned}\sum_{r=-N/4+1}^{N/4} n(2r) &= \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \int_{-\pi/2}^{\pi/2} \frac{dk}{\pi} \sum_{r=-N/4+1}^{N/4} e^{i2r(q-k)} \hat{a}^\dagger(q) a(k) \\ &= \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \int_{-\pi/2}^{\pi/2} \frac{dk}{\pi} \pi \delta_P(q-k) a^\dagger(q) a(k) \\ &= \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} a^\dagger(q) a(q) \\ &= \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} n_a(q)\end{aligned}$$

A similar result works for j odd:

$$\sum_{r=-N/4+1}^{N/4} n(2r+1) = \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} b^\dagger(q) b(q) = \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} n_b(q)$$

Similarly, we obtain

$$2 \sum_{r=-N/4+1}^{N/4} \Big[n(2r) + n(2r+1) \Big] = 2 \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} 2 \Big[a^\dagger(q) a(q) + b^\dagger(q) b(q) \Big]$$

These terms in Eq.(11) conserve the number of excitations, i.e. have as many creation ans annihilation operators.

We will next perform the Fourier transform of the off-diagonal terms in Eq.(11). However, these terms do not conserve the number of excitations. They are

$$\begin{aligned}&+ SJ \sum_{r+-N/4+1}^{N/4} \Big(a(2r) b(2r+1) + a^\dagger(2r) b^\dagger(2r+1) + a(2r) b(2r-1) + a^\dagger(2r) b^\dagger(2r-1) \Big) \\ &= SJ \sum_{r=-N/4+1}^{N/4} \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \int_{-\pi/2}^{\pi/2} \frac{dk}{\pi} \pi \delta_P(q-k) 2\cos q \Big(a(q) b(q) + a^\dagger(q) b^\dagger(q) \Big) \\ &= SJ \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} 2\cos q \Big(a(q) b(q) + a^\dagger(q) b^\dagger(q) \Big)\end{aligned}$$

Combining these expressions we find

$$\begin{aligned}H &= -JNS^2 + 2JS \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \Big[a^\dagger(q) a(q) + b^\dagger(q) b(q) + \cos q a(q) b(q) + \cos q a^\dagger(q) b^\dagger(q) \Big] \\ &= -JNS^2 - 2JS \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \pi \delta_P(0) \\ &+ 2SJ \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \Big[a^\dagger(q) a(q) + b(q) b^\dagger(q) + \cos(q) a(q) b(q) + 2\cos(q) a^\dagger(q) b^\dagger(q) \Big] \\ &= -JNS^2 \left(1 + \frac{1}{S}\right) + 2JS \int_{-\pi/2}^{\pi/2} \frac{dq}{\pi} \begin{bmatrix} \hat{a}^\dagger(q) & b(q) \end{bmatrix} \begin{bmatrix} 1 & \cos q \\ \cos q & 1 \end{bmatrix} \begin{bmatrix} a(q) \\ b^\dagger(q) \end{bmatrix}\end{aligned} \tag{20}$$

where in the second line we reordered the operators of a term in the first line and where we used that $\delta_P(0) = N/(2\pi)$. We can diagonalize the Hamiltonian by introducing the following Bogoliubov transformation[4]:

$$\begin{bmatrix} c(q) \\ d^\dagger(q) \end{bmatrix} = \begin{bmatrix} \cosh\big(\theta(q)\big) & \sinh\big(\theta(q)\big) \\ \sinh\big(\theta(q)\big) & \cosh\big(\theta(q)\big) \end{bmatrix} \begin{bmatrix} a(q) \\ b^\dagger(q) \end{bmatrix}$$

and its inverse transformation

$$\begin{bmatrix} a(q) \\ b^\dagger(q) \end{bmatrix} = \begin{bmatrix} \cosh\big(\theta(q)\big) & -\sinh\big(\theta(q)\big) \\ -\sinh\big(\theta(q)\big) & \cosh\big(\theta(q)\big) \end{bmatrix} \begin{bmatrix} c(q) \\ d^\dagger(q) \end{bmatrix}$$

This transformation preserves our commutation relations[5]! These new operators are interpreted as *quasi-particles.* ***We restate that the quasi particle operators, $\mathbf{c(q)}$ and $\mathbf{d(q)}$, satisfy the same algebra as $\mathbf{a(q)}$ and $\mathbf{b(q)}$.***

Now we are interested in the following multiplication:

$$\begin{bmatrix} \cosh\big(\theta(q)\big) & -\sinh\big(\theta(q)\big) \\ -\sinh\big(\theta(q)\big) & \cosh\big(\theta(q)\big) \end{bmatrix} \begin{bmatrix} 1 & \cos(q) \\ \cos(q) & 1 \end{bmatrix} \begin{bmatrix} \cosh\big(\theta(q)\big) & -\sinh\big(\theta(q)\big) \\ -\sinh\big(\theta(q)\big) & \cosh\big(\theta(q)\big) \end{bmatrix}$$

$$= \begin{bmatrix} \cosh\big(2\theta(q)\big) - \sinh\big(2\theta(q)\big)\cos(q) & \cosh\big(2\theta(q)\big)\cos(q) - \sinh\big(2\theta(q)\big) \\ \cosh\big(2\theta(q)\big)\cos(q) - \sinh\big(2\theta(q)\big) & \cosh\big(2\theta(q)\big) - \sinh\big(2\theta(q)\big)\cos(q) \end{bmatrix}$$

To diagonalize the Hamiltonian we choose:

$$\theta(q) = \frac{1}{2}\tanh^{-1}(\cos(q))$$

Notice this means:

$$\begin{aligned} \operatorname{sech}^2\big(2\theta(q)\big) &= 1 - \tanh^2\big(2\theta(q)\big) \\ &= 1 - \cos^2(q) \\ &= \sin^2(q) \end{aligned}$$

and,

$$\begin{aligned} \sinh\big(2\theta(q)\big) &= \cosh\big(2\theta(q)\big)\tanh\big(2\theta(q)\big) \\ &= \cosh\big(2\theta(q)\big)\cos(q) \end{aligned}$$

The diagonal terms are now a breeze to find:

$$\begin{aligned} \cosh\big(2\theta(q)\big) - \sinh\big(2\theta(q)\big)\cos(q) &= \frac{1}{\operatorname{sech}\big(2\theta(q)\big)}\Big(1 - \cos^2(q)\Big) \\ &= |\sin(q)| \end{aligned}$$

[4]See Appendix A if you prefer foiling out the expressions over matrix multiplication. The latter method is much easier, but the former is here in case you did/tried this problem that way and want to check it.

[5]This is a direct consequence of the identity: $\cosh^2(x) - \sinh^2(x) = 1$.

In terms of our quasi-particles our Hamiltonian becomes:

$$\begin{aligned}
\hat{H} &= -JNS^2\left(1+\frac{1}{S}\right) + 2SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}\begin{bmatrix} a^\dagger(q) & b(q)\end{bmatrix}\begin{bmatrix} 1 & \cos(q) \\ \cos(q) & 1\end{bmatrix}\begin{bmatrix} a(q) \\ b^\dagger(q)\end{bmatrix} \\
&= -JNS^2\left(1+\frac{1}{S}\right) + 2SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}|\sin(q)|\begin{bmatrix} c^\dagger(q) & d(q)\end{bmatrix}\begin{bmatrix} c(q) \\ d^\dagger(q)\end{bmatrix} \\
&= -JNS^2\left(1+\frac{1}{S}\right) + 2SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}|\sin(q)|\Big(c^\dagger(q)c(q) + d(q)d^\dagger(q)\Big) \\
&= -JNS^2\left(1+\frac{1}{S}\right) + 2SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}|\sin(q)|\pi\delta_P(0) + SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}|\sin(q)|\Big(c^\dagger(q)c(q) + d^\dagger(q)d(q)\Big)
\end{aligned}$$

Thus, in this limit the Hamiltonian takes the simpler form

$$H = E_0 + 2SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}|\sin(q)|\Big(\hat{c}^\dagger(q)\hat{c}(q) + \hat{d}^\dagger(q)\hat{d}(q)\Big) \tag{21}$$

Where:

$$E_0 = -JNS^2 + JNS + 2SJ\int_{-\pi/2}^{\pi/2}\frac{dq}{\pi}|\sin(q)|\pi\delta_P(0) = -JNS^2\left[1+\frac{1}{S}\left(1-\frac{2}{\pi}\right)\right]$$

The second term is the leading quantum correction (to order $1/S$) to the classical ground state energy. From this we can see that ***the Hamiltonian is quadratic and diagonal in the creation and annihilation operators for our quasi particles***.

7. We can define the ground state, $|0\rangle$, to be the state which satisfies: $\hat{n}_c(q)|0\rangle = \hat{n}_d(q)|0\rangle = 0$. Thus, the energy of the ground state is:

$$E_0 = -JNS^2\left[1+\frac{1}{S}\left(1-\frac{2}{\pi}\right)\right]$$

8. There are two types of a single particle states:

$$|n_c(k)=1\rangle = c^\dagger(k)|0\rangle, \qquad \text{and} \qquad |n_d(k)=1\rangle = d^\dagger(k)|0\rangle$$

Let's find the energy of these states using our above Hamiltonian. We just need to note:

$$n_c(q)c^\dagger(k)|0\rangle = c^\dagger(q)c(q)c^\dagger(k)|0\rangle = c^\dagger(q)\big[\pi\delta_P(k-q) + c^\dagger(k)c(q)\big]|0\rangle = \pi c^\dagger(q)\delta_P(k-q)|0\rangle$$

so after subtracting off the ground state energy and integrating, *the excitation energies of our single particle states* are of the form:

$$E(k) = 2JS|\sin(k)|$$

Note that: $\boldsymbol{k \to 0 \Rightarrow E(k) \to 0}$ (restricting ourselves to the first Brillouin zone, i.e. neglect $k = n\pi$ with $n \neq 0$), so *in the long wave length limit,* $k \to 0$, *both types of single particle states have energies equal to the ground state: they are gapless*!

Furthermore, *we can expand our energy in the immediate neighborhood of* $k = 0$ to obtain:

$$E(k) \approx 2JS|k|$$

Thus, we see ***the energy vanishes linearly with the momentum k***! Finally, the velocity of these excitations as $k \to 0$ is given by:

$$v_s = \left.\frac{dE(k)}{d|k|}\right|_{k=0} = 2JS$$

This is the *spin wave velocity*.

2 Two component complex scalar field

Consider the Lagrangian:

$$\mathcal{L} = \left(\partial_\mu \phi_a\right)^* \partial^\mu \phi_a - V(|\phi_a|^2)$$

Where: $V(|\phi_a|^2) = m^2 \phi_a \phi_a^*$

1. Let's begin by finding:

   (a) The canonical momentum Π_a conjugate to ϕ_a. This is simple enough:

   $$\Pi_a = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a} = \partial^0 \phi_a^*$$

   Since the field is complex, we can not forget about:

   $$\Pi_a^* = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a^*} = \partial^0 \phi_a$$

   As a quick note, we see that we neglect any sort of index which indicates if we are talking about a co- or contra-variant canonical momentum. We just make the correct choice where it matters, and forgo any additional labels to avoid too much clutter.

   (b) With σ representing the extra degree of freedom we get from the complex fields we see the Hamiltonian of this system is:

   $$\mathcal{H} = \sum_{a,\sigma} (\partial_0 \phi_a^\sigma) \Pi_a^\sigma - \mathcal{L} = 2 \sum_a \Pi_a^* \Pi_a - \mathcal{L}$$

   From here on out I drop all summation symbols with the understanding we are summing over the components of the field. With this in mind we see:

   $$\mathcal{H} = \left(\Pi_a^* \Pi_a + \nabla \phi_a \cdot \nabla \phi_a^*\right) + V(|\phi|^2)$$

   So then:

   $$H = \int d^3x \mathcal{H}$$

   (c) Now for the total linear momentum the momentum density is given by:

   $$\mathcal{P}^k = \Pi_a \partial^k \phi_a + h.c.$$

   We could arrive here from the energy-momentum tensor as we did in the last homework. The Hermitian conjugate is needed to keep things real. From this we have:

   $$P^k = \int d^3x \ \mathcal{P}^k$$

2. We begin by considering a two component complex scalar field:

   $$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

   clearly:

   $$\forall \, U \in SU(2), \ |U\Phi|^2 = |\Phi|^2$$

Furthermore, *in the case of a global symmetry:* $\mathcal{L}(U\Phi) = \mathcal{L}(\Phi)$ (see Appendix B for the case of a local symmetry). We know from Noether's theorem that a symmetry implies the existence of conserved currents. To find these currents we express the elements of our group in the following form[6]:

$$
\begin{aligned}
[U(x)]_{ab} &= \left[e^{i\lambda^k\theta^k(x)}\right]_{ab} \approx \delta_{ab} + i\lambda^k_{ab}\theta^k(x) \\
[U^\dagger(x)]_{ab} &= \left[e^{-i\lambda^k\theta^k(x)}\right]_{ab} \approx \delta_{ab} - i\lambda^k_{ab}\theta^k(x)
\end{aligned}
$$

Of course, we're working with small $\theta^k(x)$'s. Since we are considering a 2-component complex scalar field, we need the following two transformation rules:

$$
\begin{aligned}
\left[U(x)\Phi\right]_a &= [U(x)]_{ab}\phi_b \\
\left[(U(x)\Phi)^\dagger\right]_a &= \phi^*_b[U^\dagger(x)]_{ba}
\end{aligned}
$$

Taking our representations of the spin-1/2 doublet representation of $SU(2)$ to be the Pauli matrices, i.e. $\lambda^k_{ab} = \sigma^k_{ab}$, the explicit transformation for our complex scalar field is given by:

$$
\delta\phi_a = \big([U(x)]_{ab} - \delta_{ab}\big)\phi_b \approx i\sigma^k_{ab}\phi_b\theta^k(x)
$$

Note, we are summing over b. Also, for the complex conjugate:

$$
\delta\phi^*_a = \phi^*_b\big([U^\dagger(x)]_{ba} - \delta_{ab}\big) \approx -i\phi^*_b\sigma^k_{ba}\theta^k(x)
$$

We are interested in a global symmetry; thus, our θ^k's are constant. Applying this variation to our Lagrangian, exploiting the Euler-Lagrange equation, and finally taking the necessary functional derivatives we find[7]:

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\partial_\nu\phi_a + \phi_a \leftrightarrow \phi^*_a \\
&= \partial_\nu\Big[\frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\phi_a + \delta\phi^*_a\frac{\delta\mathcal{L}}{\delta\partial_\nu\phi^*_a}\Big] \\
&= i\partial_\nu\Big[\sigma^k_{ab}\big(\partial^\nu\phi_a\big)^*\phi_b - \phi^*_b\sigma^k_{ba}\big(\partial^\nu\phi_a\big)\Big]\theta^k \\
&= i\partial_\nu\Big[\big(\partial^\nu\Phi^T\big)^*\sigma^k\Phi - \big(\Phi^T\big)^*\sigma^k\partial^\nu\Phi\Big]\theta^k
\end{aligned} \tag{22}
$$

So ***we have 3 conserved 4-currents (one for each Pauli Matrix) indexed with k***. Because we want to (eventually) show our corresponding conserved charges are the generators of $SU(2)$, we introduce an additional minus sign, and define/rewrite as:

$$
j^k_\nu \equiv -i\Big(\big(\partial_\nu\Phi^\dagger\big)\sigma^k\Phi - \Phi^\dagger\sigma^k\partial_\nu\Phi\Big)
$$

This is clearly conserved by the above relation.

[6]The specific representation is not yet determined, but they are Hermitian.

[7]Notice the order of products in the second term of the second line. This can be seen in a few ways. For one, the h.c. of ϕ lives in the dual space, so it must left multiply. You can also write this line as: $\partial_\nu\Big[\frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\phi_a + h.c.\Big]$. Finally, just plugging in $\phi + \delta\phi$ into our action achieves the same ordering of terms.

We can express our constants of motion in terms of these conserved currents in the usual way:

$$
\begin{aligned}
Q^k = \int d^3x j_0^k &= -i \int d^3x \Big((\partial_0 \Phi^\dagger) \sigma^k \Phi - \Phi^\dagger \sigma^k \partial_0 \Phi \Big) \\
&= -i \int d^3x \Big(\Pi^T \sigma^k \Phi - \Phi^\dagger \sigma^k \, \Pi^* \Big) \\
&= -i \sigma^k_{ab} \int d^3x \Big(\Pi_a \phi_b - \phi_a^* \, \Pi_b^* \Big)
\end{aligned}
$$

This means there are 3 classical constants of motion associated with global SU(2) transformations.

3. ***Now let's go ahead and Quantize our theory. All we need to do is promote our fields to operators, and then impose equal time commutation relations***. The result of which is:

$$
\hat{H} = \int d^3x \Big(\hat{\Pi}_a \hat{\Pi}_a^* + \nabla \hat{\phi}_a \cdot \nabla \hat{\phi}_a^{\ *} + V(|\hat{\phi}|^2) \Big)
$$

$$
[\hat{\phi}_a(\vec{x}, t), \hat{\Pi}_b(\vec{x}', t)] = [\hat{\phi}_a^*(\vec{x}, t), \hat{\Pi}_b^*(\vec{x}', t)] = i \delta_{ab} \delta^3(\vec{x} - \vec{x}')
$$

All other commutators are zero.

We can also see that the total momentum operator becomes:

$$
\hat{P}^k = \int d^3x \hat{\Pi}_a \partial^k \hat{\phi}_a + h.c.
$$

4. ***The quantum mechanical generators of global infinitesimal $SU(2)$ symmetry are related to the classical conserved quantities above, Q^k, in a very intimate way; indeed, these are just the off-springs of our classical conserved charge***. We promote the conserved charge to an operator by using the quantized field operators; thus:

$$
\begin{aligned}
\hat{Q}^k = \int d^3x \hat{j}^{0k} &= -i \int d^3x \Big(\big(\partial^0 \hat{\Phi}^T\big)^* \sigma^k \hat{\Phi} - \big(\hat{\Phi}^T\big)^* \sigma^k \partial^0 \hat{\Phi} \Big) \\
&= -i \int d^3x \Big(\hat{\Pi}^T \sigma^k \hat{\Phi} - \big(\hat{\Phi}^T\big)^* \sigma^k \, \hat{\Pi}^* \Big) \\
&= -i \sigma^k_{ab} \int d^3x \Big(\hat{\Pi}_a \hat{\phi}_b - \hat{\phi}_a^* \, \hat{\Pi}_b^* \Big)
\end{aligned}
\tag{23}
$$

These guys are indeed the generators we are after which is evident from their commutation relations: $[\hat{Q}^i, \hat{Q}^j] = 2i\varepsilon_{ijk}\hat{Q}^k$. These are clearly generators of our $SU(2)$ symmetry since we have the same structure constant as the Pauli matrices!

To work out these commutation relations we note at equal times:

$$
\big[\hat{\Pi}_a \hat{\phi}_b(\vec{x}), \hat{\Pi}_c \hat{\phi}_d(\vec{x}')\big] = i \Big(\hat{\Pi}_a \hat{\phi}_d(\vec{x}) \delta_{bc} - \hat{\Pi}_c \hat{\phi}_b(\vec{x}) \delta_{ad} \Big) \delta(\vec{x} - \vec{x}')
$$

Which implies:

$$
\begin{aligned}
-\big[\hat{\Pi}_a \sigma^i_{ab} \hat{\phi}_b(\vec{x}), \hat{\Pi}_c \sigma^j_{cd} \hat{\phi}_d(\vec{x}')\big] &= -i \Big(\hat{\Pi}_a \hat{\phi}_d(\vec{x}) \sigma^i_{ab} \sigma^j_{bd} - \hat{\Pi}_c \hat{\phi}_b(\vec{x}) \sigma^i_{ab} \sigma^j_{ca} \Big) \delta(\vec{x} - \vec{x}') \\
&= -i \Big(\hat{\Pi}_a \hat{\phi}_d(\vec{x}) \big[\sigma^i \sigma^j\big]_{ad} - \hat{\Pi}_c \hat{\phi}_b(\vec{x}) \big[\sigma^j \sigma^i\big]_{bc} \Big) \delta(\vec{x} - \vec{x}') \\
&= -i \hat{\Pi}_a \Big[\big[\sigma^i, \sigma^j\big] \Big]_{ad} \hat{\phi}_d(\vec{x}) \delta(\vec{x} - \vec{x}') \\
&= 2i \epsilon^{ijk} \Big(-i \hat{\Pi}_a \sigma^k_{ad} \hat{\phi}_d(\vec{x}) \Big) \delta(\vec{x} - \vec{x}')
\end{aligned}
$$

We can finally show our desired result:

$$[\hat{Q}^i, \hat{Q}^j] = \int d^3x d^3x' 2i\epsilon^{ijk}\Big[i\Big(\hat{\phi}^*_a \sigma^k_{ad}\hat{\Pi}^*_a(\vec{x}) - \hat{\Pi}_a\sigma^k_{ad}\hat{\phi}_d(\vec{x})\Big)\Big]\delta(\vec{x}-\vec{x}')$$
$$= 2i\epsilon^{ijk}\hat{Q}^k$$

Again, these conserved charges satisfy the same algebra as the Pauli matrices; therefore, they are generators of $SU(2)$ transformations!

5. Let's now use the Heisenberg equation of motion to find dynamics of our field operators:

   (a)
   $$\partial_0\hat{\phi}_a(\vec{x},t) = \frac{1}{i}[\hat{\phi}_a(\vec{x},t), \hat{H}(\vec{x}',t)]$$

   Clearly $\hat{\phi}_a$ only doesn't commute with the momentum operator; thus, we just need to compute:

   $$[\hat{\phi}_a(\vec{x},t), \hat{\Pi}_b(\vec{x}',t)\hat{\Pi}^*_b(\vec{x}',t)] = [\hat{\phi}_a(\vec{x},t), \hat{\Pi}_b(\vec{x}',t)]\hat{\Pi}^*_b(\vec{x}',t) = i\delta_{ab}\delta^3(\vec{x}-\vec{x}')\hat{\Pi}^*_b(\vec{x}',t)$$

   It follows:
   $$\partial_0\hat{\phi}_a(\vec{x},t) = \hat{\Pi}^*_a(\vec{x},t)$$

   As it should be!

   (b) The equation of motion for the complex conjugate field follows the exact same procedure, and is just the complex conjugate of the above equation:

   $$\partial_0\hat{\phi}^*_a(\vec{x},t) = \hat{\Pi}_a(\vec{x},t)$$

   (c)
   $$\partial_0\hat{\Pi}_a(\vec{x},t) = \frac{1}{i}[\hat{\Pi}_a(\vec{x},t), \hat{H}(\vec{x}',t)]$$

   The commutator with the potential is the exact same procedure as (a) and (b) except the canonical commutation relation is reversed, so there will be an overall minus sign. Because of this, I will not do it explicitly.

   The following commutator requires an integration by parts[8]:

   $$\begin{aligned}&\int d^3x'[\hat{\Pi}_a(\vec{x},t), \nabla'\hat{\phi}_b(\vec{x'},t)\nabla'\hat{\phi}^*_b(\vec{x'},t)]\\ =&\int d^3x'[\hat{\Pi}_a(\vec{x},t), \nabla'\hat{\phi}_b(\vec{x'},t)]\nabla'\hat{\phi}^*_b(\vec{x'},t)\\ =&-\int d^3x'[\hat{\Pi}_a(\vec{x},t), \hat{\phi}_b(\vec{x'},t)]\nabla'^2\hat{\phi}^*_b(\vec{x'},t)\\ =&i\nabla^2\hat{\phi}^*_a(\vec{x},t)\end{aligned} \tag{24}$$

   The last line comes from commutation relations, and then using both of the delta functions. As a result, we find:
   $$\partial_0\hat{\Pi}_a(\vec{x},t) = \nabla^2\hat{\phi}^*_a(\vec{x},t) - m^2\hat{\phi}^*_a$$

[8] I throw out all boundary terms immediately because our field operators evaluate to zero there.

(d) Now we can rinse and repeat... Or take a complex conjugate of the above expression; I opt for the latter:

$$\partial_0 \hat{\Pi}^*_a(\vec{x},t) = \frac{1}{i}[\hat{\Pi}^*_a(\vec{x},t), \hat{H}(\vec{x}',t)]$$
$$= \nabla^2 \hat{\phi}_a(\vec{x},t) - m^2 \hat{\phi}_a(\vec{x},t)$$

Finally, we can use our two equations for each of the fields above (four in total) and write:

$$\begin{aligned}(\partial^2 + m^2)\hat{\phi}_a(x) = 0\\ (\partial^2 + m^2)\hat{\phi}^*_a(x) = 0\end{aligned} \tag{25}$$

6. We're now in a position to introduce a set of creation and annihilation operators, which will show our analogy between fields and harmonic oscillators still holds in the complex case. Since we have already quantized the real scalar field, we quickly note some similarities and some differences.

We first consider the case of a single component complex scalar field, then we generalize our result to multiple components. ***Even with a single component however we will see a complex scalar field requires two sets of creation and annihilation operators opposed to one as was the case for a real scalar field***; this has to do with the fact we "secretly" have a set of two quantum fields opposed to just one.

Following Peskin and Schroeder we expand our field operator out in terms of its Fourier modes, treating each mode as an independent oscillator, which is the same idea we had for the real scalar field:

$$\hat{\phi}(\vec{x},x_0) = \int \frac{d^3k}{(2\pi)^3}\left[\hat{\phi}_+(\vec{k},x_0)e^{i\vec{k}\cdot\vec{x}} + \hat{\phi}_-(\vec{k},x_0)e^{-i\vec{k}\cdot\vec{x}}\right]$$

From our equations of motion in 2.5 (eq. (21)) we know that each part must satisfy the Klein-Gordon equation:

$$\partial_0^2\hat{\phi}_\pm(\vec{k},x_0) + (k^2+m^2)\hat{\phi}_\pm(\vec{k},x_0) = 0$$

We deduce the following time dependence:

$$\hat{\phi}_\pm(\vec{k},x_0) = \hat{\phi}_\pm(\vec{k})e^{\mp i\omega(\vec{k})x_0}$$

where $\omega(\vec{k}) = \sqrt{k^2+m^2}$. There are 4 combinations total, but we only keep the solutions which are Lorentz invariant; that is, exponents which are multiples of: $\omega x_0 - \vec{k}\cdot\vec{x} = k\cdot x$. Thus:

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}\left[\hat{\phi}_+(\vec{k})e^{-ik\cdot x} + \hat{\phi}_-(\vec{k})e^{ik\cdot x}\right]$$

We now recall that for the case of a real scalar field we had the conditions: $\hat{\phi}_+(\vec{k}) = \hat{\phi}^\dagger_-(-\vec{k})$ and $\hat{\phi}_-(\vec{k}) = \hat{\phi}^\dagger_+(-\vec{k})$. These conditions are no longer necessary because our complex scalar field is not necessarily Hermitian. This reaffirms the prediction that the complex field will have a set of two creation and annihilation operators! Furthermore, we can again see the necessity of a set of two operators after considering the mode expansion of the complex conjugate of the above field:

$$\hat{\phi}^*(\vec{x},x_0) = \int \frac{d^3k}{(2\pi)^3}\left[\hat{\phi}^*_+(\vec{k},x_0)e^{i\vec{k}\cdot\vec{x}} + \hat{\phi}^*_-(\vec{k},x_0)e^{-i\vec{k}\cdot\vec{x}}\right]$$

and so,

$$\partial_0^2\hat{\phi}^*_\pm(\vec{k},x_0)+(k^2+m^2)\hat{\phi}^*_\pm(\vec{k},x_0)=0$$

which leads us to:

$$\hat{\phi}^*(x)=\int\frac{d^3k}{(2\pi)^3}\left[\hat{\phi}^*_+(\vec{k})e^{-ik\cdot x}+\hat{\phi}^*_-(\vec{k})e^{ik\cdot x}\right]$$

We are now in a position to define our creation and annihilation operators; ***our constraint this time is our field operators must be complex conjugates of one another***. Exploiting orthogonality to equate the coefficients of our exponentials we see[9]:

$$\hat{\phi}^\dagger=\hat{\phi}^*\to\hat{\phi}^*_+=\hat{\phi}^\dagger_-,\ \ \hat{\phi}^*_-=\hat{\phi}^\dagger_+$$

This motivates:

$$\hat{a}(\vec{k})=2\omega(\vec{k})\hat{\phi}_+$$

and,

$$\hat{b}^\dagger(\vec{k})=2\omega(\vec{k})\hat{\phi}_-$$

We can show that if these guys satisfy the following commutation relations:

$$[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k}')]=(2\pi)^3\omega(\vec{k})\delta^3(\vec{k}-\vec{k}')$$
$$[\hat{b}(\vec{k}),\hat{b}^\dagger(\vec{k}')]=(2\pi)^3\omega(\vec{k})\delta^3(\vec{k}-\vec{k}')$$

with all others zero, then our canonical commutation relations are preserved. With these the field operators become:

$$\hat{\phi}(x)=\int\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left(\hat{a}(\vec{k})e^{-ik\cdot x}+\hat{b}^\dagger(\vec{k})e^{ik\cdot x}\right)$$

and,

$$\hat{\phi}^*(x)=\int\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left(\hat{b}(\vec{k})e^{-ik\cdot x}+\hat{a}^\dagger(\vec{k})e^{ik\cdot x}\right)$$

The generalization to a 2-component complex field, let alone a n-component one, is trivial: just introduce subscripts and impose $[\hat{a}_m,\hat{a}^\dagger_n]$ etc. are all proportional to δ_{mn} (where appropriate). Then we find:

$$\begin{aligned}\hat{\phi}_m(x)&=\int\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left(\hat{a}_m(\vec{k})e^{-ik\cdot x}+\hat{b}^\dagger_m(\vec{k})e^{ik\cdot x}\right)\\ \hat{\phi}^*_m(x)&=\int\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left(\hat{b}_m(\vec{k})e^{-ik\cdot x}+\hat{a}^\dagger_m(\vec{k})e^{ik\cdot x}\right)\end{aligned}\tag{26}$$

From these relations we can easily calculate our canonical momentum operators in terms of these creation and annihilation operators:

$$\begin{aligned}\hat{\Pi}_m(x)&=i\int\frac{d^3k}{2(2\pi)^3}\left(\hat{a}^\dagger_m(\vec{k})e^{ik\cdot x}-\hat{b}_m(\vec{k})e^{-ik\cdot x}\right)\\ \hat{\Pi}^*_m(x)&=-i\int\frac{d^3k}{2(2\pi)^3}\left(\hat{a}_m(\vec{k})e^{-ik\cdot x}-\hat{b}^\dagger_m(\vec{k})e^{ik\cdot x}\right)\end{aligned}\tag{27}$$

[9]There is no real difference between $*$ and $\dagger$ here (for single component complex scalar fields). They're more like additional labels I use to relate the Fourier modes. If we didn't use a $\dagger$, the equations below would be something like: $\hat{\phi}^*_+=\hat{\phi}^*_-$ otherwise.

7. Now let's go ahead and express $\hat{H}$, $\hat{P}$ and $\hat{Q}^k$ in terms of our creation and annihilation operators. We'll start with $\hat{Q}^k$. We calculate two types of terms from the definition of $\hat{Q}^k$: $\hat{\Pi}_m\hat{\phi}_n(x)$ and $-\hat{\phi}^*_m\hat{\Pi}^*_n(x)$. We'll do this in steps:

$$-\hat{\phi}^*_m\hat{\Pi}^*_n(x) = i\int\frac{d^3k}{2(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3 2\omega(\vec{k}')}\Big(\hat{b}_m(\vec{k}')e^{-ik'\cdot x}+\hat{a}^\dagger_m(\vec{k}')e^{ik'\cdot x}\Big)\Big(\hat{a}_n(\vec{k})e^{-ik\cdot x}-\hat{b}^\dagger_n(\vec{k})e^{ik\cdot x}\Big)$$

Let's now foil everything out:

$$i\Big(\hat{b}_m\hat{a}_n(\vec{k})(\vec{k}')e^{-i(k+k)'\cdot x}-\hat{b}_m(\vec{k}')\hat{b}^\dagger_n(\vec{k})e^{i(k-k)'\cdot x}+\hat{a}^\dagger_m(\vec{k}')\hat{a}_n(\vec{k})e^{-i(k+k')\cdot x}-\hat{a}^\dagger_m(\vec{k}')\hat{b}^\dagger_n(\vec{k})e^{i(k+k')\cdot x}\Big)$$
$$\to i\Big(\hat{b}_m(\vec{k})\hat{a}_n(-\vec{k})e^{-2i\omega(\vec{k})x_o}-\hat{b}_m(\vec{k})\hat{b}^\dagger_n(\vec{k})+\hat{a}^\dagger_m(\vec{k})\hat{a}_n(\vec{k})-\hat{a}^\dagger_m(\vec{k})\hat{b}^\dagger_n(-\vec{k})e^{2i\omega(\vec{k})x_o}\Big)$$

This last simplification results from the identity:

$$\int d^3x e^{\pm i(\vec{k}\pm\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}\pm\vec{k}')$$

and the fact that we're integrating over spatial coordinates (see Eq. 23). The arrow represents a spatial integration as well us the collapse of the $\vec{k}'$ integral using the resulting delta function.

In a similar fashion:

$$\hat{\Pi}_m\hat{\phi}_n(x) = i\int\frac{d^3k}{2(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3 2\omega(\vec{k}')}\left[\Big(\hat{a}^\dagger_m(\vec{k})e^{ik\cdot x}-\hat{b}_m(\vec{k})e^{-ik\cdot x}\Big)\Big(\hat{a}_n(\vec{k}')e^{-ik'\cdot x}+\hat{b}^\dagger_n(\vec{k}')e^{ik'\cdot x}\Big)\right]$$

Expanding:

$$i\Big(\hat{a}^\dagger_m(\vec{k})\hat{a}_n(\vec{k}')e^{i(k-k')\cdot x}-\hat{b}_m(\vec{k})\hat{a}_n(\vec{k}')e^{-i(k+k')\cdot x}+\hat{a}^\dagger_m(\vec{k})\hat{b}^\dagger_n(\vec{k}')e^{-i(k+k')\cdot x}-\hat{b}_m(\vec{k})\hat{b}^\dagger_n(\vec{k}')e^{-i(k-k')\cdot x}\Big)$$
$$\to i\Big(\hat{a}^\dagger_m(\vec{k})\hat{a}_n(\vec{k})-\hat{b}_m(\vec{k})\hat{a}_n(-\vec{k})e^{-2i\omega(\vec{k})x_o}+\hat{a}^\dagger_m(\vec{k})\hat{b}^\dagger_n(-\vec{k})e^{2i\omega(\vec{k})x_o}-\hat{b}_m(\vec{k})\hat{b}^\dagger_n(\vec{k})\Big)$$

Since there is a factor of i multiplying these above two results, adding these two expressions together is a cynch:

$$\hat{Q}^k = -i\sigma^k_{mn}\int d^3x\Big(\hat{\Pi}_m\hat{\phi}_n-\hat{\phi}^*_m\hat{\Pi}^*_n\Big) = \sigma^k_{mn}\int\frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\Big(\hat{a}^\dagger_m(\vec{k})\hat{a}_n(\vec{k})-\hat{b}_m(\vec{k})\hat{b}^\dagger_n(\vec{k})\Big)$$

This is distinct, but reminiscent of, our conserved charge in the case of a $U(1)$ symmetry, which described particle number conservation. We can actually see particle conservation if we notice that there is another linearly independent matrix past our Pauli matrices: the identity matrix! Notice the cancellation of the time dependent factors, so this conserved charge is indeed time independent! That is, it's a constant of motion.

8. Let's now express the Hamiltonian in terms of our creation and annihilation operators. For simplicity I'll work out the case of a single component complex scalar field as we did in 2.6.

   We begin by noting we'll need the following sum of terms:

$$\hat{\Pi}\hat{\Pi}^*(x)+\nabla\hat{\phi}\cdot\nabla\hat{\phi}^*(x) = \int\int\frac{d^3k}{2\omega(\vec{k})(2\pi)^3}\frac{d^3k'}{2\omega(\vec{k}')(2\pi)^3}\Big(\omega(\vec{k})\omega(\vec{k}')+\vec{k}\cdot\vec{k}'\Big)$$
$$\times\Big(\hat{a}(\vec{k})e^{-ik\cdot x}-\hat{b}^\dagger(\vec{k})e^{ik\cdot x}\Big)\Big(\hat{a}^\dagger(\vec{k}')e^{ik'\cdot x}-\hat{b}(\vec{k}')e^{-ik'\cdot x}\Big)$$

We find $\nabla\hat{\phi}(x)$ in the same way we found $\hat{\Pi}(x)$. The parentheses foil out to:

$$\Bigg(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k'})e^{i(\vec{k}-\vec{k'})\cdot\vec{x}}e^{-i(\omega(\vec{k})-\omega(\vec{k'}))x_0} - \hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k'})e^{-i(\vec{k}+\vec{k'})\cdot\vec{x}}e^{i(\omega(\vec{k})+\omega(\vec{k'}))x_0}$$
$$- \hat{a}(\vec{k})\hat{b}(\vec{k'})e^{i(\vec{k}+\vec{k'})\cdot\vec{x}}e^{-i(\omega(\vec{k})+\omega(\vec{k'}))x_0} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k'})e^{-i(\vec{k}-\vec{k'})\cdot\vec{x}}e^{i(\omega(\vec{k})-\omega(\vec{k'}))x_0}\Bigg)$$

Let's just put a pin in this guy for now, and move on to the next expansion:

$$\hat{\phi}\hat{\phi}^*(x) = \int\int \frac{d^3k}{2\omega(\vec{k})(2\pi)^3}\frac{d^3k'}{2\omega(\vec{k'})(2\pi)^3}\Big(\hat{a}(\vec{k})e^{-ik\cdot x} + \hat{b}^\dagger(\vec{k})e^{ik\cdot x}\Big)\Big(\hat{a}^\dagger(\vec{k'})e^{ik'\cdot x} + \hat{b}(\vec{k'})e^{-ik'\cdot x}\Big)$$

We foil out the terms to find:

$$\Bigg(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k'})e^{i(\vec{k}-\vec{k'})\cdot\vec{x}}e^{-i(\omega(\vec{k})-\omega(\vec{k'}))x_0} + \hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k'})e^{-i(\vec{k}+\vec{k'})\cdot\vec{x}}e^{i(\omega(\vec{k})+\omega(\vec{k'}))x_0}$$
$$+ \hat{a}(\vec{k})\hat{b}(\vec{k'})e^{i(\vec{k}+\vec{k'})\cdot\vec{x}}e^{-i(\omega(\vec{k})+\omega(\vec{k'}))x_0} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k'})e^{-i(\vec{k}-\vec{k'})\cdot\vec{x}}e^{i(\omega(\vec{k})-\omega(\vec{k'}))x_0}\Bigg)$$

Now we recall to obtain the Hamiltonian we integrate the Hamiltonian density over spatial coordinates. Using the identity,

$$\int d^3x e^{i(\vec{k}\pm\vec{k'})\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{k}-\vec{k'})$$

we can seriously clean up this mess! Indeed, making the necessary substitutions, and performing the integral over d^3x we find:

$$\hat{H} = \int d^3x\Big(\hat{\Pi}\cdot\hat{\Pi}^* + \nabla\hat{\phi}\cdot\nabla\hat{\phi}^* + V(|\hat{\phi}|^2)\Big) = \int d^3x\Big(\hat{\Pi}\cdot\hat{\Pi}^* + \nabla\hat{\phi}\cdot\nabla\hat{\phi}^* + m^2|\hat{\phi}|^2\Big)$$
$$= \int\int \frac{d^3k}{2\omega(\vec{k})(2\pi)^3}\frac{d^3k'}{2\omega(\vec{k'})}$$
$$\Bigg[\delta^3(\vec{k}-\vec{k'})\Big(\omega(\vec{k})\omega(\vec{k'}) + \vec{k}\vec{k'} + m^2\Big)\Big(\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k'})e^{i(\omega(\vec{k})-\omega(\vec{k'}))x_o} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k'})e^{-i(\omega(\vec{k})-\omega(\vec{k'}))x_o}\Big)$$
$$+ \delta^3(\vec{k}+\vec{k'})\Big(-\omega(\vec{k})\omega(\vec{k'}) - \vec{k}\vec{k'} + m^2\Big)\Big(\hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k'})e^{-i(\omega(\vec{k})+\omega(\vec{k'}))x_o} + \hat{a}(\vec{k})\hat{b}(\vec{k'})e^{i(\omega(\vec{k})+\omega(\vec{k'}))x_o}\Big)\Bigg]$$

After collapsing the delta functions, the second term goes away. This is due to the fact the delta function sends $\vec{k'}\to-\vec{k}$, and our frequency is just the energy, which we know from our equations of motion is given by: $\omega^2(\vec{k}) = \vec{k}^2 + m^2 \to -\omega^2(\vec{k}) + \vec{k}^2 + m^2 = 0$. Similarly, for the first term we find that $\vec{k'}\to\vec{k}$ so then $\omega(\vec{k})\omega(\vec{k'}) + \vec{k}\vec{k'} + m^2 \to 2\omega^2(\vec{k})$. Thus, we find:

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\omega(\vec{k})\Big[\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\Big]$$

Finally, for a n-component complex scalar field we find:

$$\begin{aligned}
\hat{H} &= \int d^3x\Big(\hat{\Pi}_m \cdot \hat{\Pi}^*_m + \nabla\hat{\phi}_m \cdot \nabla\hat{\phi}^*_m + V(|\hat{\phi}_m|^2)\Big) \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\omega(\vec{k})\left[\hat{a}_m(\vec{k})\hat{a}^\dagger_m(\vec{k}) + \hat{b}^\dagger_m(\vec{k})\hat{b}_m(\vec{k})\right] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\omega(\vec{k})\left[\hat{a}^\dagger_m(\vec{k})\hat{a}_m(\vec{k}) + \hat{b}^\dagger_m(\vec{k})\hat{b}_m(\vec{k})\right] + \sum_m \int d^3k\frac{\omega(\vec{k})}{2}\delta^3(0)
\end{aligned}$$

The delta function is related to the infinite volume of space: $\delta^3(0) = V/(2\pi)^3$. Note, we are summing over m!

Thus, the normal ordered Hamiltonian is:

$$:\hat{H}: = \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\omega(\vec{k})\left[\hat{a}^\dagger_m(\vec{k})\hat{a}_m(\vec{k}) + \hat{b}^\dagger_m(\vec{k})\hat{b}_m(\vec{k})\right] \tag{28}$$

The ground state is defined by:

$$\hat{a}^\dagger_m(\vec{k})\hat{a}_m(\vec{k})|0\rangle = \hat{b}^\dagger_m(\vec{k})\hat{b}_m(\vec{k})|0\rangle = 0$$

This tells us that the ground state energy of our system is $E_o = 2\int d^3k\frac{\omega(\vec{k})}{2}\delta^3(0)$, and has occupation numbers $n_{a\vec{k}} = n_{b\vec{k}} = 0$. With normal ordering we shift the ground state energy to zero.

<u>What about the momentum</u>? Well we go back through the riveting procedure of substituting in our creation and annihilation operators once more! Recall:

$$\begin{aligned}
\hat{P} &= \int d^3x\hat{\Pi}\nabla\hat{\phi} + h.c. \\
&= \int d^3x \int\int \frac{d^3k}{2(2\pi)^3}\frac{d^3k'}{(2\pi)^3 2\omega(\vec{k'})}\vec{k'}\Big(\hat{a}(\vec{k})e^{ik\cdot x} - \hat{b}^\dagger(\vec{k})e^{-ik\cdot x}\Big)\Big(\hat{a}^\dagger(\vec{k'})e^{-ik'\cdot x} - \hat{b}(\vec{k'})e^{ik'\cdot x}\Big) + h.c.
\end{aligned}$$

So foiling again:

$$\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k'})e^{i(k-k')\cdot x} - \hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k'})e^{-i(k+k')\cdot x} - \hat{a}(\vec{k})\hat{b}(\vec{k'})e^{i(k+k')\cdot x} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k'})e^{-i(k-k')\cdot x}$$

Which becomes on substituting back into our integral:

$$\begin{aligned}
\hat{P} = \int\int \frac{d^3k}{2(2\pi)^3}\frac{d^3k'}{2\omega(\vec{k'})}\vec{k'}\Big[&\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k'})\delta^3(\vec{k}-\vec{k'})e^{i(\omega(\vec{k})-\omega(\vec{k'}))\cdot x_o} - \hat{b}^\dagger(\vec{k})\hat{a}^\dagger(\vec{k'})\delta^3(\vec{k}+\vec{k'})e^{-i(\omega(\vec{k})+\omega(\vec{k'}))\cdot x_o} \\
&- \hat{a}(\vec{k})\hat{b}(\vec{k'})\delta^3(\vec{k}+\vec{k'})e^{i(\omega(\vec{k})+\omega(\vec{k'}))\cdot x_o} + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k'})\delta^3(\vec{k}-\vec{k'})e^{-i(\omega(\vec{k})-\omega(\vec{k'}))\cdot x_o}\Big] + h.c. \\
= \int \frac{d^3k}{2(2\pi)^3 2\omega(\vec{k})}\vec{k}\Big[&\hat{a}(\vec{k})\hat{a}(\vec{k})^\dagger + \hat{b}^\dagger(\vec{k})\hat{a}(-\vec{k})e^{-i(\omega(\vec{k})-\omega(-\vec{k}))\cdot x_o} \\
&+ \hat{a}(\vec{k})\hat{b}^\dagger(-\vec{k})e^{i(\omega(\vec{k})-\omega(-\vec{k}))\cdot x_o} + \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k})\Big] + h.c.
\end{aligned}$$

Since ω depends on the square of k, we can drop all the exponentials. Furthermore, when we do this, the remaining term:

$$\vec{k}\left[\hat{a}(\vec{k})\hat{b}^{\dagger}(-\vec{k}) + \hat{a}(-\vec{k})\hat{b}^{\dagger}(\vec{k})\right]$$

is odd under $\vec{k}$, so it integrates to zero. Thus, we finally find:

$$\begin{aligned}\hat{P} &= \int \frac{d^3k}{2(2\pi)^3 2\omega(\vec{k})}\vec{k}\left[\hat{a}(\vec{k})\hat{a}^{\dagger}(\vec{k}) + \hat{b}(\vec{k})\hat{b}^{\dagger}(\vec{k})\right] + h.c. \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\vec{k}\left[\hat{a}^{\dagger}(\vec{k})\hat{a}(\vec{k}) + \hat{b}^{\dagger}(\vec{k})\hat{b}(\vec{k})\right] \\ &= :\hat{P}:\end{aligned} \tag{29}$$

The last equality follows from the creation and annihilation operator commutation relations and:

$$\int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\vec{k}\delta^3(0) = 0$$

as well as adding the Hermitian conjugate. Our momentum operator is already normal ordered! The expected result follows:

$$\hat{P}|0\rangle =: \hat{P}|0\rangle := 0$$

Finally we normal order our $SU(2)$ generators in respect to this very same ground state:

$$\begin{aligned}\hat{Q}^k &= \sigma^k_{mn}\int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left(\hat{a}^{\dagger}_m(\vec{k})\hat{a}_n(\vec{k}) - \hat{b}_m(\vec{k})\hat{b}^{\dagger}_n(\vec{k})\right) \\ \rightarrow: \hat{Q}^k : &= \sigma^k_{mn}\int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left(\hat{a}^{\dagger}_m(\vec{k})\hat{a}_n(\vec{k}) - \hat{b}^{\dagger}_n(\vec{k})\hat{b}_m(\vec{k})\right)\end{aligned} \tag{30}$$

Clearly:

$$:\hat{Q}^k:|0\rangle = 0$$

9. Using our creation operators we can build up our single particle states from the vacuum in the usual fashion:

$$\begin{aligned}\hat{a}^{\dagger}_i(\vec{q})|0\rangle &= |n_{ai}(\vec{q}) = 1\rangle \equiv |a_i(\vec{q})\rangle \\ \hat{b}^{\dagger}_i(\vec{q})|0\rangle &= |n_{bi}(\vec{q}) = 1\rangle \equiv |b_i(\vec{q})\rangle\end{aligned}$$

Using the commutation relations of our creation and annihilation operators we find the excitation spectrum for these single particles states to be:

$$\langle:\hat{H}:\rangle = \omega(\vec{q}) = \sqrt{|\vec{q}|^2 + m^2}$$

Note this is independent of i (species) as well as if we are talking about particles or anti-particles; thus, ***there is a four-fold degeneracy***. If we focus on just the particles or the anti-particles alone, we see that the degeneracy is 2. This means the single particle states of a single "type" (same i but complex conjugates) form a $j = 1/2$ representation of $SU(2)$. Indeed, one can show:

$$\begin{aligned}:\hat{Q}^k:|a_i(\vec{q})\rangle &= \sigma^k_{ii}|a_i(\vec{q})\rangle \\ :\hat{Q}^k:|b_i(\vec{q})\rangle &= -\sigma^k_{ii}|b_i(\vec{q})\rangle\end{aligned}$$

That is the particles are the spin-up states while the anti-particles are the spin down states[10], which is indeed what you'd expect for the $j = 1/2$ states! We could also show:

$$:\hat{P}: |a_i(\vec{q})\rangle = \vec{q}|a_i(\vec{q})\rangle$$
$$:\hat{P}: |b_i(\vec{q})\rangle = \vec{q}|b_i(\vec{q})\rangle$$

This completes our list of Quantum numbers as well as this assignment!

[10]We should note that we have particles and antiparticles in another sense due to the underlying $U(1)$ symmetry, which was previously pointed out.

Appendix

A Foiling out Bogoliubov Transform

We begin by multiplying everything out:

$$\hat{c}(q) = \cosh\big(\theta(q)\big)\hat{a}(q) + \sinh\big(\theta(q)\big)\hat{b}^\dagger(q) \rightarrow \hat{c}^\dagger(q) = \cosh\big(\theta(q)\big)\hat{a}^\dagger(q) + \sinh\big(\theta(q)\big)\hat{b}(q)$$

$$\hat{d}(q) = \cosh\big(\theta(q)\big)\hat{b}(q) + \sinh\big(\theta(q)\big)\hat{a}^\dagger(q) \rightarrow \hat{d}^\dagger(q) = \cosh\big(\theta(q)\big)\hat{b}^\dagger(q) + \sinh\big(\theta(q)\big)\hat{a}(q)$$

Let's invert these equations so that we have an explicit expression for $\hat{a}$ and $\hat{b}$ in terms of $\hat{c}$ and $\hat{d}$. This is easy enough if we recall: $\cosh^2(x) - \sinh^2(x) = 1$. Note this identity also preserves our desired commutation relations! We find:

$$\hat{a}(q) = \cosh\big(\theta(q)\big)\hat{c}(q) - \sinh\big(\theta(q)\big)\hat{d}^\dagger(q) \rightarrow \hat{a}^\dagger(q) = \cosh\big(\theta(q)\big)\hat{c}^\dagger(q) - \sinh\big(\theta(q)\big)\hat{d}(q)$$

$$\hat{b}(q) = \cosh\big(\theta(q)\big)\hat{d}(q) - \sinh\big(\theta(q)\big)\hat{c}^\dagger(q) \rightarrow \hat{b}^\dagger(q) = \cosh\big(\theta(q)\big)\hat{d}^\dagger(q) - \sinh\big(\theta(q)\big)\hat{c}(q)$$

An astute choice of $\theta(q)$ will cancel all cross terms resulting in a diagonalized Hamiltonian. Let's expand each term in our summation individually, then we can group like terms, and see where this brings us:

1. $\hat{a}^\dagger(q)\hat{a}(q) = \big[\cosh\big(\theta(q)\big)\hat{c}^\dagger(q) - \sinh\big(\theta(q)\big)\hat{d}(q)\big]\big[\cosh\big(\theta(q)\big)\hat{c}(q) - \sinh\big(\theta(q)\big)\hat{d}^\dagger(q)\big]$

$$= \cosh^2\big(\theta(q)\big)\hat{c}^\dagger(q)\hat{c}(q) + \sinh^2\big(\theta(q)\big)\hat{d}(q)\hat{d}^\dagger(q) - \cosh\big(\theta(q)\big)\sinh\big(\theta(q)\big)\Big(\hat{c}^\dagger(q)\hat{d}^\dagger(q) + \hat{d}(q)\hat{c}(q)\Big)$$

2. $\hat{b}^\dagger(q)\hat{b}(q) = \big[\cosh\big(\theta(q)\big)\hat{d}^\dagger(q) - \sinh\big(\theta(q)\big)\hat{c}(q)\big]\big[\cosh\big(\theta(q)\big)\hat{d}(q) - \sinh\big(\theta(q)\big)\hat{c}^\dagger(q)\big]$

$$= \cosh^2\big(\theta(q)\big)\hat{d}^\dagger(q)\hat{d}(q) + \sinh^2\big(\theta(q)\big)\hat{c}(q)\hat{c}^\dagger(q) - \cosh\big(\theta(q)\big)\sinh\big(\theta(q)\big)\Big(\hat{c}^\dagger(q)\hat{d}^\dagger(q) + \hat{d}(q)\hat{c}(q)\Big)$$

3. $\hat{a}(q)\hat{b}(q) = \big[\cosh\big(\theta(q)\big)\hat{c}(q) - \sinh\big(\theta(q)\big)\hat{d}^\dagger(q)\big]\big[\cosh\big(\theta(q)\big)\hat{d}(q) - \sinh\big(\theta(q)\big)\hat{c}^\dagger(q)\big]$

$$= \cosh^2\big(\theta(q)\big)\hat{c}(q)\hat{d}(q) + \sinh^2\big(\theta(q)\big)\hat{c}^\dagger(q)\hat{d}^\dagger(q)) - \cosh\big(\theta(q)\big)\sinh\big(\theta(q)\big)\Big(\hat{c}(q)\hat{c}^\dagger(q) + \hat{d}^\dagger(q)\hat{d}(q)\Big)$$

4. $\hat{a}^\dagger(q)\hat{b}^\dagger(q) = \big[\cosh\big(\theta(q)\big)\hat{c}^\dagger(q) - \sinh\big(\theta(q)\big)\hat{d}(q)\big]\big[\cosh\big(\theta(q)\big)\hat{d}^\dagger(q) - \sinh\big(\theta(q)\big)\hat{c}(q)\Big]$

$$= \cosh^2\big(\theta(q)\big)\hat{c}^\dagger(q)\hat{d}^\dagger(q) + \sinh^2\big(\theta(q)\big)\hat{c}(q)\hat{d}(q)) - \cosh\big(\theta(q)\big)\sinh\big(\big(\theta(q)\big)\Big(\hat{c}^\dagger(q)\hat{c}(q) + \hat{d}(q)\hat{d}^\dagger(q)\Big)$$

Substituting these expressions into our Hamiltonian and normal ordering we find (this is why you should use a matrix!):

$$\Big[\hat{a}^\dagger(q)\hat{a}(q) + \hat{b}^\dagger(q)\hat{b}(q) + \cos(q)\hat{a}(q)\hat{b}(q) + \cos(q)\hat{a}^\dagger(q)\hat{b}^\dagger(q)\Big]$$
$$= 2\Big[\cosh\big(2\theta(q)\big) - \sinh\big(2\theta(q)\big)\cos(q)\Big]\Big(1 + \hat{c}^\dagger(q)\hat{c}(q) + \hat{d}^\dagger(q)\hat{d}(q)\Big)$$
$$+\Big[\big[\cosh^2\big(\theta(q)\big) + \sinh^2\big(\theta(q)\big)\big]\cos(q) - 2\cosh\big(\theta(q)\big)\sinh\big(\theta(q)\big)\Big]\Big(\hat{c}(q)\hat{d}(q) + \hat{c}^\dagger(q)\hat{d}^\dagger(q)\Big)$$

This shows us what choice we need to make on our angle $\theta(q)$; indeed, setting the coefficient of the cross term to zero gives:

$$\begin{aligned}\cos(q) &= \frac{2\cosh\big(\theta(q)\big)\sinh\big(\theta(q)\big)}{\cosh^2\big(\theta(q)\big)+\sinh^2\big(\theta(q)}\\ &= \frac{2\tanh\big(\theta(q)\big)}{1+\tanh^2\big(\theta(q)}\\ &= \tanh\big(2\theta(q)\big)\end{aligned} \tag{31}$$

This implies:

$$\theta(q) = \frac{1}{2}\tanh^{-1}(\cos(q))$$

This is what we got before with 10 times the work!

B Local SU(2) Transformation

We are interested in the following symmetry: $\mathcal{L}(U\Phi) = \mathcal{L}(\Phi)$. Our Lagrangian density above is only invariant for constant transforms; i.e. $\partial_\mu U_{ab}(x) = 0\ \forall a, b$. In order to promote our global symmetry to a local one we will need to define a covariant derivative. Inspired by our simpler case of a $U(1)$ symmetry we define:

$$D_\mu = I\partial_\mu - igA_\mu$$

where I in the 2x2 identity matrix and A_μ is a 2x2 matrix valued vector field. We can figure out how A_μ transforms if we invoke:

$$\big(D_\mu\Phi\big)' = U\big(D_\mu\Phi\big)$$

This tells us that the field itself transforms as the covaraint derivative does provided we choose the transformation properties of our gauge field wisely. A simple calculation shows:

$$\big(I\partial_\mu - igA'_\mu\big)U\Phi = U\Big[I\partial_\mu\Phi + iU^{-1}\big(I(\partial_\mu U) - igA'_\mu U\big)\Phi\Big]$$

$$= U\big(\partial_\mu\Phi - igA_\mu\big)\Phi$$

From this we deduce, for our transformation properties to be realized, that:

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} = UA_\mu U^{-1} + \frac{i}{g}U(\partial_\mu U^{-1})$$

We can actually expand this vector field using the generators of SU(2); with the above generators of our group we find:

$$(A_\mu(x))_{ab} = A^k_\mu(x)\lambda^k_{ab}$$

Here a, b and k all run over N. Following the work in problem 2.2 we see that under a local symmetry we'd have also need to consider how the gauge field varies. In the adjoint representation the gauge fields transforms like:

$$\delta A^k_\mu(x) \approx if^{ksj}A^j_\mu(x)\theta^s(x) + I\frac{1}{g}\partial_\mu\theta^k(x)$$

So since $\big[\sigma^i, \sigma^j\big] = 2i\varepsilon_{ijk}\sigma^k$ for SU(2), we see:

$$\delta\phi_a \approx i\sigma^k_{ab}\phi_b\theta^k(x)$$

and,

$$\delta A_\mu^k(x) \approx 2i\varepsilon^{ksj} A_\mu^j(x)\theta^s(x) + I\frac{1}{g}\partial_\mu \theta^k(x)$$

We use these relations to derive conserved currents, and thus, charges. We know already:

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\partial_\nu\phi_a + \phi_a \leftrightarrow \phi_a^* + \frac{\delta\mathcal{L}}{\delta A_\mu^k}\delta A_\mu^k + \frac{\delta\mathcal{L}}{\delta\partial_\nu A_\mu^k}\delta\partial_\nu A_\mu^k$$

Using our equations of motion we can see:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\partial_\nu\phi_a + \phi_a \leftrightarrow \phi_a^* &= \partial_\nu\Big[\frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_a^*}\delta\phi_a^*\Big] \\ &= \frac{i}{2}\partial_\nu\Big[\Big(\sigma_{ab}^k\big(D^\nu\phi_a\big)^*\phi_b - \sigma_{ba}^k\big(D^\nu\phi_a\big)\phi_b^*\Big)\theta^k(x)\Big] \\ &\equiv \partial_\nu\Big[j^{\nu k}\theta^k(x)\Big]\end{aligned}$$

We can see the Lagrangian does not depend on $\partial_\nu A_\mu$; thus:

$$\frac{\delta\mathcal{L}}{\delta A_\mu^k}\delta A_\mu^k + \frac{\delta\mathcal{L}}{\delta\partial_\nu A_\mu^k}\delta\partial_\nu A_\mu^k = \frac{\delta\mathcal{L}}{\delta A_\mu^k}\delta A_\mu^k$$

we put everything together and find:

$$\delta\mathcal{L} = \partial_\nu\Big[j^{\nu k}\theta^k(x)\Big] + \Big[2i\varepsilon^{ksj}A_\mu^j(x)\theta^s(x) + I\frac{1}{g}\partial_\mu\theta^k(x)\Big]\frac{\delta\mathcal{L}}{\delta A_\mu^k}$$

For θ constant this reduces to the answer in problem 2.2 (after choosing a gauge where $A_\mu^k = 0$).